This column is devoted to mathematics for fun. What better purpose is there for mathematics? To appear here, a theorem or problem or remark does not need to be profound (but it is allowed to be); it may not be directed only at specialists; it must attract and fascinate.

We welcome, encourage, and frequently publish contributions from readers—either new notes, or replies to past columns.

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Two More Probabilistic Arguments

After the column about probabilistic arguments was finished, I came across two problems (both from high-school mathematical competitions in Russia) that may be easily solved using nice probabilistic arguments, and I'd like to share these.

1. The sets $S_1, S_2, \ldots, S_k$ are different subsets of a set S that has 200 elements. Moreover, $S_i \not\subset S_j$ for any $i \neq j$. Prove that $k \leq \binom{200}{100}$.

Here is the solution. Consider the following process: We start with an empty set and add random elements of S one by one until (after 200 steps) we get the whole set S. For a fixed subset A, let us compute the probability $\Pr[A]$ that A will appear during this process. For example, $\Pr[\varnothing] = \Pr[S] = 1$; for any $s \in S$, the probability $\Pr[\{s\}]$ is equal to 1/200 (all elements of S can be chosen and added to $\varnothing$ with equal probabilities). Moreover, any subset $A \subset S$ of a given cardinality a has the same chance to appear during this process, and only one subset of cardinality a may appear, so $\Pr[A] = 1/\binom{200}{a}$.

Consider k random variables $\sigma_1, \ldots, \sigma_k$; the value of σ_i is equal to 1 if the given set S_i appears during the process; otherwise, σ_i is equal to 0. The expected value of σ_i is $1/\binom{200}{s_i}$, where s_i is the number of elements in S_i, so this expected value is at least $1/\binom{200}{100}$ (each row in the Pascal triangle has a maximum in the center).

Now, consider the random variable $\sigma = \sigma_1 + \cdots + \sigma_k$. This sum cannot exceed 1, as two different sets S_i and S_j may not appear in the process (if S_i precedes S_j in the process, then $S_i \not\subset S_j$). So, the expected value of σ does not exceed 1, and each term has expected value at least $1/\binom{200}{100}$. Therefore, the number of terms k does not exceed $\binom{200}{100}$.

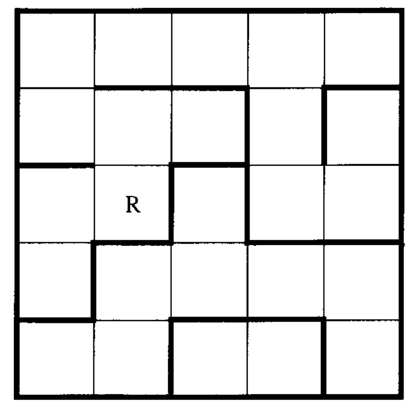

Figure 1. Robot in the labyrinth

2. A robot R placed in the labyrinth (as in Fig. 1) is equipped with a program. The labyrinth is a square $n \times n$ where some walls are placed between cells (in addition to the external walls around the square). The program is a sequence of commands `left`, `right`, `up`, and `down` (no loops or branches). Executing each command, the robot moves in the prescribed direction if possible (and does nothing when there is a wall in this direction). Prove that for any n, there exists a program that works correctly for all labyrinths of size $n \times n$ (independently of the positions of walls inside the square and the robot's initial position). Here, "works correctly" means that the robot visits all reachable cells.

To solve this problem, we prove that a sufficiently long random program will work with positive probability. For each $n \times n$ labyrinth, there is a program of size $4n^3$ that works for it, as each cell is reachable in at most $4n$ steps (round-trip) and there are at most n^2 admissible cells. Therefore, a random program of size $N = 4n^3$ will work with probability at least $\varepsilon = (1/4)^{4n^3}$ and fail with probability at most $1 - \varepsilon$. A random program of size $2N$ will fail with probability at most $(1-\varepsilon)^2$; a random program of size kN will fail with probability at most $(1-\varepsilon)^k$. This probability is computed for a fixed labyrinth; if k is large enough, $(1-\varepsilon)^k$ is smaller than 1 divided by the number of different labyrinths of size $n \times n$, and a random program of size kN works for all of them with positive probability. Q.E.D.

Poncelet Theorem Revisited

Consider two circles C_1 and C_2 (Fig. 2). The well-known Poncelet theorem guarantees that if there exists a triangle inscribed in C_1 and circumscribed around C_2, then there are infinitely many triangles with this property.

Poncelet's theorem can be reformulated as follows. Consider the mapping $f: C_1 \to C_1$ defined as shown in Figure 3.

> If $f(f(f(A))) = A$ for some point A on C_1, then $f(f(f(X))) = X$ for any point X on C_1.

There is a nice proof of this statement (it is explained, for example, in Prasolov and Tikhomirov's textbook on geometry): one can define a measure on C_1 in such a way that the measure of the arc $X - f(X)$ is a constant that does not depend on the choice of X. Then, $f(f(f(A))) = A$ means that this constant equals one-third of the measure of C_1.

The same argument allows us to prove the Poncelet theorem not only for triangles but for arbitrary n-gons [if $f^{(n)}(A) = A$, then this constant equals $(1/n)$th fraction of the measure of C_1 and $f^{(n)}(X) = X$ for any X].

OK, but why should such a measure exist? After we decide to look for it, finding such a measure is rather easy. Assume that the measure is $\rho(X)ds$, where $\rho(X)$ is some (yet unknown) density function and s is the natural parameter. To find conditions on ρ that guarantee the desired property, consider two infinitesimally close tangents to C_2. The measures of infintesimal arcs A_1 and A_2 cut by these lines are to be made equal (Fig. 4).

The lengths of arcs A_1 and A_2 are proportional to the segments l_1 and l_2. Therefore, if we define $\rho(X)$ for $X \in C_1$ as the reciprocal of the length of the tangent from X to the circle C_2, arcs A_1 and A_2 will have equal measures, and we are done.

What properties of curves C_1 and C_2 were used in this proof? For C_2, we need to know that two tangents to C_2 going from the same point X are equal (Fig. 5).

If the tangents were of different lengths, the density $\rho(X)$ wouldn't be well defined.

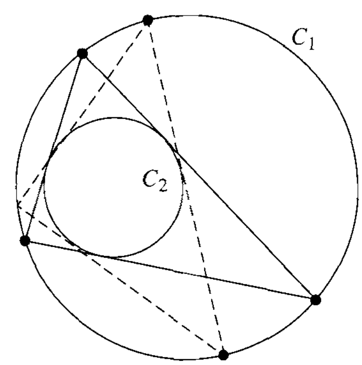

Figure 2. Two circles and triangles.

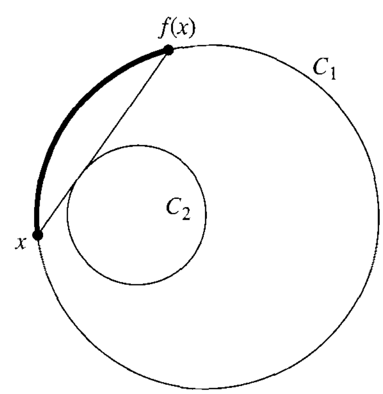

Figure 3. Poncelet mapping.

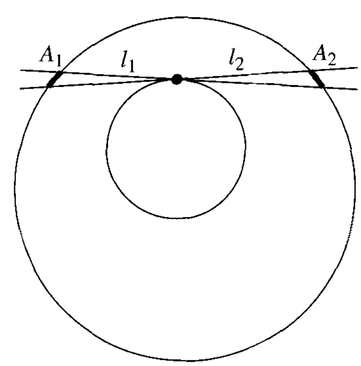

Figure 4. Two infinitesimal arcs should be equal.

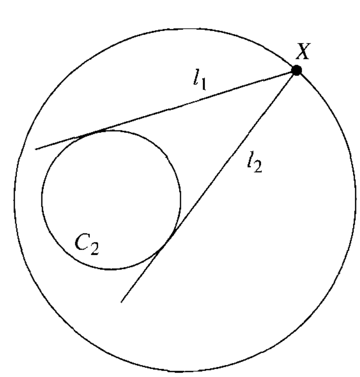

Figure 5. Two equal tangents to C_2: $l_1 = l_2$.

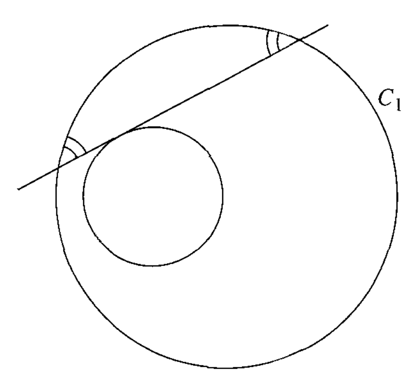

Figure 6. Two equal intersection angles.

For C_1, we need another property of a circle: any line intersecting a circle at two points, forms equal angles with the circle in both intersection points (Fig. 6).

This property guarantees that the arcs A_1 and A_2 (Fig. 4) are proportional to l_1 and l_2 (infinitesimal triangles are similar).

The Poncelet theorem is valid not only for circles but for any conic sections. However, this proof seems to be not applicable in the general case. Prasolov and Tikhomirov say (after explaining the proof for the case of two circles), "We won't prove this theorem in the general case since all known proofs are complicated."

However, the Moscow mathematician A.A. Panov found that this proof can be generalized. His argument is explained below. The inspiration comes from classical mechanics, so let us recall some facts.

It is well known that there is no gravity inside the sphere. A similar two-dimensional statement is also true if the gravitational force is propor-

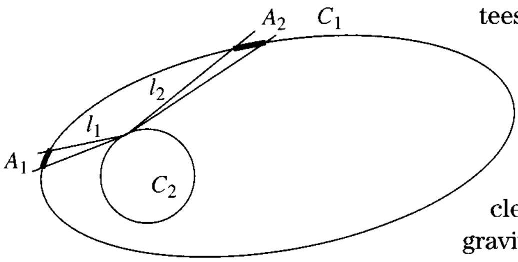

Figure 7. Two elliptic arcs have the same measure.

tional to the inverse distance (not the squared inverse distance, as in the three-dimensional case). To see why, look again at Figure 4: forces coming from arcs A_1 and A_2 compensate each other, because distances are proportional to masses.

Now, what can be said about the gravity inside an ellipsoid? Or inside an ellipse in the two-dimensional case? Of course, the answer depends on the mass distribution. I will show that there exists a distribution that guarantees the absence of gravity inside the ellipse. Indeed, imagine that a circle is drawn on a weightless elastic film using heavy ink, and then this film is stretched together with the circle (so the circle becomes an ellipse). Then, the gravity is still absent inside the ellipse. Here is why. Although the lengths l_1 and l_2 in Figure 4 do change when we stretch the film, their ratio remains the same, as do the masses on arcs A_1 and A_2, so the gravitational forces from A_1 and A_2 still compensate each other. Thus, we have constructed a distribution of masses along the ellipse (we denote this distribution by $d\varphi$ in the sequel) that generates no gravity inside the ellipse.

Returning to Poncelet's theorem, let us prove it for the case when C_1 is an ellipse and C_2 is a circle. Consider a distribution $d\varphi/l(x)$ on C_1, where $l(x)$ is the length of the tangent from $x \in C_1$ to the cirlce C_2 (Fig. 7).

The same argument as before shows that the measures of arcs A_1 and A_2 are equal. Therefore, all tangents to the circle C_2 cut the same fraction of ellipse C_1 (when measured according to the distribution $d\varphi/l(x)$), and the Poncelet theorem is proved.

What if both C_1 and C_2 are ellipses? Then, we stretch the picture to convert C_2 into a circle. The statement of the Poncelet theorem is invariant under affine transformations, so we are done. It is also invariant under projective transformations, so the statement is true for any conic sections.

Remark: As A.A. Panov points out, in fact any two conic sections could be transformed to circles by one projective transformation; this observation gives us another way to prove Poncelet's theorem for any two conic sections after we have proved it for circles.

I close with an "archaeological discovery" from David Gale of Berkeley.

Euclid's Last (or Lost) Theorem
by David Gale

In a triangle called ABC,
Pick a point on AB, call it P.
  Pick a Q on BC,
  Where BQ is BP.
Ah the joys of pure geo-me-tree!

On CA pick an R, oh please do,
Where CR is exactly CQ,
  And now pick an S
  On AB, more or less,
So that "AS is AR" is true.

On BC the next letter is T,
Where BT is BS, don't you see.
  On CA pick a U,
  And you'll know what to do,
Next what's this? we've arrived back at P!

Now some proofs were soon found close at hand,
But it didn't turn out quite as planned,
  For though not very large
  (They would fit in the margin) regrettably, none of them scanned.