# Three-Dimensional Solutions for Two-Dimensional Problems

*This column is devoted to mathematics for fun. What better purpose is there for mathematics? To appear here, a theorem or problem or remark does not need to be profound (but it is allowed to be); it may not be directed only at specialists; it must attract and fascinate.*

*We welcome, encourage, and frequently publish contributions from readers—either new notes, or replies to past columns.*

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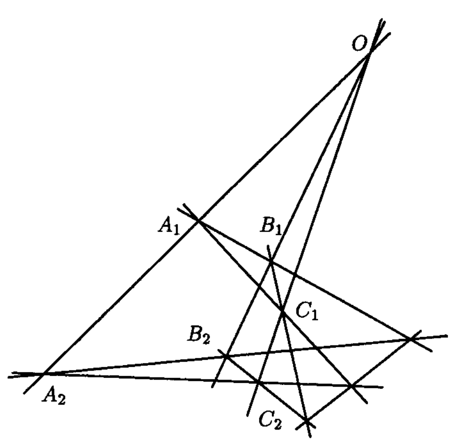
In this issue we continue our collection of nice proofs. I present several examples where a simple but unexpected construction in a three-dimensional space provides a short solution of a plane problem which is rather difficult.

**1.** The first example is so famous that most of you surely know it. However, it is too nice to omit. It is the Desargues theorem.

Theorem. Consider two triangles $A_1B_1C_1$ and $A_2B_2C_2$. Assume that the straight lines $A_1A_2$, $B_1B_2$, and $C_1C_2$ go through a single point $O$. In this case the three intersection points of the corresponding sides (of $A_1B_1$ and $A_2B_2$, of $B_1C_1$ and $B_2C_2$, and, finally, of $A_1C_1$ and $A_2C_2$) lie on a straight line (Fig. 1).

FIGURE 1



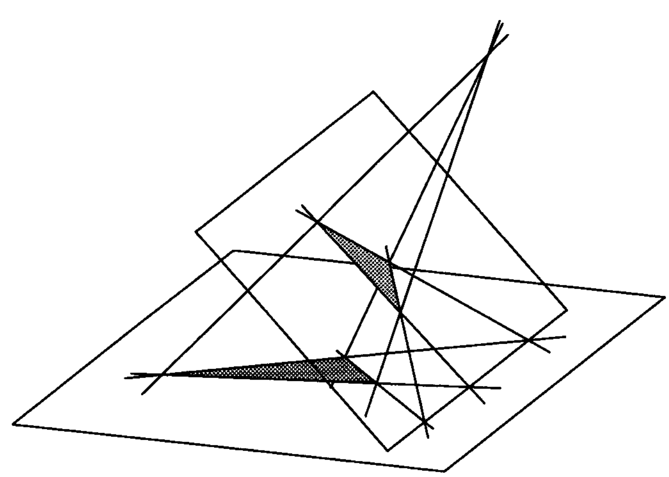

Let us see why this theorem is evident. Imagine a transparency on which the triangle $A_1B_1C_1$ is drawn. This transparency is inclined in such a way that one side of it lies on a horizontal table. A lamp casts light on the transparency, and we see the shadow of the triangle on the table. This shadow is the triangle $A_2B_2C_2$. (This is without loss of generality: any two triangles can be related in this way.) The sides of this triangle are shadows of the sides of the original triangle and intersect them where the transparency touches the tables, i.e., on the line of intersection of two planes (transparency and table). See Fig. 2.

**2.** The second example is the problem about three common chords of three circles. Consider three intersecting circles in a plane. For each two of them we connect the two intersection points by a common chord. We have to prove that these three chords go through one point (Fig. 3).

To see why it is true, imagine a horizontal plane through the center of a sphere. This plane divides the sphere into two hemispheres separated by a circle. We need only the upper hemisphere. Looking at this hemisphere from above, we see a circle (Fig. 4).

Now consider two intersecting hemispheres of this type whose diameter circles lie on the same horizontal

FIGURE 2



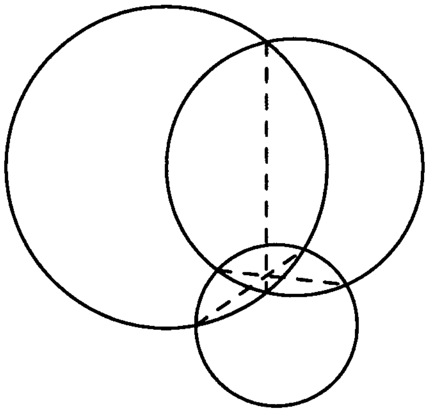


FIGURE 3

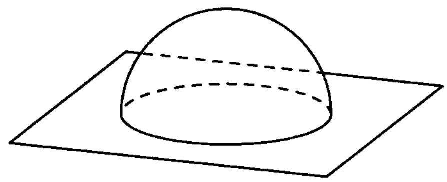


FIGURE 4

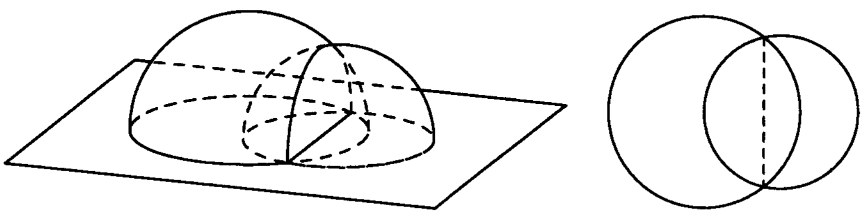
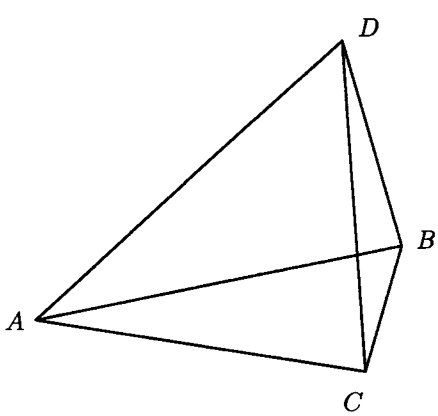


FIGURE 5

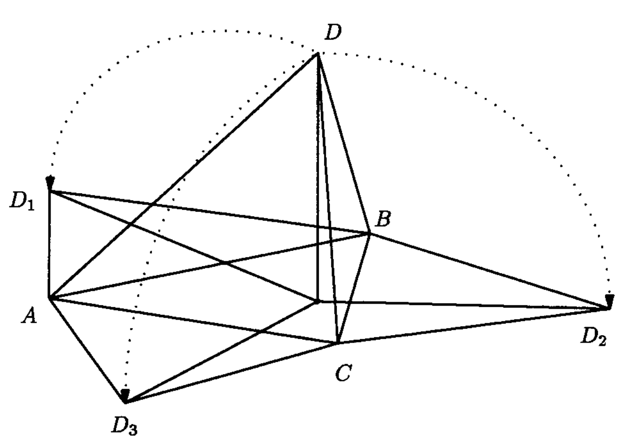
plane and intersect. Looking from above, we see two intersecting circles. A closer look reveals the common chord of these two circles. Indeed, the spheres intersect each other along a circle that is orthogonal to the horizontal plane and therefore is visible from above as a straight line (Fig. 5)

FIGURE 6




Now our problem becomes easy. Imagine three hemispheres based on a horizontal plane. Looking from above, we see three circles (which are diameter circles of those hemispheres). Consider the point where all three hemispheres intersect—in other words, the point where the circle that is the intersection of two spheres, intersects the third one. Looking from above, we see this point as the point of intersection of three common chords, so the problem is solved.

FIGURE 7




**3.** In our third example we start with a space construction and transform it into a plane problem. Consider a paper tetrahedron $ABCD$, with face $ABC$ horizontal (Fig. 6). Let us cut the tetrahedron along the lines $AD$, $BD$ and $CD$ and turn the side faces about the horizontal edges until they are horizontal. We get a plane hexagon $AD_1BD_2CD_3$ (Fig. 7) whose vertices include three copies $D_1$, $D_2$, and $D_3$ of the vertex $D$.

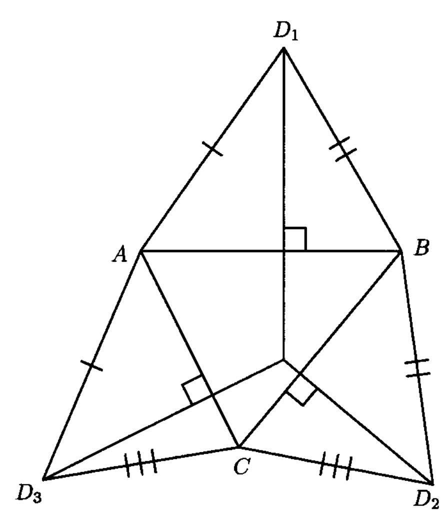
Let us follow the movement of these three copies while the side faces $ABD$, $BCD$, and $ACD$ are turned around the horizontal sides. Each copy moves along a circle that is orthogonal to the horizontal plane and to one of the sides of the triangle $ABC$. Therefore, in the top view, the vertices $D_1$, $D_2$, and $D_3$ move along straight lines orthogonal to the sides of triangle $ABC$ (Fig. 8).

We arrive at the solution of the following problem:

Consider a triangle $ABC$ and three other triangles $ABD_1$, $BCD_2$, $ACD_3$ that have common sides with it. Assume that the sides adjacent to any vertex of the given triangle are equal ($AD_1 = AD_3, BD_1 = BD_2, CD_2 = CD_3$). Consider the altitudes of the three triangles orthogonal to the sides of $ABC$ and going through $D_1$, $D_2$, $D_3$. Prove that these altitudes, continued, meet in a point.

**4.** This example is also about circles. Consider a black disc of diameter $d$ drawn on a plane and a number of long white paper strips of different widths. Our goal is to cover the disc by these strips so that no black spot is visible.

FIGURE 8



If the total width of the strips is at least $d$, this is trivial—just put all the strips side-to-side. It turns out that if the total width is smaller than $d$, such covering is impossible.

Why? To explain this, recall the following geometric fact. The area of the part of a sphere which lies between two parallel planes (intersecting the sphere) depends on the radius of the sphere and the distance between the planes, but not on the position of the planes.

In other words, if we cut a spherical lemon into slices of equal thickness, the amount of skin will be the same for all slices (Fig. 9).

What is the connection between this fact and our circle and strips problem? Imagine that the circle is a top view of a hemisphere. Then each strip that goes across the circle becomes part of the hemisphere lying between two parallel planes (visible as border lines of the strip); the area of this part is proportional to the width of the strip (Fig. 10).

If the strips cover the circle, the corresponding parts of the hemisphere cover the hemisphere. Therefore, their total area is not less than the area of the hemisphere. And the whole hemisphere corresponds to a strip of width $d$. Therefore, the total width of all strips is at least $d$. Q.E.D.

**5.** Our last example is the classical problem of Apollonius: to construct a circle tangent to three given circles (Fig. 11). All the information below is taken from the paper by A.V. Khabelishvili in the Russian Journal *Istoriko-matematicheskie Issledovaniya* ser. 2, vol. 1 (36), number 2 (1996), 6–81. I cannot vouch for the history, but the proof suggested in this paper is indeed nice.

According to the paper, the problem of constructing a circle tangent to three given circles was stated by the Greek mathematician Apollonius (260–170 B.C.) in his treatise "On tangents" in two volumes. However, this treatise as a whole and the solution given by Apollonius were lost.

Many famous mathematicians worked on this problem later (including F. Viète, R. Descartes, I. Newton, L. Euler); however, all the solutions proposed involve some notions not available to Apollonius. (The best-known solution uses inversion. If two given circles intersect in the center of inversion, then after the inversion they become straight lines; this special case is easy.)

A.V. Khabelishvili suggests a solution that he believes may be the original solution found by Apollonius, who is famous as an expert in conic sections.

Here it is. Assume the three given circles are drawn in a horizontal plane. Imagine three similar cones with vertical axes that intersect the plane along these circles (Fig. 12), like three conical volcanos of different height in the middle of the sea; the given circles are shorelines of these mountains.

Consider one more cone. This cone is similar to the three given cones, and also has vertical axis, but its vertex is pointing down. Let us put this cone in between the three cones and then move it down until it touches them. At that point the intersection of this cone with the horizontal plane will be the required circle, and the vertex of this cone will coincide with the point of intersection of the given cones. So if we consider finding the point of intersection of three cones as a legal operation, the Apollonius problem is easy.

However, we are looking for a ruler and compass construction, so we continue to follow Khabelishvili's argument. Consider the plane that goes through the vertices of three given cones. In its final (desired) position, the fourth cone intersects this plane in an ellipse through the vertices of the given cones.

FIGURE 9

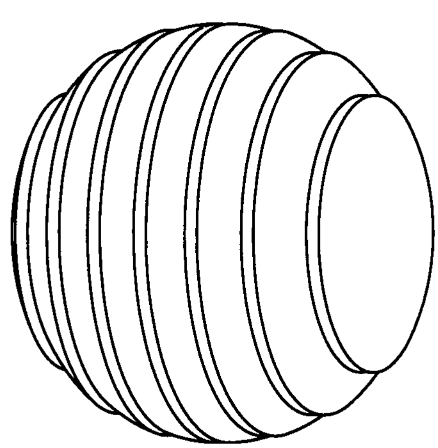


FIGURE 10

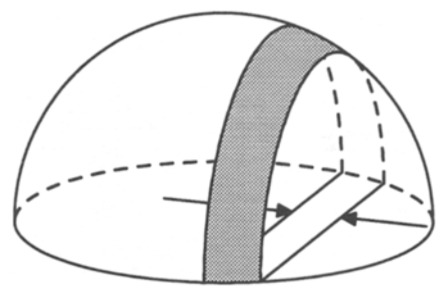


FIGURE 11

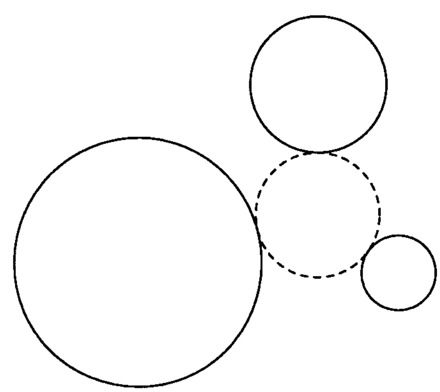


FIGURE 12

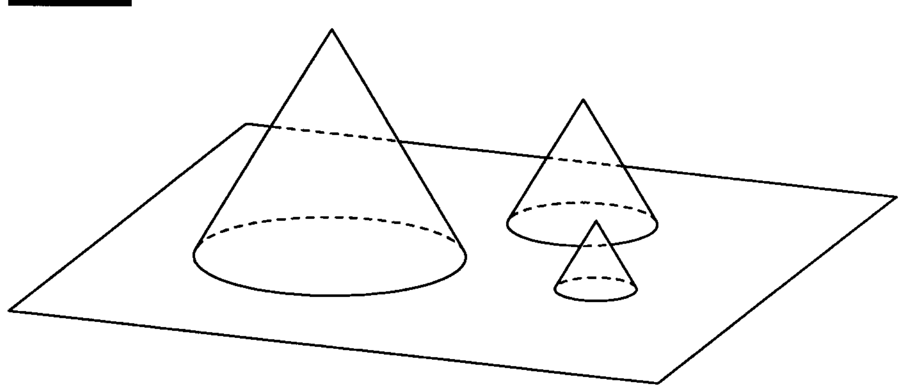
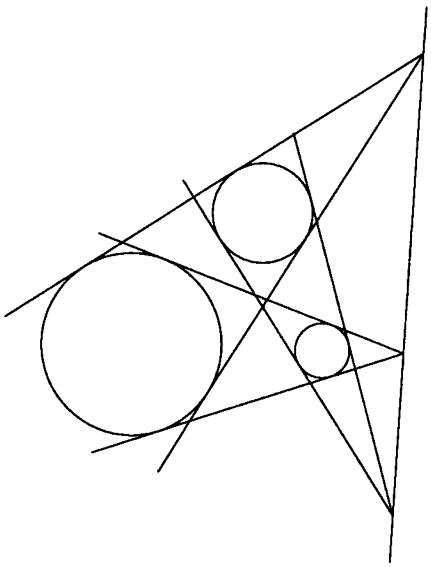


FIGURE 13



For any three points there are many ellipses going through them, so we need some additional information. Please note that all ellipses that are intersections of the moving fourth cone and the plane, are similar to each other and have the same ratio of long and short axes. Therefore, using a suitable projection (we will say now: affine transformation) we reduce the problem to the following one: construct a circle going through three given points.

It remains to show that all required constructions may be performed using compass and ruler only (on a suitable plane). We will not go into details and mention only two basic facts needed:

**A.** To find on the horizontal plane (that contains the three given circles) the line of intersection with the plane going through the cone vertices, we do the following: for each two circles we draw two common tangents and find their intersection; these three intersection points lie on the straight line in question (Fig. 13).

**B.** Looking at the side view of a cone and a plane intersecting this cone, it is easy to construct the major and minor axes of the intersection ellipse and determine the ratio in which the cone's axis divides the long axis of the ellipse.

Dear reader, what do you think? Was this the original solution of Apollonius? It would be interesting to hear from historians of mathematics.

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