Coherence spaces

A concrete interpretation of proofs

$$\begin{array}{c} \vdots \pi \\ \vdots \\ A \vdash B \end{array} \qquad\qquad A \xrightarrow{[\![\pi]\!]} B$$

denotational:

$$\pi \xrightarrow{\beta} \pi'$$

$$[\![\pi]\!] = [\![\pi']\!]$$

Abstractly: construction of a cartesian closed category

- product $A \times B$ & projections, pairs
- internal Hom $(A \to B)$ object B^A

Concrete construction → "coherence"

History

second order

J.-Y. Girard 1986 The system F of
Variable types fifteen years later
(from ordinals, Π^1_2 logic)

Jean-Yves Girard 1987 Linear Logic

Propositions/Types	Coherence space
A atomic	arbitrary A
A & B	product A & B
A -> B	stable functions from A to B viewed as a coherence space

Proof

$$A \overset{\pi}{\vdash} B$$

$$\overset{\pi}{\vdash} B$$

$$A_1, \ldots, A_n \overset{\pi}{\vdash} B$$

Stable map

$[\![\pi]\!] : A \to B$
or object (clique) in $[\![A \to B]\!]$

$[\![\pi]\!] : 1 \to B$
or object (clique) in $[\![B]\!]$

$[\![\pi]\!]$ stable function $(A_1 \& \ldots \& A_n) \to B$
or object of $[\![(A_1 \& \ldots \& A_n) \to B]\!]$

what makes it work:

$$\frac{t : T \quad f : T \to W}{f(t) : W}$$

(closure to &)

$f \in [T \to W]$

$f : [T] \longrightarrow [W]$

Remark GIRARD invented (1980) coherence space for second order

map (function) $\overset{\text{coherence space}}{X} \longrightarrow \overset{\text{other coherence space}}{T(X)}$

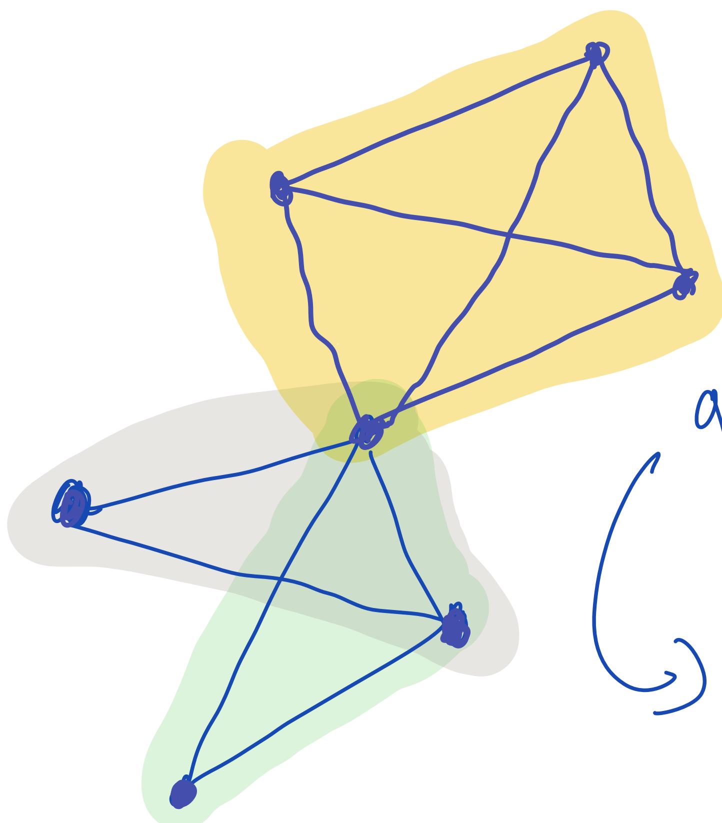
eg $X \to (X \to X)$

can be represented as a coherence space!

it interprets $\forall X\, T[X]$

object $\longrightarrow$ 2nd order proofs

WEB |A| of a coherence space A



1 : • no edge

graph finite / countable

maximal CLIQUES

word from graph theory that is suggested to JYG

TWO VIEWS of a coherence space

particular family of sets $\longleftrightarrow$ cliques of a graph

if $\left(\begin{array}{l} a \in A,\ a' \subset a \\ \text{then } a' \in A \end{array} \right) \longrightarrow$ vertices : singletons $\{\alpha\}$

if $\left(\begin{array}{l} a_i \in A \quad (i \in I) \\ a_i \cup a_j \quad (i, j \in I) \end{array} \right)$ then $\left(\bigcup_{i \in I} a_i \right) \in A$ $\longrightarrow$ $\alpha \bullet\!\!-\!\!-\!\!\bullet\ \alpha'$ when $\{\alpha, \alpha'\} \in A$

A&B product

$$\mathrm{Web}\,|A \& B| = |A| \uplus |B|$$

graph!

A graph

B graph

all edges inbetween

$A \to B$ stable functions

$\llbracket A \to B \rrbracket$ later

stable maps (Berry) Girard)

- denumerable web
- computable from finite approximants

stable functions from $\overset{\text{set of cliques}}{A}$ to B

- if $a \subset a'$ then $F(a) \subset F(a')$
- when a_i are all in a clique
$$F\left(\bigcup^{\uparrow} a_i\right) = \bigcup^{\uparrow} F(a_i)$$
- $a \cup a' \in A$ $\qquad F(a \cap a') = F(a) \cap F(a')$

Property of F stable

- if $\beta \in F(a)$
  then $\exists a_0 \subset a$, a_0 finite, $\beta \in F(a_0)$
- a_0 minimal $\longrightarrow$ a_0 unique

PROOF

- $\beta \in F(a)$ $\quad a = \bigcup a_i \quad a_i$ finite
  $$\beta \in F(a) = F\left(\bigcup a_i\right) = \bigcup_{i \in I} \left(F a_i\right)$$
  $\exists i \quad \beta \in F(a_i)$ finite
- if a_0 minimal with $a_0 \subset a$ $\beta \in F(a)$
  if $a' \subset a$ s.t. $\beta \in F(a')$
  $\beta \in F(a') \cap F(a_0) = F(a' \cap a_0)$ $\quad a' \cap a_0 \subset a_0$
  $a' = a_0$

F stable $\qquad\qquad\qquad\qquad \mathrm{Tr}(F)$

$$F(a) = b \xrightarrow{\text{cl. property}} (a_0, \beta) \quad \beta \in b$$

$\downarrow$

finite

minimum

for $\beta \in F(a)$

$$F(a) = \{\beta \mid \exists \underset{\text{finite}}{a_0 \subset a} \ (a_0, \beta) \in T\}$$

$A \to B$ = stables functions $A \to B$

pairs (a_{fin}, β)

in a one to one correspondence with the cliques of a coherence space

$$(!A) \multimap B$$

linear maps from $!A$ to B

NOTATION

strict cohence

$$\alpha \frown \alpha' [A] \quad \text{edge } \alpha - \alpha' \text{ in } A$$

tokens from the web $|A|$ of A

VARIANT

$$\alpha \asymp \alpha' [A]$$

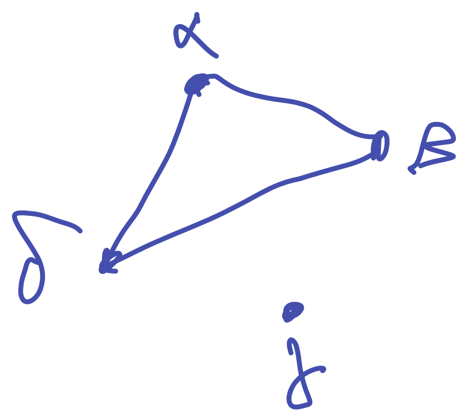
$$\alpha \frown \alpha' [A]$$

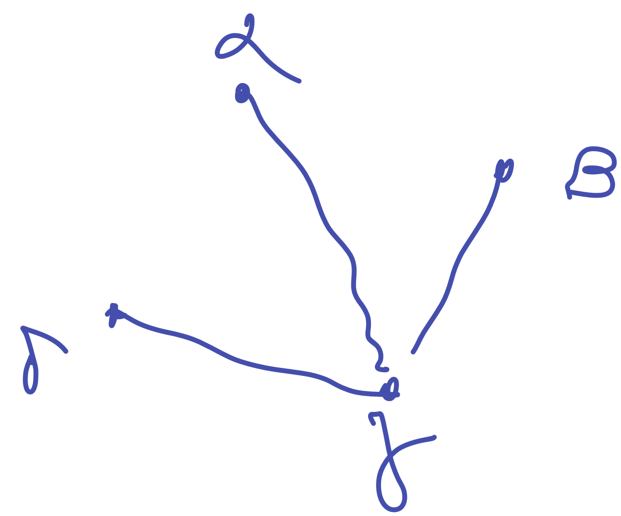
or $\alpha = \alpha'$

Constructing coherence spaces

negation $A^{\perp}$

complement graph





$$(A^{\perp})^{\perp} = A$$

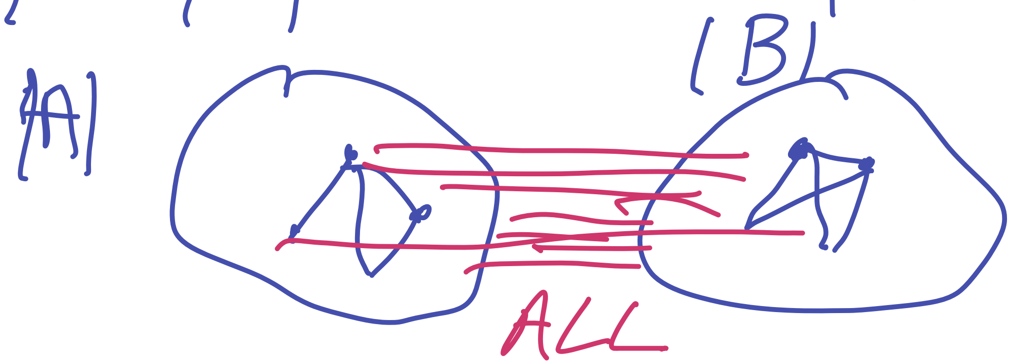
$A \& B$: (déjà vu)

WEB : $|A| \uplus |B|$ additive

$\alpha, \alpha' \in |A|$ $\quad \alpha \overset{\frown}{\smile} \alpha' \, [A \& B]$ iff $\alpha \overset{\frown}{\smile} \alpha' \, [A]$

$\beta, \beta' \in |B|$ $\quad \beta \overset{\frown}{\smile} \beta' \, [A \& B]$ iff $\beta \overset{\frown}{\smile} \beta' \, [B]$

$\alpha \in |A| \quad \beta \in |B|$ $\quad \alpha \overset{\frown}{\smile} \beta \, [A \& B]$



$$\overset{\text{OR}}{} \quad |A \oplus B| = |A| \uplus |B|$$

$$A \,\&\, B \simeq (A^{\perp} \oplus B^{\perp})^{\perp}$$

$\alpha, \alpha' \in |A|$ $\quad \alpha \supset \alpha' [A \oplus B]$ iff $\alpha \supset \alpha' [A]$

$\beta, \beta' \in |B|$ $\quad \beta \supset \beta' [A \oplus B]$ iff $\beta \supset \beta' [B]$

$\alpha \in |A|$, $\beta \in |B|$ $\quad \cancel{\alpha \supset \beta [A \oplus B]}$ NEVER

tensor / times

web $|A \otimes B| = |A| \times |B|$

multiplicative

$$(\overset{A}{\alpha}, \overset{B}{\beta}) \mathrel{\substack{\frown\\\smile}} (\overset{A}{\alpha'}, \overset{B}{\beta'})$$

whenever $\alpha \mathrel{\substack{\frown\\\smile}} \alpha'$ and $\beta \mathrel{\substack{\frown\\\smile}} \beta'$

$A \backslash B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\smile$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\smile$	$\frown$	$\frown$

par $A \parr B = (A^\perp \otimes B^\perp)^\perp$

$(\alpha, \beta) \sqsupset (\alpha', \beta')$ whenever $\alpha \sqsupset \alpha' \, [A]$

$\beta \sqsupset \beta' \, [B]$

B \ A	∪	=	∩
∪	∪	∪	∩
=	∪	=	∩
∩	∩	∩	∩

that's the only two commutative connective

1 multiplicative unit

1 $*$ no edge

$* \supset *$ that's all

one clique $\{*\}$

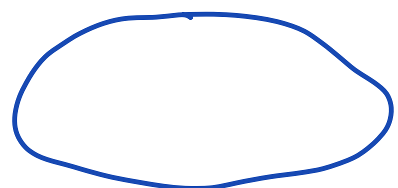
$A \otimes 1 \cong A \cong A ⅋ 1$

$1^\perp = 1$ (Mix rule)

0 additive unit

web $\emptyset$

no clique but $\emptyset$



(had to draw)

$A \oplus 0 \simeq A \& 0 \simeq A$

$!A$ WEB finite cliques of A

$a \frown a'$ when $a \cup a' \in A$

included in a bigger clique

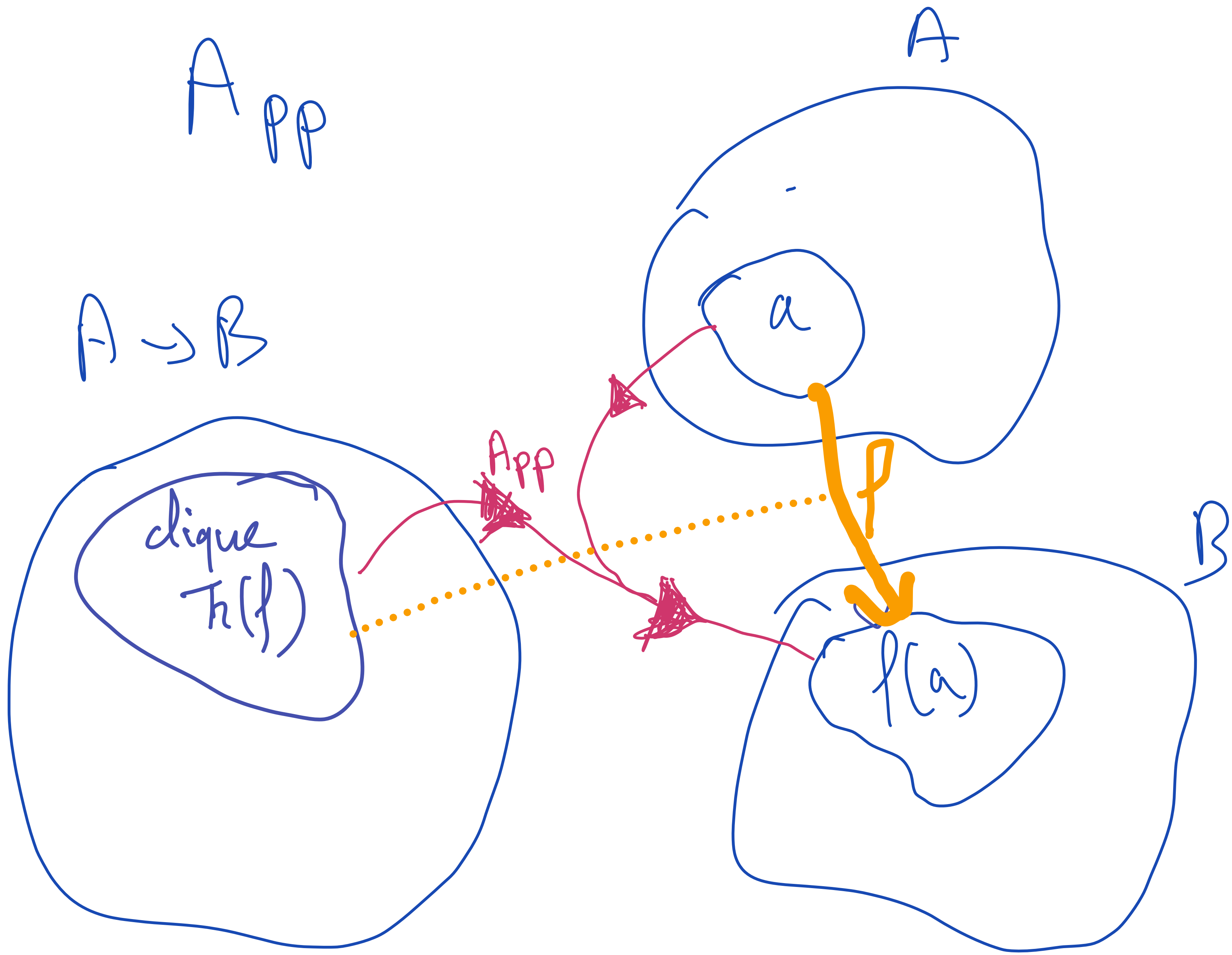
$$?A = (!A^{\perp})^{\perp}$$

$$A \to B = \, !A \multimap B$$

$$= (!A)^{\perp} \mathbin{\wp} B$$

coherence space
of stable maps
from A to B

App



Interpreting proofs of
NJ &
AND $\to$ implication

as simply typed
λ calculus

$$x : A$$

$$\frac{t_1 : A \qquad t_2 : B}{\langle t_1, t_2 \rangle : A \,\&\, B}$$

$$\frac{t : A \,\&\, B}{\pi_1(t) : A} \qquad \frac{t : A \,\&\, B}{\pi_2(t) : B}$$

$$\cancel{x : A} \qquad \cancel{x : A}$$

$$\vdots$$

$$\frac{t[x] : B}{\lambda x\, t : A \supset B} \to_i$$

$$\frac{f : A \supset B \qquad u : A}{f(u) : B} \to_e$$

$\Pi : A_1 \& \cdots \& A_n \longrightarrow C$

stable function from $A_1 \; \& \quad \& A_n$ to C

product

$A_1 \quad A_n$ free hypotheses

variables

(always possible to add some)

$x : A$

$\lambda y^B x^A : B \to A$

$A \,\&\, A_1 \,\&$ $\qquad \& \, A_n$

projection (stable)

in particular if no free variable

$\lambda x^A . x^A$: Identity $(\{\alpha\}, \alpha)$

$a \quad F(a) = a \qquad$ STABLE

rules for & easy

function with several arguments $t_1 : A_1 \ldots t_n : A_n$

function with ONE argument,

$\langle t_1 : A_1, \ldots, t_n : A_n \rangle : A_1 \& \cdots \& A_n$

pairing and projections
are stable

$$\frac{f : A \to B \qquad t : A}{f(t) : B}$$

App

trace $\to$ stable function

apply the stable function

$$\dfrac{\begin{array}{c} x : A \\ \vdots \\ t : B \end{array}}{(\lambda x^A . t^B) : B}$$

$$(A) \mathbin{\&} \cancel{A_1 \mathbin{\&} \cdots \mathbin{\&} A_n} \xrightarrow{\text{stable}} B$$

$$\cancel{A_1 \mathbin{\&} \cdots \mathbin{\&} A_n} \xrightarrow{\text{stable}} [\![A \to B]\!]$$

trace w.r.t. A

No A_i: $f : A \xrightarrow{\text{stable}} B \rightsquigarrow \mathrm{Tr}(f) \in [\![A \to B]\!]_3$

Denotational?

$$[\![(\lambda x^A\, t^B)\, u^A]\!]$$

$$= App\ [\![\lambda x^A\, t^B]\!]\, [\![u^A]\!]$$

$$= [\![t[u^A/x^A]]\!]$$

We may observe properties of proofs:
if there is no stable function such that ...
then there is no such proof

What have we seen so far?

- fundamentals of proof theory
  + some light model theory
  arithmetic, completeness
- proof theoretical semantics
  interpreting formulas
- semantics of proofs
- new syntax for proofs

meaning, understanding

Semantics is about translation

lots of them this week!

- into a language you better understand
- into different structure that bring a new viewpoint

Jacques LACAN L'étourdit 1973

Le sens ne se produit jamais que de la traduction d'un discours dans un autre.

For the rest of the week:

Proof theoretic semantics

understanding proofs by PTS

→ result on logic(s)

→ new syntax

→ new logics (?)

(see ~~eg~~ Thusday)

An interesting question
full abstraction (completeness)

are all semantic objects
the interpretation
of some proof ?

new viewpoint on the SAME objects

Maximality of cliques $\neq$ totality
unbounded logical complexity $\forall\exists\forall$

A remark on full abstraction
NO QUOTIENT please
(otherwise proof/cut elim)

Multiplicative Linear Logic
with a coherence spaces
(Ralph Loader 1994 (?)

Meaning of P as justifications of P

argumentative dialogues

(computable ≠ model theoretic view)

- Inferentialism
  ludics and dialogue (Myriam Quatrini Friday)

but also the syntactic
(λ calculus) part
of Montague semantics

Before model theoretic interpretation
Montague semantic translation
is • unfolding of the
logical structure
of a sentence
using
- syntactic structure
- word effect on logical structure

Limits

Proof theoretical semantic is simple only for intuitionistic logic(s)

(Lafont? Girard?)

$$
\dfrac{\dfrac{\dfrac{\Pi_1}{\vdash A}}{\vdash A, K}\,w \qquad \dfrac{\dfrac{\Pi_2}{\vdash A}}{\vdash A, K^{\perp}}\,w}{\dfrac{\vdash A, A}{\vdash A}\,\mathrm{contr}}\,\mathrm{cut}
$$

$$
\dfrac{\dfrac{\dfrac{\Pi_1}{\vdash A}}{\vdash A, A}\,w}{\vdash A}\,\mathrm{ctr}
\qquad\qquad
\dfrac{\dfrac{\dfrac{\Pi_2}{\vdash A}}{\vdash A, A}\,w}{\vdash A}\,\mathrm{ctr}
$$

$(A \to 0) \xrightarrow{\text{one}} 0 \sim A$

Joyal no (simple) semantics of proofs
(functions, products,
internalised function space)
for classical logic ($A \to B \sim \neg B \to \neg A$, $\neg\neg A = A$)
(an argument for Linear Logic!)

Indeed at most one proof / function $A \to B$
up to normalisation
(my first talk at Cirm 1992? Girard Takeuti 60)

A deeper objection

$\forall n > 2 \ \forall a, b, c \in \mathbb{N}$

if $a, b, c \neq 0$ then $a^n + b^n \neq c^n$

do we understand better the meaning of this formula from Wiles' proof ???

Perhaps a proof is unreadable
(eg discussion with Wiles)
we actually understand better
or not.

At least for
logically simple
formulas of ordinary language
argumentation, dialogue
improve our understanding

Another embarrassing question:

how to interpret a formula A as the set of its proofs when there is no proof of A???

$$[\![(A \to A) \to A]\!] = \emptyset \quad ???$$

models, valuation etc interpret it

To circumvent this problem

pseudo proofs (endless, cyclic, etc.)

- look like proofs
- interact with other proofs