This column is devoted to mathematics for fun. What better purpose is there for mathematics? To appear here, a theorem or problem or remark does not need to be profound (but it is allowed to be); it may not be directed only at specialists; it must attract and fascinate.

We welcome, encourage, and frequently publish contributions from readers—either new notes, or replies to past columns.

Please send all submissions to the Mathematical Entertainments Editor, **Alexander Shen,** Institute for Problems of Information Transmission, Ermolovoi 19, K-51 Moscow GSP-4, 101447 Russia; e-mail:shen@landau.ac.ru

Unexpected Proofs

One of the nice things about mathematics is that sometimes a question looks very simple but the answer uses an unexpected and elegant argument. Let me show two examples.

Boxes in a Train

Rules of the Moscow underground say that you are allowed to bring on a rectangular box of size $w \times h \times d$ only if $w + h + d$ does not exceed 150 cm. Question: Is it possible to cheat by packing one box into another? The answer is no:

> If a rectangular box $w_1 \times h_1 \times d_1$ can be placed inside another one of size $w_2 \times h_2 \times d_2$, then $w_1 + h_1 + d_1 \leq w_2 + h_2 + d_2$.

We present two completely different proofs of this fact. The first considers the ε-neighborhood of a box (including the interior part). Its volume $V(\varepsilon)$ is defined for non-negative ε. It is easy to see that $V(\varepsilon)$ is a polynomial in ε:

$$V(\varepsilon) = V + S\varepsilon + \pi l\varepsilon^2 + (4/3)\pi\varepsilon^3.$$

Here, V is the volume of the box, S is the area of its surface, and l is the sum of the dimensions $(w + h + d)$. Indeed, the neighborhood consists of

- the box itself (V)
- six rectangular boxes (of thickness ε) covering the faces and having total volume $S\varepsilon$
- twelve pieces near the edges that can be combined into three cylinders of radius ε and lengths w, h, and d; total volume $\pi\varepsilon^2(w + h + d)$
- eight pieces near the vertices that form a ball of radius ε having total volume $(4/3)\pi\varepsilon^3$.

Now, assume we have two boxes, one inside another. Then, the ε-neighborhood of the first box will be inside the ε-neighborhood of the second, so

$$V_1 + S_1\varepsilon + \pi l_1\varepsilon^2 + (4/3)\pi\varepsilon^3 \leq V_2 + S_2\varepsilon + \pi l_2\varepsilon^2 + (4/3)\pi\varepsilon^3.$$

This is true for any ε, even for a large one when the ε^2 term is the main term (note that the ε^3 terms are the same for both neighborhoods and cancel each other). Therefore, $l_1 = w_1 + h_1 + d_1$ does not exceed $l_2 = w_2 + h_2 + d_2$.

The second proof uses randomness. Let X be a convex set in $\mathbb{R}^3$. Consider a random line m in $\mathbb{R}^3$. The orthogonal projection of X onto m is a segment. Let us denote by $d(X)$ the expected length of this segment.

Let X_m be a segment of length m. Then, $d(X_m)$ is proportional to m, i.e., $d(X_m) = cm$ for some c. (In fact, $c = 1/2$, but the exact value is not important now.)

Now, let X be a box of size $w \times h \times d$. For each line m, the projection of X onto m has length $p_w + p_h + p_d$, where p_w, p_h, and p_d are projections of segments of length w, h, and d, the edges of the box. By averaging, we get

$$d(X) = c(w + h + d).$$

If a box (X_1) is placed inside another one (X_2), then the projection of X_1 onto a line m is included in the projection of X_2 onto m, so $d(X_1) \leq d(X_2)$. Combining this observation with the preceding one, we see that

$$w_1 + h_1 + d_1 \leq w_2 + h_2 + d_2.$$

(End of the second proof.)

Square Split into Triangles

It is easy to split a square into n equal triangles if n is even. However,

> *it is impossible to split a square into n triangles of equal area if n is odd.*

However, the proof of this fact is not straightforward and uses some topology and algebra.

We start with a special case where (a) the triangles form a triangulation and (b) all vertices have rational coordinates. (Later, we'll see how these assumptions can be removed.) For any rational number r, define its 2-valuation $\|r\|$ as follows: if $r = 2^k(p/q)$, where p and q are odd, $\|r\|$ is 2^{-k}. By definition, $\|0\| = 0$. In a sense, $\|r\|$ measures "oddness" of r: for example, 3/2 is "odder" than 1, and 2 is "odder" than 4. Now, divide all rational points (x, y) (both x and y are rational) into three classes. If both x and y (represented as irreducible fractions) have even numerators, the point (x, y) belongs to class A. If at least one of x and y has an odd numerator, compare the "oddness" of x and y: when x is "more odd," we get a B point, otherwise a C point. Formally,

A: $\|x\| < 1$ and $\|y\| < 1$
B: $\|x\| > \|y\|$ and $\|x\| \geq 1$
C: $\|x\| \leq \|y\|$ and $\|y\| \geq 1$

Let us return to our square $I^2 = [0, 1] \times [0, 1]$ and its triangulation with rational vertices.

Lemma. *There exists a triangle in the triangulation whose vertices are labeled with all three labels A, B, and C.*

Proof. Our classification can be considered as a mapping α from the set of vertices into the set $\{A, B, C\}$. Imagine that A, B, and C are vertices of some triangle ABC. Then, α can be uniquely extended to a mapping of the whole square into the triangle ABC that is piecewise affine (affine on each triangle of the triangulation).

Now the statement of the lemma can be reformulated as follows: *α covers the interior part of the triangle ABC.* To prove this statement (a version of Sperner's lemma), let us consider the restriction of α to the boundary of the unit square. We know its values on the square's vertices: (0, 0) has type A, while (1, 0) has type B, and both (1, 1) and (0, 1) have type C (see Fig. 1).

Moreover, it is easy to see that any vertex on the lower side of the square has type A or B and any vertex on the left side has type A or C, whereas all

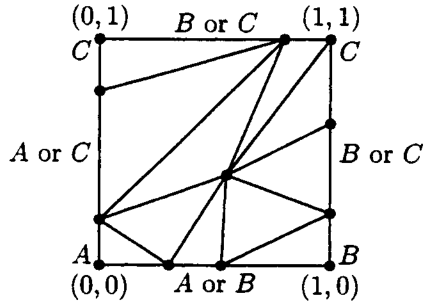

Figure 1

vertices on the remaining two sides have type B or C. Therefore, the restriction $\alpha|\partial I^2$ of α to the boundary of the square I^2 maps it into the boundary of the triangle ABC and has degree 1. Therefore, $\alpha|\partial I^2$ is not homotopic to a constant mapping. On the other hand, if the image $\alpha(I^2)$ were contained in the boundary of triangle ABC, α would provide a homotopy between $\alpha|\partial I^2$ and a constant mapping. (End of the proof of the lemma.)

Now we know that our triangulation contains a triangle whose vertices are labeled A, B, and C. Let their coordinates be (a_1, a_2), (b_1, b_2), and (c_1, c_2), respectively. This triangle has area

$$S = \frac{1}{2} \det \begin{vmatrix} b_1 - a_1 & b_2 - a_2 \\ c_1 - a_1 & c_2 - a_2 \end{vmatrix},$$

and $\|S\| > 1$ (as we'll see). On the other hand, $S = 1/n$, because all n triangles of the triangulation have the same area. Therefore, n is even.

It remains to prove that $\|S\| > 1$. Recall two main properties of the 2-valuation:

- $\|ab\| = \|a\| \cdot \|b\|$
- $\|a + b\| \leq \max(\|a\|, \|b\|)$; this inequality turns into equality if $\|a\| \neq \|b\|$

Using these properties, it is easy to check that the point $(b_1', b_2') = (b_1 - a_1, b_2 - a_2)$ belongs to type B and the point $(c_1', c_2') = (c_1 - a_1, c_2 - a_2)$ belongs to type C (b_1 is "more odd" than a_1, so subtracting a_1 we do not change "oddness" of b_1, etc.) By definition of types B and C, we have

$$\|b_1'\| > \|b_2'\|; \qquad \|b_1'\| \geq 1;$$
$$\|c_2'\| \geq \|c_1'\|; \qquad \|c_2'\| \geq 1.$$

Therefore, $\|b_1'c_2'\| > \|b_2'c_1'\|$ and $\|b_1'c_2'\| \geq 1$, so $\|2S\| = \|b_1'c_2' - b_2'c_1'\| = \|b_1'c_2'\| \geq 1$ and $\|S\| = 2\|2S\| > 1$.

So the statement is proved for the case of triangulation with rational vertices. Let me say, briefly, what could be done for the general case.

Assume that the triangles do not form a triangulation, e.g., vertex Q of one triangle lies on side PR of another one. (See Fig. 2.) What can we do? We can admit "degenerate" triangles like PQR, get a triangulation, apply our argument, and find a triangle that is ABC-labeled. This triangle cannot be degenerate since for its area S, we have proved that $\|S\| > 1$.

What should we do if the coordinates of the vertices are irrational? In this case, one can extend the 2-valuation to an extension of $\mathbb{Q}$ that contains all the coordinates, and use the same argument. (I omit the details.)

I found this problem (and its solution) in an article of B. Bekker and N. Netsvetaev; they attribute it to John Thomas (A dissection problem, *Math. Mag.* 41 (1968), 187–190) for the case of rational coordinates and Paul Monsky (On dividing a square into triangles, *Am. Math. Monthly* 77 (1970), 161–164) for the general case.

Letters

Concerning Poncelet's theorem and your article in the *Intelligencer*, I wonder if you know this. Poncelet's initial theorem concerned a pencil of circles, and he stated it like this: let I, II, and III be three circles in a pencil. Start from a point m in I, draw the tangent to II, get a second point n in I; from n draw a tangent to III, get another point p in I. Then, the line mp, when m runs through I, envelops a circle IV (from the initial pencil). All closure theorems follow from this one. Now, the proof of the *Intelligencer* applies to this, one has only to remark that the lengths of tangents drawn from points of I to circles in the pencil are proportional with universal constants. Then, the associ-

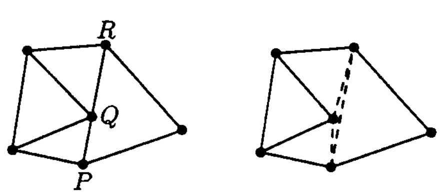

Figure 2

ated measures on the circle I, given by II, III, etc., are proportional. Otherwise stated: for such a measure the line joining two points differing by a translation of this measure always envelops some circle of the pencil.

Elliptic functions are at the core of Poncelet's theorem; here the elliptic function is the new measure.

Marcel Berger
Institut des Hautes Etudes Scientifiques
91440 Bures-sur-Yvette
France
e-mail: berger@ihes.fr

Preparing an article on the story of Poncelet's theorem, I read some of the original papers. My following notes sketch the historical background.

1. Poncelet's original theorem is not about a triangle inscribed in a circle and circumscribed around another circle, but about an n-gon inscribed in a conic section and circumscribed around another conic section. Moreover, the theorem in Poncelet's approach is a consequence of a more general theorem. This general theorem is about a pencil of conic sections. Let $C, c_1, c_2, \ldots, c_{n-1}$ be the elements of this pencil. Poncelet states: there is a conic section c_n such that whenever points $A_1, A_2, \ldots, A_n$ are on C, and line A_1A_2 touches c_1, line A_2A_3 touches $c_2, \ldots, A_{n-1}, A_n$ touches c_{n-1}, then A_nA_1 will touch c_n. The first publication was in 1822.
2. Poncelet was an officer of Napoleon. He was imprisoned in Russia, in Saratov, for more than a year. At this time, without books and equipment, he created many notions of projective geometry: ideal and imaginary points, for example. One of his practical results: Circles are exactly the conic sections containing the points $(1, i, 0)$ and $(1, -i, 0)$. One can transform two conics into two circles simply by projecting two of their common points to $(1, i, 0)$ and $(1, -i, 0)$.
3. The proof you found in Prasolov and Tikhomirov's textbook goes back to Jacobi (*Crelle J. Math.* 3 (1828), 376). Here is the short history of his proof:

In Bd. 2 of Crelle's *Journal* (1827), Steiner proposed the problem of finding the algebraic relation of the radii of circles c_1 and c_2, and the distance of their centers, if there is a 4-gon, 5-gon, ..., 8-gon inscribed in c_1 and circumscribed around c_2. (In fact, these problems had been partly solved previously by Fuss, the academic secretary of St. Petersburg. Euler solved the problem for $n = 3$, his student Fuss for $n = 4$, and Fuss was able to solve the problem for $n = 5, 6, 7, 8$ if the n-gon is symmetric about the center of the circles.)

Steiner gave the appropriate equations without proof in Crelle's *Journal* of the same year. In this issue, Abel and Jacobi had many articles on elliptic integrals and on their inverse, the elliptic functions. In Bd. 3, Jacobi wrote three articles on this topic. Then, he wrote a fourth one: he proved Poncelet's theorem for two circles by integrals and he could even check the equations of Steiner (and Fuss).

When the old Poncelet refers shortly to Jacobi's proof, he uses essentially your arguments. Jacobi's article is longer.

As I can see, Jacobi tried to solve geometrically the problem proposed by Steiner. He could set up equations, and these equations reminded him of Legendre's addition formulas of elliptic integrals. If Jacobi could find the connection, he could set up an elliptic integral related to Poncelet's theorem. It was only after this that he perceived the geometric meaning of the integrand: the reciprocal of the length of the tangent.

András Hraskó
Tornóc u. 17
1141 Budapest
Hungary
e-mail: hraskoa@ceu.hu

Poetry Quiz

Recall a familiar quatrain:

I May, I Might, I Must

If you will tell me why the fen
appears impassable, I then
will tell you why I think that I
can get across it if I try.

The first question is, Who wrote it?
(a) Karl Popper
(b) George Pólya
(c) Watty Piper
(d) none of the above.

See p. 79.