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A Strange Application of Kolmogorov Complexity*

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Abstract. This paper deals with two similar inequalities:

$$2K(\langle A, B, C\rangle) \leq K(\langle A, B\rangle) + K(\langle A, C\rangle) + O(\log n), \quad (1)$$

$$2KP(\langle A, B, C\rangle) \leq KP(\langle A, B\rangle) + KP(\langle A, C\rangle) + KP(\langle B, C\rangle) + O(1), \quad (2)$$

where K denotes simple Kolmogorov entropy (i.e., the very first version of Kolmogorov complexity having been introduced by Kolmogorov himself) and KP denotes prefix entropy (self-delimiting complexity by the terminology of Li and Vitanyi [1]). It turns out that from (1) the following well-known geometric fact can be inferred:

$$|V|^2 \leq |S_{xy}| \cdot |S_{yz}| \cdot |S_{xz}|,$$

where V is a set in three-dimensional space, S_{xy}, S_{yz}, S_{xz} are its three two-dimensional projections, and $|W|$ is the volume (or the area) of W. Inequality (2), in its turn, is a corollary of the well-known Cauchy–Schwarz inequality. So the connection between geometry and Kolmogorov complexity works in both directions.

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1. Inequality

For any binary words A, B, and C whose length does not exceed n we have

$$2K(\langle A, B, C\rangle) \leq K(\langle A, B\rangle) + K(\langle A, C\rangle) + K(\langle B, C\rangle) + O(\log n).$$

(By K we denote the Kolmogorov complexity as defined in the original Kolmogorov article, see, e.g., p. 198 of [1]; it is sometimes called the “simple Kolmogorov entropy” and is denoted by KS. Because of the term $O(\log n)$ other versions may be used, see above.)

2. Proof

We rewrite the inequality as

$$\big(K(\langle A, B, C\rangle) - K(\langle A, B\rangle)\big) + \big(K(\langle A, B, C\rangle) - K(\langle A, C\rangle)\big)$$
$$\leq K(\langle B, C\rangle) + O(\log n).$$

Now we use the equality $K(\langle X, Y\rangle) = K(X) + K(Y \mid X) + O(\log n)$ where $K(Y \mid X)$ is the conditional complexity of Y when X is known and get

$$K(C \mid \langle A, B\rangle) + K(B \mid \langle A, C\rangle) \leq K(\langle B, C\rangle) + O(\log n).$$

Now it remains to use that $K(C \mid \langle A, B\rangle) \leq K(C \mid B)$, $K(B \mid \langle A, C\rangle) \leq K(B)$, and refer to the equality mentioned above.

3. Application

We use this inequality to prove a (well-known) geometric fact which seems to have nothing in common with the Kolmogorov complexity. Assume that we have a set V in a three-dimensional space with coordinates x, y, z. Consider three projections S_{xy}, S_{yz} and S_{xz}. By $|W|$ we mean the volume (or area) of W.

Theorem (A Continuous version).

$$|V|^2 \leq |S_{xy}| \cdot |S_{yz}| \cdot |S_{xz}|.$$

To avoid difficulties (nonmeasurable sets, etc.) we consider a finite version:

Theorem (A Discrete Version). *Let X, Y, Z be finite sets, $V \subset X \times Y \times Z$, and $S_{xy} \subset X \times Y$, $S_{xz} \subset X \times Z$, and $S_{yz} \subset Y \times Z$ are projections of V. Then*

$$(\#V)^2 \leq (\#S_{xy}) \cdot (\#S_{yz}) \cdot (\#S_{xz}).$$

Proof. Choose a random sequence $v = \langle v_1 \cdots v_n\rangle \in V^n$. Its complexity is $n \cdot (\log \#V) + O(\log n)$. Now remember that v_i is a triple $\langle x_i, y_i, z_i\rangle$, therefore v can be considered as a triple $\langle x, y, z\rangle$ where $x = \langle x_1 \cdots x_n\rangle$, $y = \langle y_1 \cdots y_n\rangle$, and $z = \langle z_1 \cdots z_n\rangle$. Now use our inequality:

$$2K(v) \le K(\langle x, y\rangle) + K(\langle y, z\rangle) + K(\langle x, z\rangle) + O(\log n).$$

Remember that $\langle x, y\rangle$ can also be considered as a sequence of n elements of S_{xy}, therefore $K(\langle x, y\rangle) \le n \cdot \log \# S_{xy} + O(\log n)$. Using similar inequalities for $\langle y, z\rangle$ and $\langle x, z\rangle$ we get

$$2n \log \# V = 2K(v) \le n \cdot (\log \# S_{xy} + \log \# S_{yz} + \log \# S_{xz}) + O(\log n).$$

Dividing by n and using that $(\log n)/n \to 0$ we get

$$2 \log \# V \le \log \# S_{xy} + \log \# S_{yz} + \log \# S_{xz}. \qquad \square$$

4. Prefix Entropy and L^2-Inequality

Because of the $O(\log n)$ term the inequality of Section 1 is valid for all usual variants of Kolmogorov complexity (entropy) such as decision entropy, monotone entropy, and prefix entropy. However, the case of prefix entropy (also called the self-delimiting complexity, see p. 209 of [1]) is somehow special because the logarithmic term may be omitted:

$$2KP(\langle A, B, C\rangle) \le KP(\langle A, B\rangle) + KP(\langle A, C\rangle) + KP(\langle B, C\rangle) + O(1).$$

It turns out that this inequality can be proved analytically, using the Cauchy–Schwarz inequality. Indeed, recall that KP may be defined as a $-\log_2 P$ where P is an *a priori* probability on the set of natural numbers. Therefore, we should prove that

$$P^2(\langle A, B, C\rangle) \ge c \cdot P(\langle A, B\rangle) \cdot P(\langle A, C\rangle) \cdot P(\langle B, C\rangle)$$

for some positive constant c. The *a priori* probability is defined as a maximum enumerable from below function on natural numbers with a finite sum; therefore it is enough to show that

$$\sum_{A,B,C} \sqrt{P(\langle A, B\rangle) \cdot P(\langle A, C\rangle) \cdot P(\langle B, C\rangle)} < +\infty.$$

This fact is a consequence of the following inequality which we prefer to write using integrals instead of sums:

$$\begin{aligned}
&\iiint f(x, y) g(x, z) h(y, z)\, dx\, dy\, dz \\
&\quad \le \left(\iint f^2(x, y)\, dx\, dy\right)^{1/2} \left(\iint g^2(x, z)\, dx\, dz\right)^{1/2} \\
&\qquad \times \left(\iint h^2(x, y)\, dy\, dy\right)^{1/2}.
\end{aligned}$$

This is a version of the Cauchy–Schwarz inequality containing three L^2-norms in the right-hand side instead of two; it can be easily reduced to an ordinary Cauchy–Schwarz inequality:

$$
\begin{aligned}
&\iiint f(x,y)g(x,z)h(y,z)\,dx\,dy\,dz \\
&\quad= \iint h(y,z)\left(\int f(x,y)g(x,z)\,dx\right) dy\,dz \\
&\quad\le \iint h(y,z)\left(\int f^2(x,y)\,dx\right)^{1/2}\left(\int g^2(x,z)\,dx\right)^{1/2} dy\,dz \\
&\quad\le \left(\iint h^2(y,z)\,dy\,dz\right)^{1/2} \\
&\qquad\times\left(\iint \underbrace{\left(\int f^2(x,y)\,dx\right)}_{\varphi(y)}\underbrace{\left(\int g^2(x,z)\,dx\right)}_{\psi(z)} dy\,dz\right)^{1/2} \\
&\quad= \left(\iint h^2(y,z)\,dy\,dz\right)^{1/2}\left(\iint \varphi(y)\psi(z)\,dy\,dz\right)^{1/2} \\
&\quad= \left(\iint h^2(y,z)\,dy\,dz\right)^{1/2}\left(\int \varphi(y)\,dy\int \psi(z)\,dz\right)^{1/2} \\
&\quad= \left(\iint h^2(y,z)\,dy\,dz\right)^{1/2}\left(\iint f^2(x,y)\,dx\,dy\right)^{1/2} \\
&\qquad\times\left(\iint g^2(x,z)\,dx\,dz\right)^{1/2}.
\end{aligned}
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Thus, the connection between geometrical facts and Kolmogorov complexity works in both directions.

Reference

[1] M. Li and P. M. B. Vitanyi, Kolmogorov complexity and its applications, in: J. van Leeuwen (ed.), *Handbook of Theoretical Computer Science*, vol. A, Elsevier, Amsterdam, 1990, pp. 187–254.

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