This column is devoted to mathematics for fun. What better purpose is there for mathematics? To appear here, a theorem or problem or remark does not need to be profound (but it is allowed to be); it may not be directed only at specialists; it must attract and fascinate.

We welcome, encourage, and frequently publish contributions from readers—either new notes, or replies to past columns.

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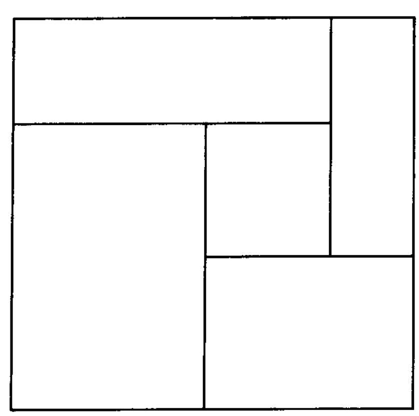
It is always nice to see a simple solution for a difficult problem. Simple but hard to find—otherwise the problem wouldn't be that difficult. The solution may require an unexpected construction, or use some seemingly unrelated theory. In the latter case we often feel uncomfortable: Is this theory really relevant? Or is there some other simple solution which does not use it at all? Here are some examples of such "mysterious solutions."

Semi-Integer Rectangles

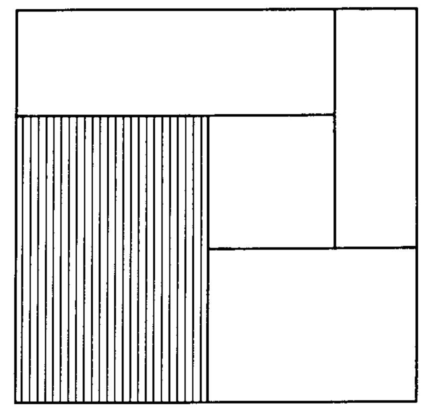
This problem is well known among people running the Moscow Olympiad. We call a rectangle *semi-integer* (for lack of a better name) if one of its sides has integer length.

PROBLEM. A rectangle is cut into several semi-integer rectangles (whose sides are parallel to the sides of the initial rectangle). Prove that the initial rectangle is also a semi-integer one.

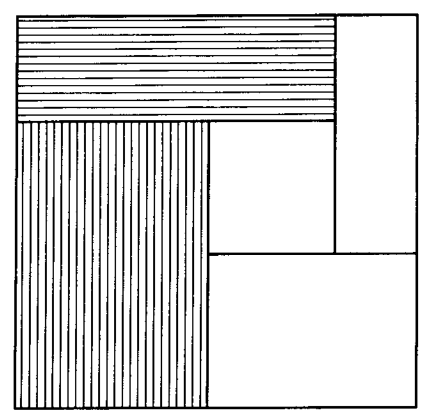
Here is one example.



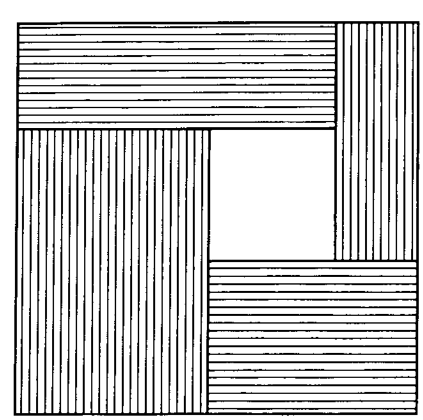
If all five small rectangles in the picture are semi-integer, then the big one is semi-integer too. Let us check that. Consider the lower-left rectangle. Assume that its vertical side is an integer. (On the picture, we indicate a semi-integer rectangle by covering it with lines parallel to the integer side.)



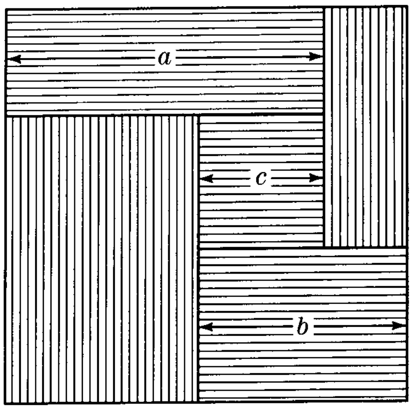
If the upper-left rectangle also has integer vertical side, we are done: the vertical side of the initial rectangle is a sum of two integers and therefore is an integer. Otherwise, we get the following picture:



The same reasoning shows that the only nontrivial case is the following one:



Now we recall that the interior rectangle is a semi-integer one; assume, for example, that its horizontal side is an integer:



Then the horizontal side of the big rectangle has length $a + b - c$, which is an integer.

Now I give a (rather unexpected) proof for the general case. The proof uses periodic functions $f: \mathbb{R} \to \mathbb{R}$ with period 1 and mean value 0 (so $\int_a^{a+1} f(x)\,dx = 0$ for any a). Let us call them *test* functions. If f is a test function and $T \subset \mathbb{R}$ is an interval of integer length, then $\int_T f(x)\,dx = 0$. This property is characteristic for intervals of integer length: if U is an interval whose length is not an integer, it is easy to construct a test function f such that $\int_U f(x)\,dx \neq 0$. For example, if $U = [a, a + h]$ where $h \notin \mathbb{Z}$, take $f(x) = \sin 2\pi(x - a)$; then

$$\int_U f(x)dx = \int_0^h \sin 2\pi t\,dt = \frac{1}{2\pi}(1 - \cos 2\pi h) > 0$$

Now we approach the main point of the proof. Consider two test functions f and g, and combine them into a function of two variables:

$$\phi(x,y) = f(x) \cdot g(y)$$

Such a function will also be called a *test* function (of two variables, so no confusion arises).

LEMMA 1 *For any semi-integer rectangle $D \subset \mathbb{R}^2$ (with sides parallel to OX and OY) and for any test function ϕ of two variables,*

$$\int_D \phi(x,y)\,dx\,dy = 0.$$

Indeed, if $D = S \times T$ where $S,T \subset \mathbb{R}$ are intervals, then

$$\int_D \phi(x,y)\,dx\,dy = \int_{S\times T} f(x)g(y)\,dx\,dy = \left(\int_S f(x)\,dx\right)\left(\int_T g(y)\,dy\right)$$

One of the factors in the right-hand side is 0 because one of the intervals S and T has integer length. (End of proof.)

Now, if some rectangle H can be cut into semi-integer rectangles, the statement of the lemma remains valid: for any test function ϕ of two variables we have

$$\int_H \phi(x,y)\,dx\,dy = 0.$$

Indeed, the integral is the sum of integrals over small rectangles. Therefore, it remains to prove the following

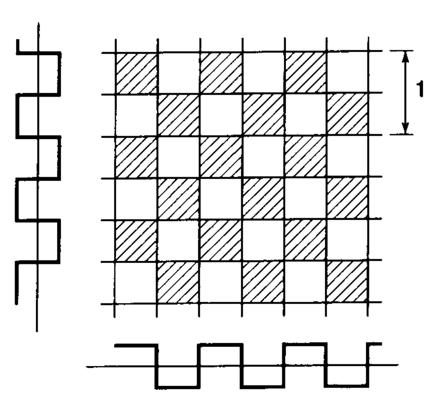
LEMMA 2 *A rectangle H is given. If for any test function ϕ of two variables we have $\int_H \phi(x,y)\,dx\,dy = 0$, then H is a semi-integer rectangle.*

Indeed, if $D = S \times T$ and $\phi(x,y) = f(x) \cdot g(y)$, then

$$\int_D \phi(x,y)\,dx\,dy = \int_S f(x)\,dx \cdot \int_T g(y)\,dy$$

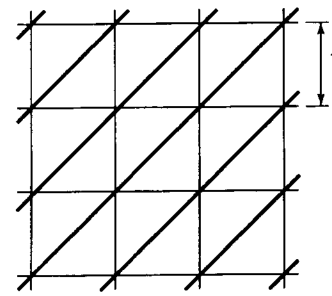
If neither of the intervals S and T has integer length, one can find test functions f and g such that both factors in the right-hand side are non-zero. (See the discussion above.)

Remark. We can make an elementary proof out of this one in the following way. Using "meanders" as test functions f and g, we get a black and white board made of $1/2 \times 1/2$ squares:



LEMMA 3 *The rectangle is a semi-integer one if and only if it covers equal amounts of black and white for any placement of the rectangle on the board. (We assume that in any such placement the rectangle sides are parallel to the board lines.)*

There is another elementary proof similar in spirit. Instead of a black and white board lemma, consider a series of black straight lines with slope 1 going through all integer points:



LEMMA 4 *The semi-integer rectangle covers the same amount of black lines independent of the placement; this property is characteristic for semi-integer rectangles.*

However, all these proofs make us (at least me) uncomfortable: the result is obtained by a simple trick which does not seem to be relevant to the "essence of the problem."

Question: Can one find a natural, "comfortable" solution for this problem?

Another question: Can we replace $\mathbb{Z}$ by an arbitrary subgroup G of the additive group $\mathbb{R}$ and consider "semi-G-rectangles" (one of the sides has length in G) instead of semi-integer rectangles?

Reformulation: assume that a rectangle is cut into smaller rectangles; we select one of the sides for each of small rectangles. Prove that at least one of the sides of the big rectangle is an integer linear combination of the sides selected.

Cube and Tetrahedron

While the problem in the preceding section is a pastime, the next example is famous theorem:

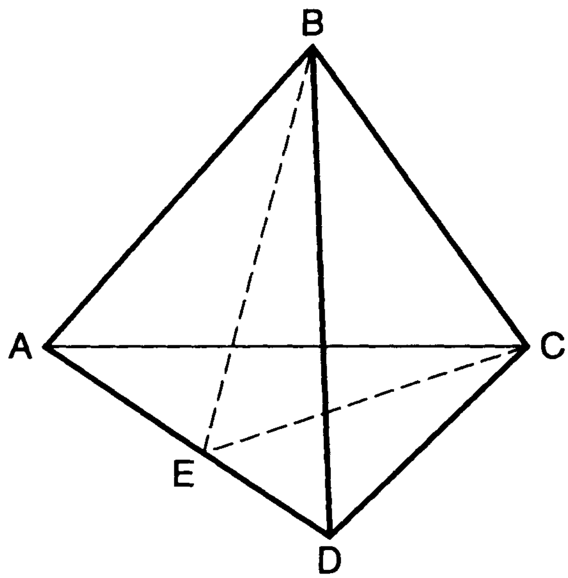
THEOREM 1 *A cube cannot be decomposed into polyhedral parts which can form a regular tetrahedron of the same volume.*

(This statement is in contrast with the two-dimensional situation, where for any two polygons of the same area such a polygonal decomposition is possible.)

The simple (but rather strange) proof goes as follows. We find a "quasi-volume" invariant, i.e., a function v defined on polyhedra which has properties similar to the volume function:

(1) if polyhedra A and B are congruent, then $v(A) = v(B)$;
(2) v is additive: if a polyhedron A is cut into pieces $A_1, \ldots, A_n$, then

$$v(A) = v(A_1) + \cdots + v(A_n)$$



If a cube C and a tetrahedron T were cut into congruent pieces, then not only would their volumes be equal, but also $v(C)$ and $v(T)$ would be equal for the same reason.

Therefore, it is enough to find a function v satisfying (1) and (2) such that $v(C) \neq v(T)$. This function is constructed as follows.

Let a polyhedron P be given. Consider all edges $e_1, \ldots, e_k$ of the polyhedron P. For each edge e_i consider the angle α_i at e_i (i.e., the angle formed by the planes that meet at e_i). Then we define

$$v(P) = \sum e_i f(\alpha_i),$$

where f is some function which will be specified later. It is evident that $v(A) = v(B)$ for congruent A and B (for any function f). However, to meet the requirement (2) we need a very special function. It should satisfy the following conditions:

(a) $f(\alpha + \beta) = f(\alpha) + f(\beta)$;
(b) $f(\pi) = 0$

(Why must such a function exist at all? This question will be discussed later.) Let me explain why v is additive. For example, consider a tetrahedron $ABCD$ which is divided by a plane BCE into two tetrahedra $ABCE$ and $BCDE$.

Let us compare $v(ABCD)$ and $v(ABCE) + v(BCDE)$. What is the difference between these two sums? The edge AD in the first one is replaced by two edges AE and ED, but the angle is the same, so the corresponding terms sum up quite nicely. The angle α at the edge BC is now divided into two angles β and γ, but $\alpha = \beta + \gamma$ and therefore $f(\alpha) = f(\beta) + f(\gamma)$. Two new edges BE and EC appeared. Each of them appears twice, both in $v(ABCE)$ and $v(BCDE)$, and the sum of the corresponding angles is equal to π. Recalling that $f(\alpha) + f(\beta)$ is equal to $f(\alpha + \beta)$ and therefore is equal to 0 when $\alpha + \beta = \pi$, we are done.

Of course, the additive property of v should be checked for the general case, but the idea is more or less clear, so we stop here.

Now, how to construct a function f with the required properties (a) and (b) above? It is easy to see that these conditions imply $f(2\pi) = 0, f(\pi/3) = 0$ and, in general, $f(r\pi) = 0$ for any rational r. Therefore, the only continuous function with this property is the zero function, and we have to look for a discontinuous one.

In fact, the word "construct" is misleading; we cannot construct such a function, we can only prove its existence using the axiom of choice. Consider $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$ (having infinite dimension).

Using the axiom of choice, we prove that any $\mathbb{Q}$-independent subset of $\mathbb{R}$ can be extended to a $\mathbb{Q}$-basis. Then the values of f on a basis may be chosen arbitrarily; after that f is extended uniquely onto $\mathbb{R}$ using $\mathbb{Q}$-linearity. It remains to apply this procedure using $\{\pi\}$ as the $\mathbb{Q}$-independent set and to demand that $f(\pi) = 0$; we still have a lot of freedom choosing values of f on the other basis vectors.

Now return to our cube C and tetrahedron T. All the angles at the cube edges are right angles, so $v(C) = 0$. (Indeed, $f(\pi/2) + f(\pi/2) = f(\pi) = 0$, as mentioned above.) It remains to show that $f(\alpha) \neq 0$ where α is the angle at T-edges. More precisely, we have to check that f *can be chosen* in such a way that $f(\alpha) \neq 0$. To do that we need the ratio α/π to be irrational, so the set $\{\alpha, \pi\}$ will be $\mathbb{Q}$-independent, which is indeed the case.

So goes the proof. Isn't it very strange that a geometric question about polyhedra has something to do with the axiom of choice?

There are general arguments that show that the axiom of choice can be eliminated from any proof of the theorem. (Indeed, for any given number of vertices used in the decomposition, the question whether this decomposition is possible or not can be formulated in the elementary theory of reals, which is, according to a famous Tarski result, decidable. Therefore the existence of a decomposition (for a given number of vertices) can be rephrased as an arithmetic statement. And we know from Gödel that any arithmetic statement which can be proven with the axiom of choice can be proven without it.)

The question remains: why does such a nice short proof use such a plainly irrelevant tool as the axiom of choice?