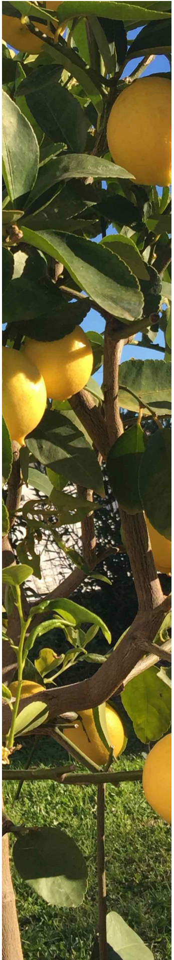
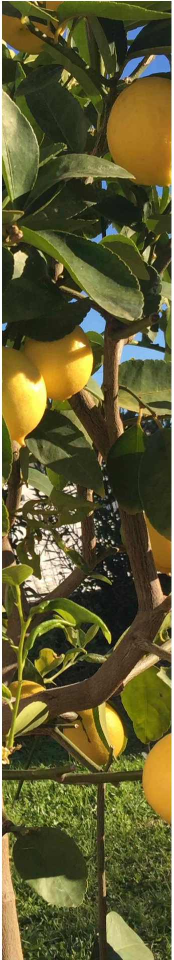


Completeness of first order intuitionistic logic w.r.t. (pre)sheaves of classical models

Christian Retoré (LIRMM, Université de Montpellier)
with Jacques van de Wiele (CNRS U. Paris 7) 1987
Ivano Ciardelli (U. München) 2010-2011,
David Théret (IMAG, U. Montpellier) 2016-2019,
...

LORIA, 15 mars 2024



Remarks

A beautiful subject I learnt about at Paris 7 in 1987— but not my main research area.

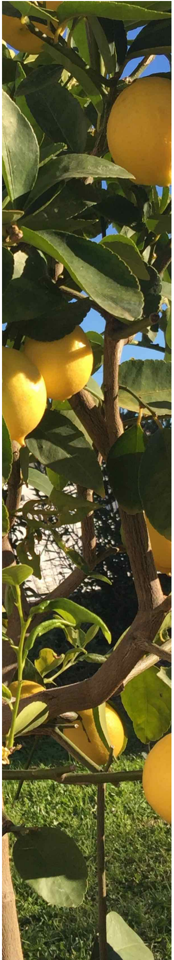
Quite difficult to know what has exactly been achieved on this question.

The presentation of Kripke-Joyal forcing and counter example is from a lecture by Jacques Van de Wiele in 1987.

The direct completeness proof is essentially due to
Ivano Ciardelli (in TACL 2011 cf. reference at the end).

It has been re-worked and extended with David Théret in 2016-2018 (?).

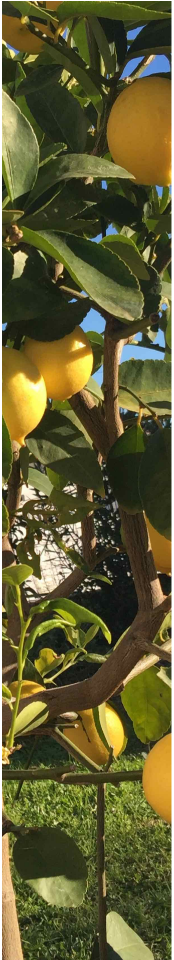
Thanks to the Topos & Logic group (Abdelkader Gouaich, Jean Malgoire, Nicolas Saby, David Theret) of the *Institut Montpelliérain Alexander Grothendieck*



A Logic?

formulas

proofs $\leftrightsquigarrow$ interpretations

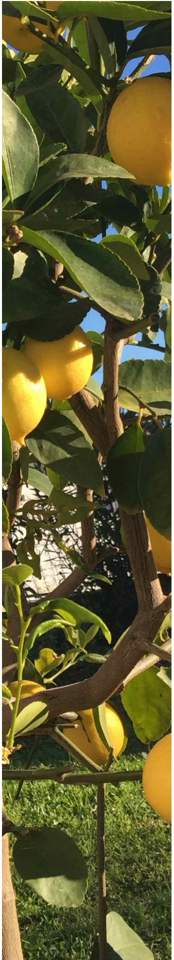


A.1. Formulas, proofs and models, classically

Formulas of a given first-order logical language, say $\mathscr{L}$ consisting of contants, p-ary function symbols, n-ary predicates can be true (or not) in a given $\mathscr{L}$- structure.

An $\mathscr{L}$- structure is simply a set M with an interpretation of:

- each constant of the language as an element in M
- each n-ary predicate of the language as an n-ary relation on M,
- each p-ary function symbol of the language as a p-ary function from M^p to M etc.

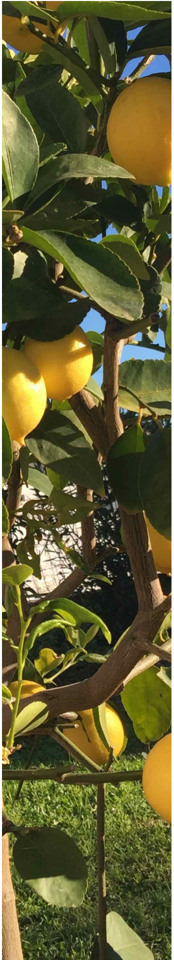


Soundness: what is provable in classical logic (LK) is true in any $\mathscr{L}$-structure.

Completeness: what is true in every $\mathscr{L}$-structure is provable (in LK).

model :
wellformed expressions of a logical language
have a meaning when the language is interpreted.

A proof may prove a formula of a given language,
or prove a formula of a given language
from assumptions (or axioms) of the same language.



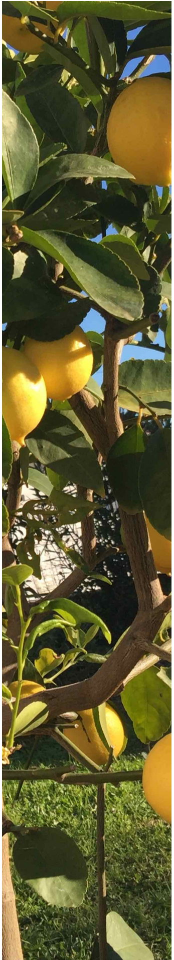
A.2. Usual / classical models of first order logic

One is given a language $\mathscr{L}$,
e.g. constants $(0, 1)$, functions $(+, *)$, and predicates $(\leqslant, =)$.

One is given a set $|M|$ (non empty).

Constants are interpreted by elements of $|M|$,
n-ary functions symbols by n-ary applications from $|M|^n$ to $|M|$, and n-ary predicates by parts of $|M|^n$. ($\mathscr{L}$-structure)

Logical connectives and quantifiers are interpreted intuitively
(Tarskian truth: ”$\wedge$” means ”and”, ”$\forall$” means ”for all” etc.).

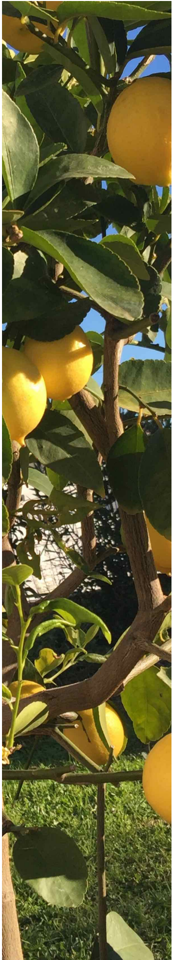


A.3. Soundness

any provable formula *G* is true for every interpretation

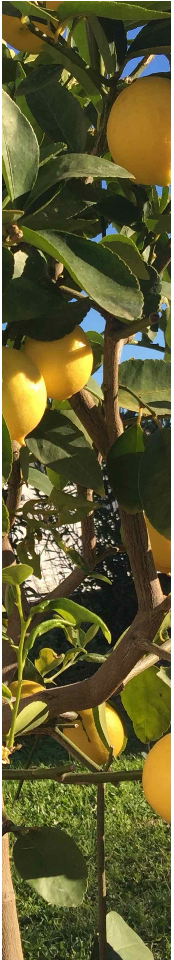
or:

when a theory *Th* entails *G* then any model that satisfies *Th* satisfies *G*

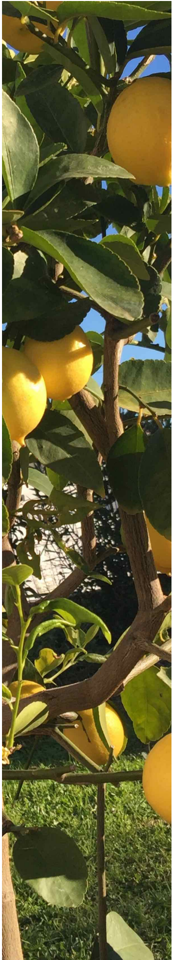


A.4. Completeness

Completeness (a word that often encompass soundness):
a formula G **that is true in every interpretation is derivable**
or
a formula G **that is true in every model of a theory** Th **is a logical consequence of** Th
e.g.
a formula F of ring theory is true in any ring
if and only if
F is provable from the axioms of ring theory
Soundness, completeness (and compacity)
are typical fo first order logic (as opposed to higher order logic).



B Intuitionistic logic



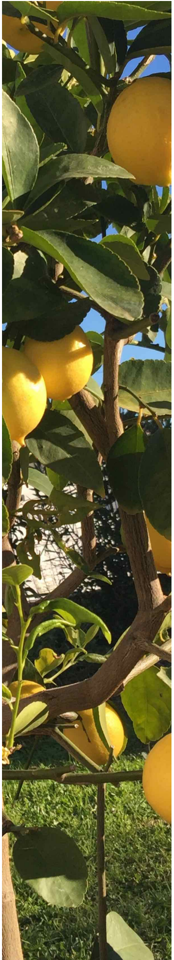
B.1. Intutionistic logic vs. classical logic (the usual logic of mathematics)

Absence of *Tertium non Datur*,
i.e. $A \vee \neg A$ does not always hold.

Disjunctive statemeents are stronger.

Existential statements are stronger.

Proof have a constructive meaning,
algorithms can be extracted from proofs.



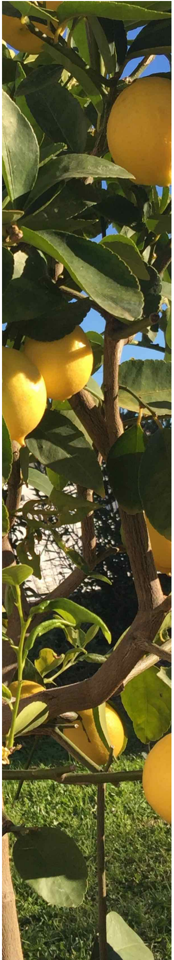
B.2. Rules of intuitionist logic: structures

Structural rules

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ E_g$$

$$\frac{\Delta \vdash C}{A, \Delta \vdash C}\ A_g$$

$$\frac{\Gamma, A, A, \Delta \vdash C}{\Gamma, A, \Delta \vdash C}\ C_g$$



B.3. Rules of intuitionistic logic: connectives

Axioms are $A \vdash A$ (if A then A...) for every A.

Negation $\neg A$ is just a short hand for $A \Rightarrow \bot$.

$$\frac{\Theta \vdash (A \wedge B)}{\Theta \vdash A} \wedge_e \qquad \frac{\Theta \vdash (A \wedge B)}{\Theta \vdash B} \wedge_e$$

$$\frac{\Theta \vdash A \qquad \Delta \vdash B}{\Theta, \Delta \vdash (A \wedge B)} \wedge_d$$

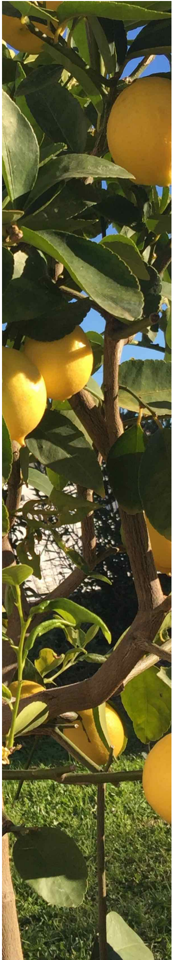
$$\frac{\Theta \vdash (A \vee B) \qquad A, \Gamma \vdash C \qquad B, \Delta \vdash C}{\Theta, \Gamma, \Delta \vdash C} \vee_e$$

$$\frac{\Theta \vdash A}{\Theta \vdash (A \vee B)} \vee_d \qquad \frac{\Theta \vdash B}{\Theta \vdash (A \vee B)} \vee_d$$

$$\frac{\Theta \vdash A \qquad \Gamma \vdash A \Rightarrow B}{\Gamma, \Theta \vdash B} \Rightarrow_e$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash (A \Rightarrow B)} \Rightarrow_d$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash C} \bot_e$$

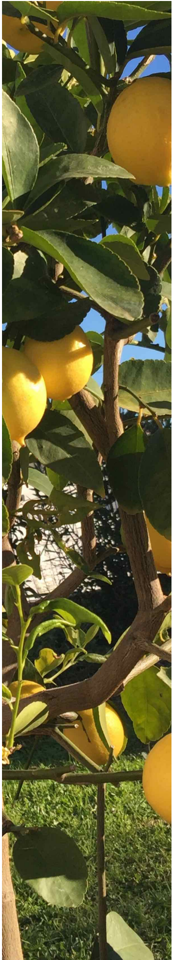


B.4. Rules of intuitionistic logic: connectives

Axioms are $A \vdash A$ (if A then A...) for every A.

Negation $\neg A$ is just a short hand for $A \Rightarrow \bot$.

$\dfrac{\Theta \vdash \forall x A}{\Theta \vdash A[x := t]}\ \wedge_e$	$\dfrac{\Theta \vdash A}{\Theta \vdash \forall x A}\ \forall_d$ (*no free x in Θ*)
$\dfrac{\Theta \vdash \exists x A \quad A, \Gamma \vdash C}{\Theta, \Gamma \vdash C}\ \exists_e$ (*no free x in Γ, C*)	$\dfrac{\Theta \vdash A[t]}{\Theta \vdash \exists x A[x]}\ \exists_d$



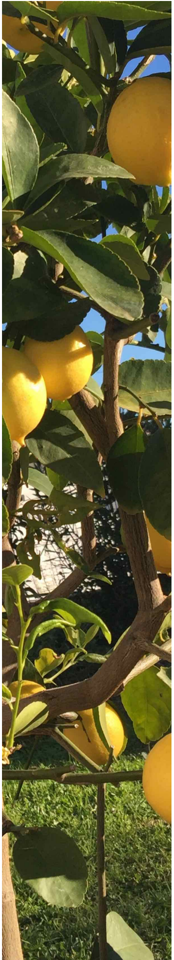
B.5. Differences

$A \vee \neg A$ does not hold for any A.
$\neg\neg B$ does not entail B.
However $\neg\neg(C \vee \neg C)$ hlods for any C.

$$\dfrac{\dfrac{[\neg(A\vee\neg A)]^2 \qquad \dfrac{\dfrac{\dfrac{[\neg(A\vee\neg A)]^2 \quad \dfrac{[A]^1}{A\vee\neg A}\vee_i}{\bot}\to_e}{\neg A}\to_i{}^1}{A\vee\neg A}\vee_i}{\bot}\to_e}{\neg\neg(A\vee\neg A)}\to_i{}^2$$

- tertium non datur
- reductio ad absurdum
- Pierce law $((p \to q) \to p) \to p$

are inutionistically equivalent.



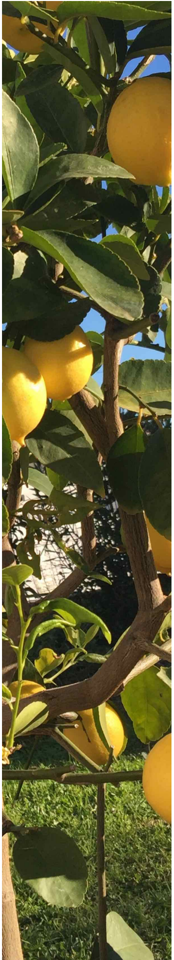
B.6. Relations to classical logic

All ”classical” proofs are valid intuitionistically.

Conversely, $\vdash^{LK} F$ iff $\vdash^{LJ} F^{\neg\neg}$ (Gödel, Glivenko, Kolmogorov).

$$\begin{aligned}
\bot^{\neg\neg} &= \bot \\
a^{\neg\neg} &= \neg\neg a \\
(A \wedge B)^{\neg\neg} &= A^{\neg\neg} \wedge B^{\neg\neg} \\
(A \to B)^{\neg\neg} &= A^{\neg\neg} \to B^{\neg\neg} \\
(\forall x.A)^{\neg\neg} &= \forall x.A^{\neg\neg} \\
(A \vee B)^{\neg\neg} &= \neg\neg(A^{\neg\neg} \vee B^{\neg\neg}) \\
(\exists x.A)^{\neg\neg} &= \neg\neg\exists x.A^{\neg\neg}
\end{aligned}$$

Richard Moot Christian Retoré Classical logic and intuitionistic logic: equivalent formulations in natural deduction, Gödel-Kolmogorov-Glivenko translation — complete proofs of well-known results that are not available elsewhere arXiv:1602.07608



B.7. Existential differences

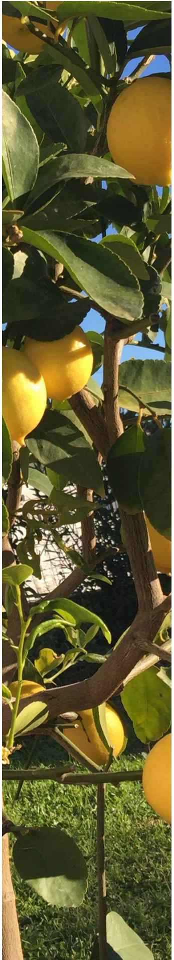
$\neg\forall x.\neg P(x)$ does not entail $\exists x.P(x)$.

A normal intuitionistic proof of $\exists x P(x)$ is a proof of $P(t)$ for some term t.

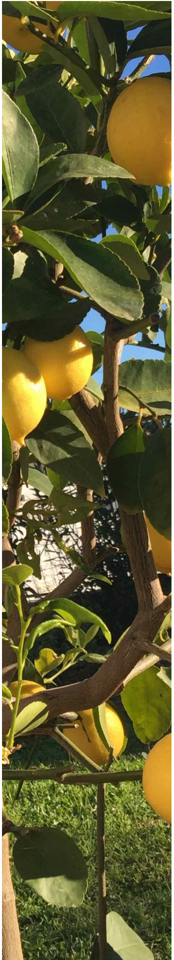
From a proof of $\forall x \exists y\ P(x, y)$ one may extract a function that computes from any x a term $t(x)$ such that $P(x, t(x))$. Extraction of certified functional programs from formal proofs of their specification.

An example, in the language of rings: $\forall x.((x = 0) \vee \neg(x = 0))$ is not provable and their are concrete counter models.

$[\neg\forall x.((x = 0) \vee \neg(x = 0))] \rightarrow [\neg\forall x.\neg\neg((x = 0) \vee \neg(x = 0))]$
is also non provable.



C (Pre)sheaf semantics a.k.a. topological models



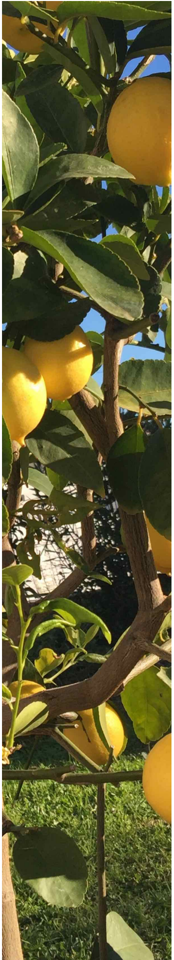
C.1. Presheaves

A pre sheaf can de defined as a contravariant functor F

- from open subsets of a topological set (this partial order can be viewed as a category)
- to a category (e.g. sets, groups, rings):

Contravariant functor: when $U \subset V$ there is a restriction map ρ_V, U from $F(V)$ to $F(U)$ and $\rho_{U_3,U_2} \circ \rho_{U1,U_2} = \rho_{U_1,U_3}$ whenever is makes sense, i.e. when $U_3 \subset U_2 \subset U_1$.

Example of pre-sheaf on the topological space R:
$U \mapsto F(U)$ the ring of bounded functions from U to R.



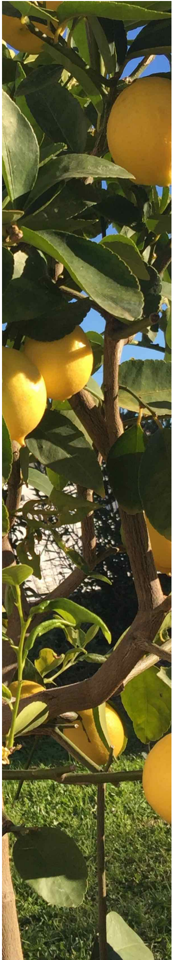
C.2. Sheaves

The presheaf is said to be a sheaf (resp separated presheaf) if every family of compatible elements has unique glueing:

> whenever U_i is cover of an open set U,
>
> with for every i an element $c_i \in F(U_i)$ such that for every pair i, j $\rho_{U_i,U_j}(c_i) = \rho_{U_j,U_i}(c_j)$
>
> there is a unique (resp. at most one) c in $F(u)$ such that $c_i = \rho_{U,U_i}(c)$.

Example of a pre-sheaf that is not a sheaf on the topological space R: $U \mapsto C(U, R)$ the ring of bounded functions from U to R.

Example of a sheaf on the topological space R: $U \mapsto C(U, R)$ the ring of continuous functions from U to R.



C.3. Pre-topology

Grothendieck generalized the notion of topological space, using coverings.

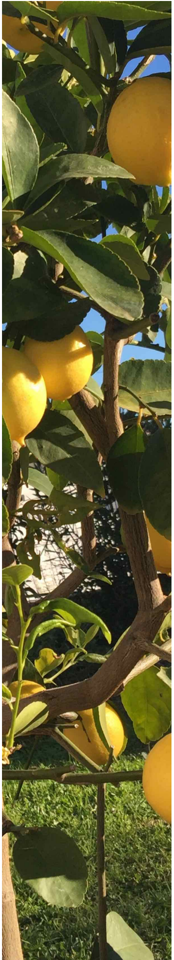
A site is a category with every object endowed with various coverings.

A covering of an object φ consists in a set of arrows $f_i, i \in \mathscr{I}$ with codomain φ_i — when the category is a preorder it is enough to know the domain of every f_i: there is at most one arrow from φ_i to φ.

1. $\varphi \lhd \{\varphi\}$;
2. if $\psi \leqslant \varphi$ and $\varphi \lhd \{\varphi_i \,|\, i \in \mathscr{I}\}$ then $\psi \lhd \{\psi \wedge \varphi_i \,|\, i \in \mathscr{I}\}$;
3. if $\varphi \lhd \{\varphi_i \,|\, i \in \mathscr{I}\}$ and if for each $i \in \mathscr{I}$, $\varphi_i \lhd \{\psi_{i,k} \,|\, k \in \mathscr{K}_i\}$, then $\varphi \lhd \{\psi_{i,k} \,|\, i \in \mathscr{I}, k \in \mathscr{K}_i\}$.

Pre-topology: Grothendieck, SGA4, 1962

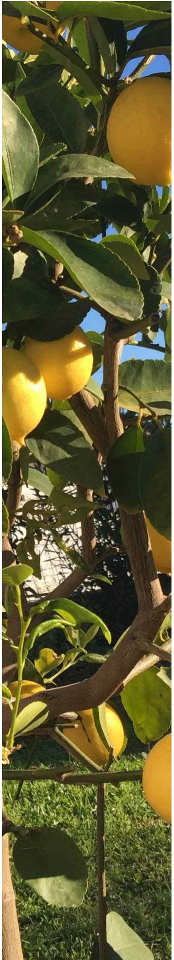
Site: late 70s early 80s Joyal, Lawvere, Lambek,...



C.4. First order language

A first order language $\mathscr{L}$ is defined by

- a collection of predicates (also called relational symbols), each of them endowed with an arity There might be a binary predicate, "=".
- a collection of functions (also called function symbols) each of them endowed with an arity — this collection may include functions of arity 0, which are called constants.

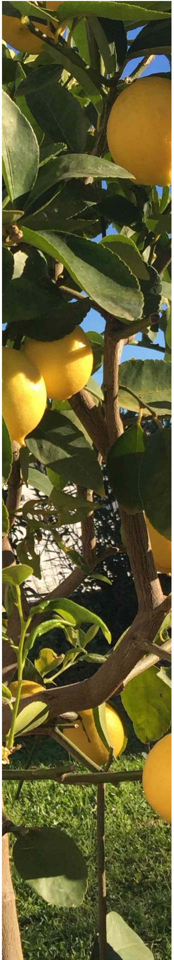


C.5. $\mathscr{L}$-terms

Terms of $\mathscr{L}$ are defined as usual from an at least countable set of variables:

- variables are terms;
- if ϑ is a k-ary function symbols and if $t_1, ..., t_k$ are k terms $\vartheta(t_1, ..., t_k)$ is a term as well — hence, constants, which are 0-ary functions are terms.

A term without variables is said to be a closed term.



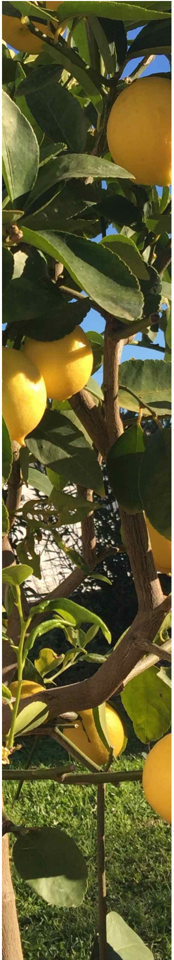
C.6. $\mathscr{L}$-formulas

Given an n-ary predicate R of $\mathscr{L}$, and n terms $t_1, ..., t_n$ of $\mathscr{L}$,

$R(t_1, ..., t_n)$ is an atomic $\mathscr{L}$-formula

Formulas of $\mathscr{L}$ are defined as follows:

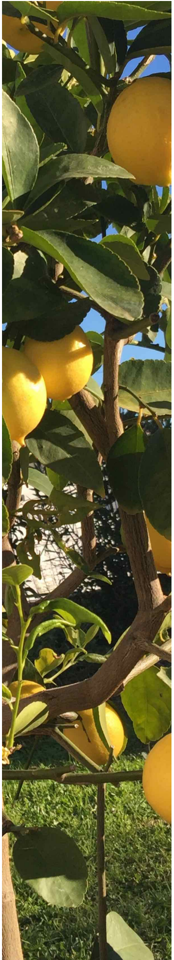
- atomic $\mathscr{L}$-formulas are formulas; among atomic formulas we have $\perp$ a proposition that is a 0 ary predicate symbol.
- if F and G are $\mathscr{L}$-formulas, $F \wedge G$, $F \vee G$ and $F \rightarrow G$ are $\mathscr{L}$-formulas.
- if F is an $\mathscr{L}$-formula, then $\neg F$ is just a short hand for the $\mathscr{L}$-formula $F \rightarrow \perp$;
- if F is an $\mathscr{L}$-formula, and if x is a variable $\forall x\ F$ and $\exists x F$ are $\mathscr{L}$-formulas.



C.7. $\mathscr{L}$-formulas: bound and free variables

Bound and free occurrences of variables are defined as expected:

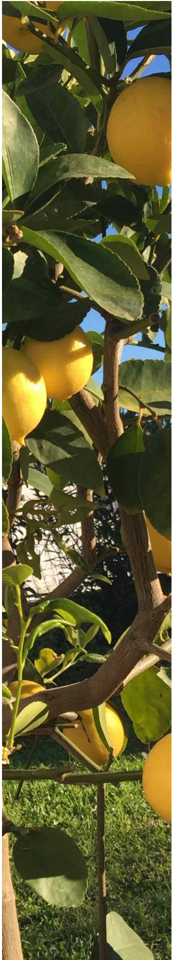
- any occurrence of a variable in an atomic formula is free;
- an occurrence of a variable in $F \wedge G$, $F \vee G$ and $F \rightarrow G$ is free (resp. bound) if and only if this occurrence is free (resp. bound) in the subformula F or in G in which the occurence is.
- an occurrence of a variable in $\neg F$ is free (resp. bound) if and only if tis is free (resp. bound) in the subformula F
- the free occurrences of x in F are bound by $\exists x$ in $\exists xF$ and they are bound by $\forall x$ in $\forall xF$; occurrences of variables other than in $\exists xF$ or $\forall xF$ are free (resp. bound) iff they are free (resp. bound) in F.



C.8. $\mathscr{L}$-structure

Given a first order language $\mathscr{L}$ an $\mathscr{L}$-structure (or a model) M_u is a non empty set $|M_u|$ and an interpretation of the symbols in the language:

- if ϑ is an *n*-ary function the interpretation ϑ_u of ϑ in M_u is a *k*-ary (total) function $\vartheta_u : |M_u|^k \mapsto |M_u|$ — in particular a constant *a* is interpreted as an element a_u of $|M_u|$ (a function from $\{*\}$ to $|M_u|$).
- if R is an *n*-ary predicate, the interpretation R_u of R in M_u is an *n*-ary relation R_u on $|M_u|$ i.e. $R_u \subset |M_u|^n$. If there is the equality "=" predicate, it is necessarily interpreted by equality in $|M_u|$ i.e. $=_u$ is $\{(x, x) | x \in |M_u|\}$.



C.9. Morphisms of $\mathscr{L}$-structures

Let M_u and M_v be two $\mathscr{L}$ structures over the same language — interpretation in u or in v of function symbols (e.g. ϑ) and predicates (e.g. R) are denoted with a subscript u or v (e.g. R_u ϑ_u: interpretations in M_u and R_v ϑ_v: interpretations in M_v).

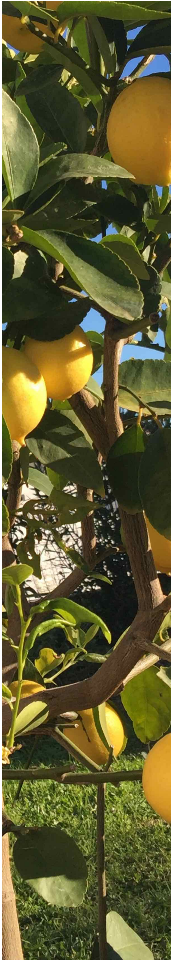
A map $\rho_{u \to v}$ from $|M_u|$ to $|M_v|$ is said to be a morphism of $\mathscr{L}$-structures when:

- For any k-ary function ϑ symbol of $\mathscr{L}$:

  $$\forall c_1, \dots, c_k \in |M_u| \quad \rho_{u\to v}(\vartheta_u(c_1, \dots, c_k)) = \vartheta_v(\rho_{u\to v}(c_1), \dots, \rho_{u\to v}(c_k))$$

- For any n-ary predicate R of $\mathscr{L}$:

  $$\forall c_1, \dots, c_n \in |M_u| \quad \mathit{if}(c_1, \dots, c_n) \in R_u \text{ then } (\rho_{u\to v}(c_1), \dots, \rho_{u\to v}(c_k)) \in R_v$$

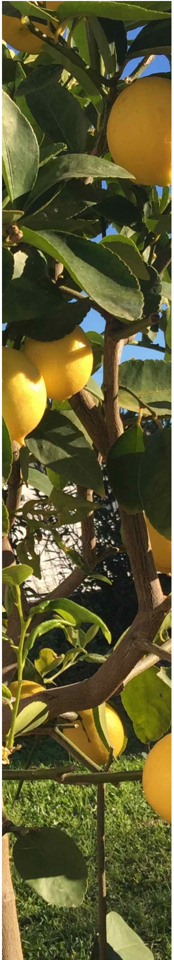


C.10. Presheaf semantics: models

A **presheaf model** M for $\mathscr{L}$ is a presheaf of first-order $\mathscr{L}-$structures over a Grothendieck site $(\mathscr{C}, \lhd)$ (or a topological space viewed as a poset for inclusion):

- for any object u an $\mathscr{L}$ structure M_u
- for any arrow $f : v \hookrightarrow u$ a morphism of $\mathscr{L}$ structures (cf. supra) $M(f) : M_u \to M_v$

satisfying the following extra conditions.
Separateness For any elements a, b of M_u,
if there is a cover $u \lhd \{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ such that for all $i \in \mathscr{I}$ we have $M(f_i)(a) = M(f_i)(b)$,
then $a = b$.
Locality For any $n-$ary relation symbol R,
for any tuple $(a_1, \ldots, a_n)$ from M_u
if there is a cover $u \lhd \{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$
such that $\forall i \in \mathscr{I}$ one has $(M(f_i)(a_1), \ldots, M(f_i)(a_n)) \in R_{u_i}$,
then $(a_1, \ldots, a_n) \in R_u$.



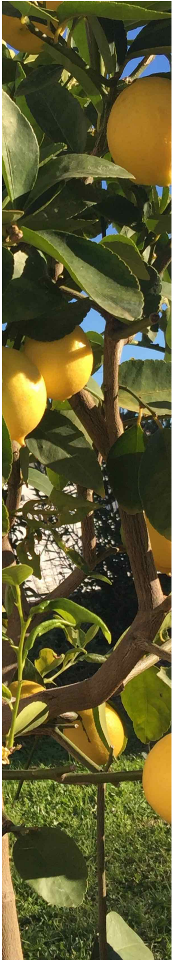
C.11. Presheaf semantics: Kripke-Joyal forcing — 1/4 assignments

Given a presheaf model M, and some open u, we inductively define for any formula G of $\mathscr{L}$ the relation $u \Vdash G$ ("meaning": G is true at u).

Assignment A usual, in order to define $u \Vdash G$, we need an assignment ν in M_u of the free variables of G, and this is written $u \Vdash_\nu G$ with $\nu = [z_1 \mapsto c_1; \cdots; z_p \mapsto c_p]$ where the z_i are the free variables in G and $c_i \in |M_u|$.

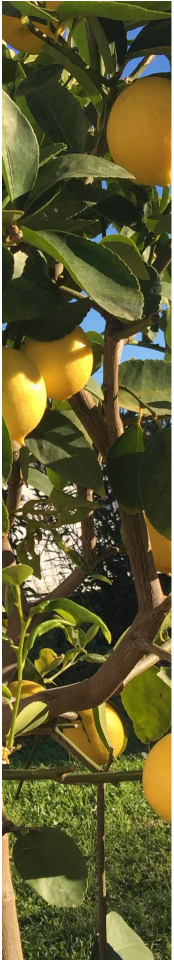
As we shall see, $u \Vdash_\nu G$ can be defined from $v \Vdash_{\nu'} G'$ with $f : v \hookrightarrow u$ and with G' having free variables among those of G (plus possibly one free variable in the $\exists$ and $\forall$ cases).

If $\nu = [z_1 \mapsto c_1; \cdots; z_p \mapsto c_p]$
we naturally define ν' by $\nu' = [z_1 \mapsto M(f)(c_1); \cdots; z_p \mapsto M(f)(c_p)]$
where $M(f)$ is the restriction $M(f) : |M_u| \to |M_v|$.



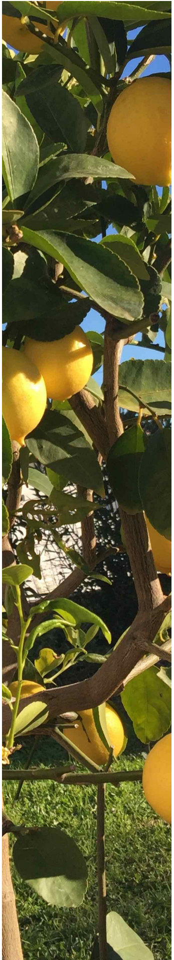
C.12. Presheaf semantics: Kripke-Joyal forcing — 2/4 atoms and conjunction

- $u \Vdash_v R(t_1, \ldots, t_n)$ iff $([t_1]_v, \ldots, [t_n]_v) \in R_u$.
- $u \Vdash_v t_1 = t_2$ iff $[t_1]_v = [t_2]_v$.
- $u \Vdash_v \bot$ iff $u = \emptyset$ It is so, because the empty covering is a covering (with 0 open) of the empty open. Hence, because of the locality condition on atoms, the empty open forces all atomic formulas including $\bot$.
- $u \Vdash_v \varphi \wedge \psi$ iff $u \Vdash_v \varphi$ and $u \Vdash_v \psi$.



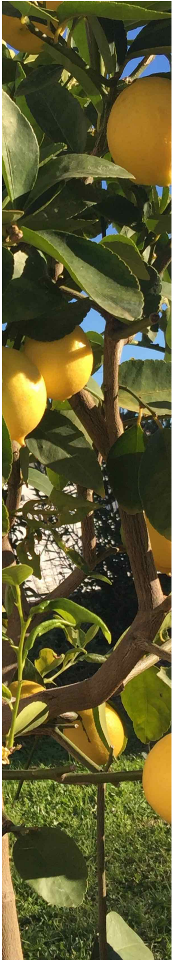
C.13. Presheaf semantics: Kripke-Joyal forcing — 3/4 disjunction and existential

- $u \Vdash_v \varphi \vee \psi$ iff there exists a covering family $\{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ such that for any $i \in \mathscr{I}$ we have $u_i \Vdash_{v_i} \varphi$ or $u_i \Vdash_{v_i} \psi$.
  Alternatively, $u \Vdash_v \varphi \vee \psi$ there exist two opens u_1, u_2 with $u_1 \cup u_2 = u$ such taht $u_1 \Vdash \varphi$ and $u_2 \Vdash \psi$.
- $u \Vdash_v \exists x \varphi$ iff there exists a covering family $\{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ and elements $a_i \in |M_{u_i}|$ for $i \in \mathscr{I}$ such that $u_i \Vdash_{v_i \cup [x \mapsto a_i]} \varphi$ for any index i.



C.14. Presheaf semantics: Kripke-Joyal forcing — 4/4 implication and universal

- $u \Vdash_\nu \varphi \to \psi$ iff for all $f : v \hookrightarrow u$, if $v \Vdash_{\nu_v} \varphi$ then $v \Vdash_{\nu_v} \psi$.
- $u \Vdash_\nu \neg\varphi$ iff for all $f : v \hookrightarrow u$, with $v \neq \emptyset$, $v \nVdash_{\nu_v} \varphi$. This is obtain from $\emptyset \Vdash \bot$ and $\to$ cases because $\neg\varphi = \varphi \to \bot$.
- $u \Vdash_\nu \forall x \varphi$ iff for all $f : v \hookrightarrow u$ and all $a \in M_v$, $v \Vdash_{\nu_v \cup [x \mapsto a]} \varphi$.

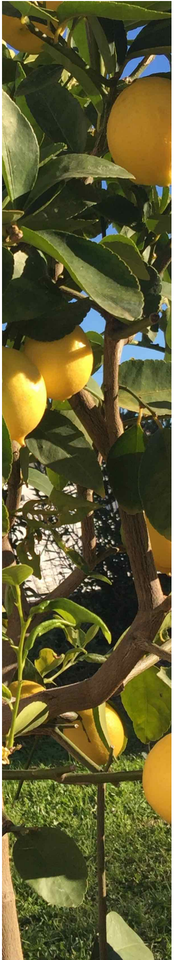


C.15. Validity

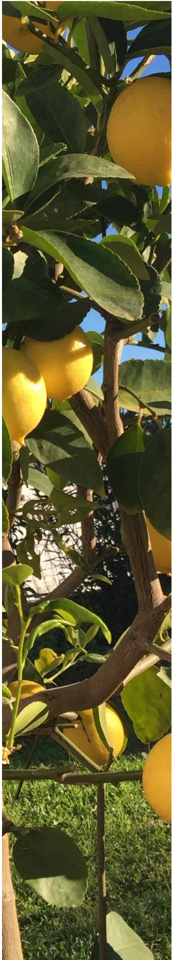
A formula G is said to be valid in a topological model in a presheaf model over a topological space $(X, O(X))$ or a pre-topology whenever

$$X \Vdash G$$

i.e. G is true at the global section.



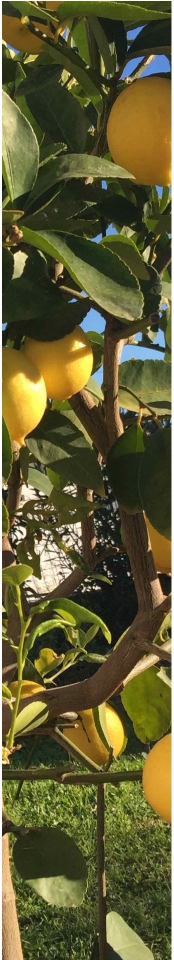
D An example



D.1. Language

Let us consider the language of ring theory:

- two constants $0, 1$
- two binary functions $+, \cdot$
- "=" equality as the only predicate



D.2. The (pre)sheaf of $\mathscr{L}$-structures

A presheaf model over the topological space $\mathbb{R}$ for this language is defined by $|M_u| = C(u, \mathbb{R})$ the continuous functions from u to $\mathbb{R}$:

$0_u(x) = 0$ and $1_u(x) = 1$ for all $x \in u$ (constant functions),

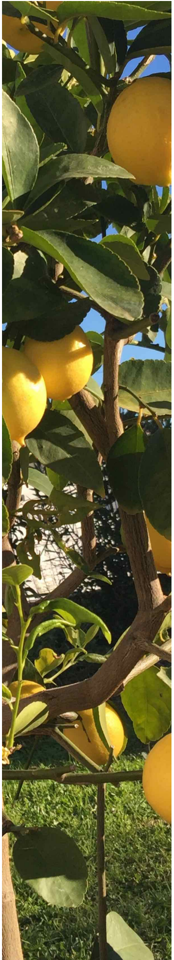
$+_u$ pointwise addition $(f +_u g)(x) = f(x) + g(x)$,

$\cdot_u$ pointwise multiplication $(f \cdot_u g)(x) = f(x) \cdot g(x)$.

The restriction $\rho_{u \to v} : |M_u| \to |M_v|$ morphism, when $v \hookrightarrow u$ is defined by: $\forall f \in C(u, \mathbb{R}) \forall x \in v \; \rho_{u \to v}(f)(x) = f(x)$.

$\rho_{u \to v}$ is a morphism, because:

- $\rho_{u \to v}(0_u) = 0_v, \quad \rho_{u \to v}(1_u) = 1_v$
- $\rho_{u \to v}(f +_u g) = \rho_{u \to v}(f) +_v \rho_{u \to v}(g), \quad \rho_{u \to v}(f ._u g) = \rho_{u \to v}(f) ._v \rho_{u \to v}(g)$
- (”=” is the only predicate) $\forall f, g \in |M_u| = C(u, \mathbb{R})$ if $f = g$ in $C(u, \mathbb{R})$ then $\rho_{u \to v}(f) = \rho_{u \to v}(g)$ in $C(v, \mathbb{R})$.

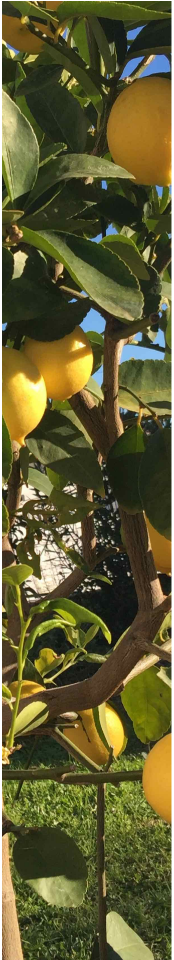


D.3. Locality and separateness conditions

Locality condition for atoms:

"=" is the only predicate so we just have to check that, given two elements a and b of M_u
if there is a cover $u \lhd \{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ such that $\forall i \in \mathscr{I}$ we have $(\rho_{u \to u_i}(a)) = \rho_{u \to u_i}(b))$ in $|M_{u_i}|$,
then $a = b$ in $|M_u|$. This is true, because two functions that are equal on each open of a covering of u are equal on u.

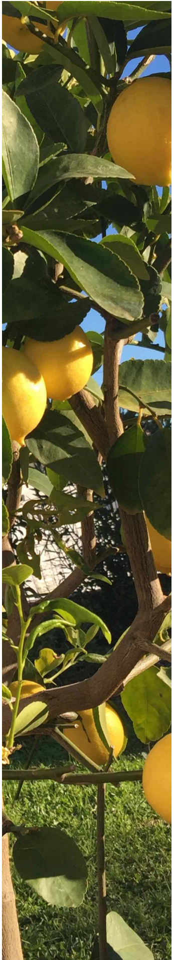
Separateness is exactly the locality condition for our unique predicate, i.e. the "=" predicate, which is interpreted as "=".



D.4. Remarks

This presheaf is a sheaf: given a cover u_i of $\mathbb{R}$ and a family $f_i \in C(u_i, \mathbb{R})$ such that any two f_j and f_k agree on $u_j \cap u_k$ for all j, k there exists a unique $f \in C(\mathbb{R}, \mathbb{R})$ s.t. $f\big|_{u_i} = f_i$.

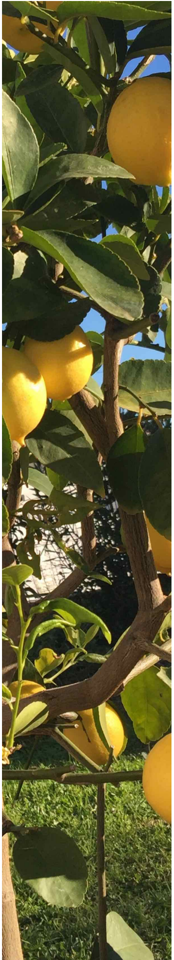
If one takes bounded functions from R to R this is not true anymore. Covering $I_n =]1/n, 1/n + 2[$ $f_n = 1/n$



D.5. Truth in a (pre)sheaf model

We say that Γ entails φ in presheaf semantics, in symbols, in case for any open u of any presheaf model and for any assignment v into $|M_u|$, if $u \Vdash_v \gamma$ for any $\gamma \in \Gamma$ then $u \Vdash_v \varphi$.

If $\Gamma \vdash \varphi$ is provable in intuitionnistic logic, then Γ entails φ in presheaf semantics.

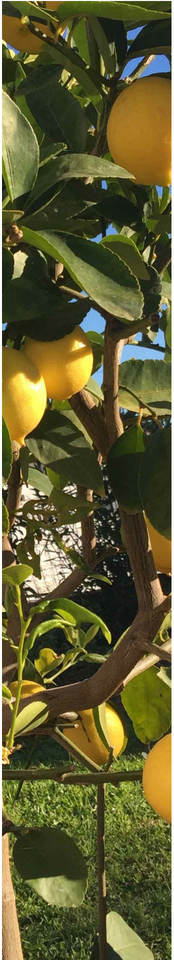


D.6. Properties of Kripke-Joyal forcing

Functoriality of $\Vdash$:
if $f_i : U_i \to U_j$ and $U_j \Vdash F(t_1, \ldots, t_n)$
then $U_i \Vdash F(t_1^i, \ldots, t_n^i)$ where t_k^i is the restriction of t_k to U_i.

We asked for the validity of atoms to be local, but Krike-Joyal forcing propagates this property to all formulae:

Locality lemma: If there exist a covering of U by $f_i : U_i \to U$ and if for all i one has $U_i \Vdash F(t_1^i, \ldots, t_n^i)$ then $U \Vdash F(t_1, \ldots, t_n)$



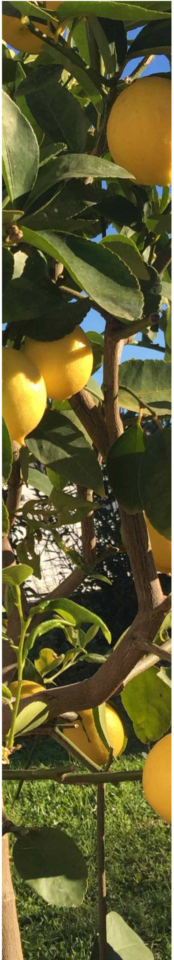
D.7. Soundness

Whenever $\Vdash F$ in IQC then any presheaf semantics satisfies F.

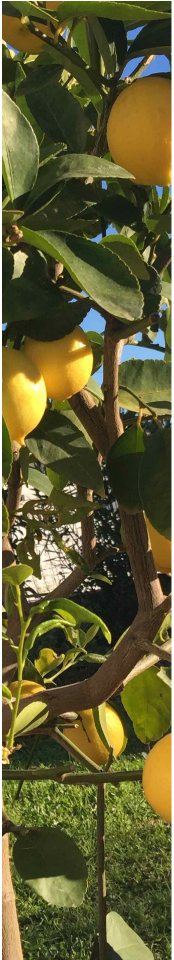
Whenever $\Gamma \Vdash F$ in IQC then any presheaf semantics that satisfies Γ satisfies F as well.

The theory of rings, whose language has two binary functions $(+, .)$ two constants $0, 1$ and equality, can be interpreted in the presheaf on the topological space R which maps U to the ring $C_{U,R}$ of continuous functions from the open set U to R.

In this model, both $[\neg\forall x.((x = 0) \vee \neg(x = 0))]$
and $[\forall x \neg\neg((x = 0) \vee \neg(x = 0))]$ are both valid.



E An formula of ring theory classically provable false in some intuitionistic models



E.1. A remark on $C(U,\mathbb{R})$ 1/3

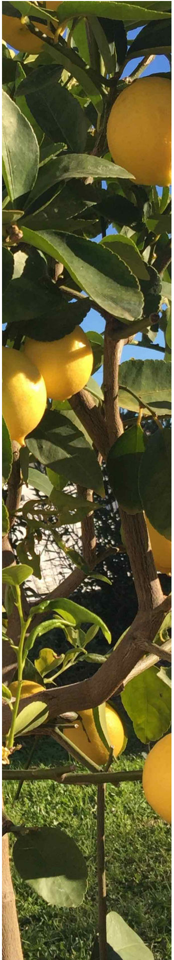
Given any non empty open subset $U \subset \mathbb{R}$ there exist

- an open subset $V =]a,b[\subset U$

- and a continuous function $\ell : V \mapsto \mathbb{R}$

such that $V \not\Vdash_{[x \mapsto \ell]} (x = 0) \vee \neg(x = 0)$ with ℓ:

$$\begin{array}{rcccl} \ell : &]a,b[& \mapsto & \mathbb{R} & \\ & x & \mapsto & 0 & \text{if } x \leqslant (a+b)/2 \\ & x & \mapsto & x-(a+b)/2 & \text{if } x \geqslant (a+b/2) \end{array}$$



E.2. A remark on $C(U, \mathbb{R})$ 2/3

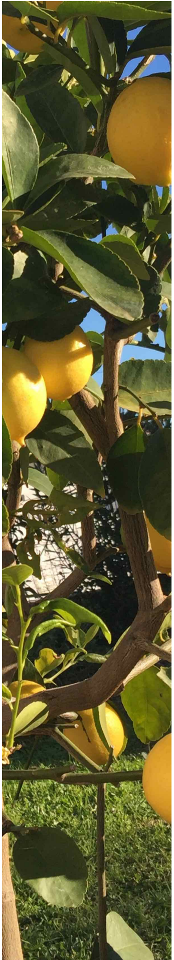
$]a, b[\not\Vdash_{[x \mapsto \ell]} (x = 0 \vee \neg(x = 0)).$

We proceed by contradiction (the meta logic is classical).

Let us assume that $]a, b[\Vdash_{[x \mapsto \ell]} (x = 0 \vee \neg(x = 0)).$

Thus, there exist two open sets u_1, u_2 with $u_1 \cup u_2 =]a, b[$, s.t.:

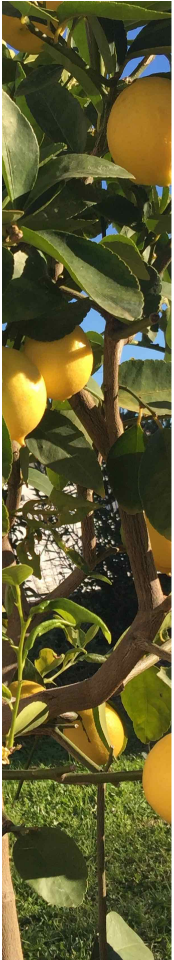
- $u_1 \Vdash_{[x \mapsto \ell_{u_1}]} x = 0$ i.e. $\forall x_1 \in u_1 \quad \ell(x_1) = 0$
- $u_2 \Vdash_{[x \mapsto \ell_{u_2}]} \neg(x = 0)$ i.e. $\forall v_2 \subset u_2, v_2 \neq \emptyset \quad v_2 \not\Vdash_{[x \mapsto \ell_{v_2}]} x = 0$
  i.e. ℓ never is constantly 0 on a non empty open $v_2 \subset u_2$.



E.3. A remark on $C(U,\mathbb{R})$ 2/3

This is impossible because $(a+b)/2$ must be in u_1 or in u_2.

- If $(a+b)/2 \in u_1$ then ℓ should be constantly 0 on a neighbourhood of $(a+b)/2$, but $\ell(x) > 0$ when $x > (a+b)/2$.
- If $(a+b)/2 \in u_2$ then ℓ should never be constantly 0 on any open $v_2 \subset u_2$ but ℓ is constantly 0 on $v_2 =](a+b)/2 - \varepsilon, (a+b)/2[\subset u_2$.



E.4. A classically valid but intuitionistically non valid formula

$C(\mathbb{R},\mathbb{R})$ validates $\neg\forall x\ (x=0)\vee\neg(x=0)$ (*).

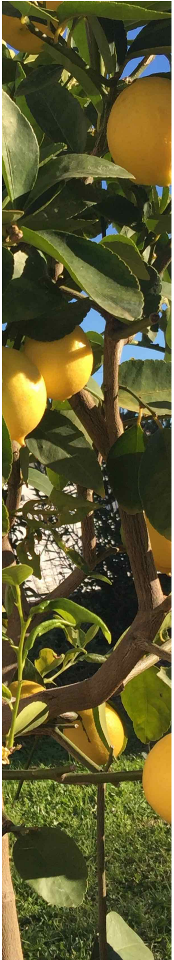
Indeed, according to Kripke-Joyal $\mathbb{R}\Vdash\neg\forall x\ (x=0)\vee\neg(x=0)$ means that for every non empty open $u\subset\mathbb{R}$, $u\nVdash\forall x\ (x=0)\vee\neg(x=0)$.

But $u\Vdash\forall x\ (x=0)\vee\neg(x=0)$ means that for every open $v\subset u$ and for every $f\in C(v,\mathbb{R})$ $v\Vdash_{[x\mapsto f]}(x=0)\vee\neg(x=0)$.

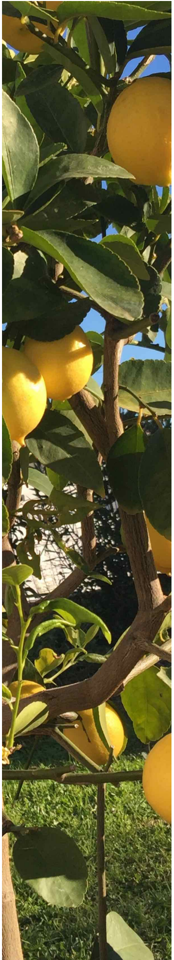
We precisely established supra (with ℓ)
that $u\nVdash\forall x\ (x=0)\vee\neg(x=0)$.

But $C(\mathbb{R},\mathbb{R})$ validates $\forall x\ \neg\neg((x=0)\vee\neg(x=0))$ (**)
— because $\vdash\neg\neg(C\vee\neg C)$ is provable for all C.

However in classical logic (*) is the negation of (**) !!!



F Completeness



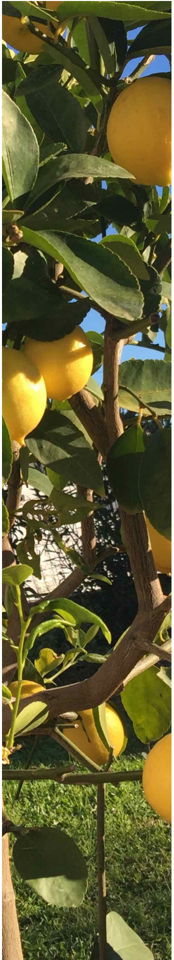
F.1. Statements

Soundness: H intuitionistically provable (under Th)
$\Rightarrow$ H is true at any open of any topological interpretation (satisfying Th).

Completeness: F true at the global section of any topological interpretation (satisfying Th)
$\Rightarrow$ F intuitionistically provable (from Th).

Two lemmas:

- (functoriality) **if** $H[c_1, ..., c_n]$ true at U
  then $H[{c_1}^V, ..., {c_n}^V]$true at any open $V \subset U$.
- (locality) The locality condition for atomic formula (cf. above) extends to *any* formula:
  if (u_i) covers u and for all i $u_i \Vdash_{x_k \mapsto c_k^{u_i}} G$
  then $u \Vdash_{x_k \mapsto c_k} G$.



F.2. Proof of soundness

Induction on the proof height, looking at every possible last rule, e.g. in natural deduction. Below: $\vee_e$ case.

$$\frac{\Theta \vdash (A \vee B) \qquad A, \Gamma \vdash C \qquad B, \Delta \vdash C}{\Theta, \Gamma, \Delta \vdash C} \vee_e$$

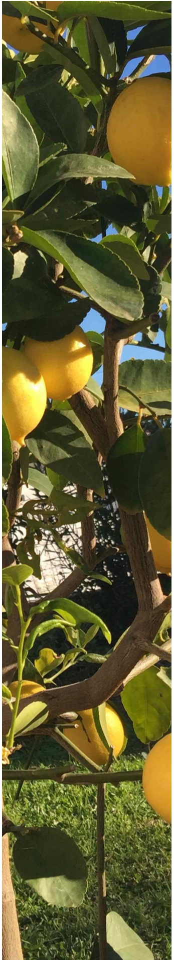
We have to show that **if** $U \Vdash \Theta, \Gamma, \Delta$ **then** $U \Vdash C$.

If $U \Vdash \Theta$ by induction hypothesis, $U \Vdash A \vee B$. Hence, there exists a covering (U_i) such that for every i $U_i \Vdash A$ or $U_i \Vdash B$.

If $U_i \Vdash A$, because $U \Vdash \Gamma$ we have $U_i \Vdash \Gamma$ (functor property), and by induction hypothesis (proof of $A, \Gamma \vdash C$) $U_i \Vdash C$.

Similarly, if $U_i \Vdash B$, then $U_i \Vdash C$.

So for all i $U_i \Vdash C$ and by locality lemma $U \Vdash C$.



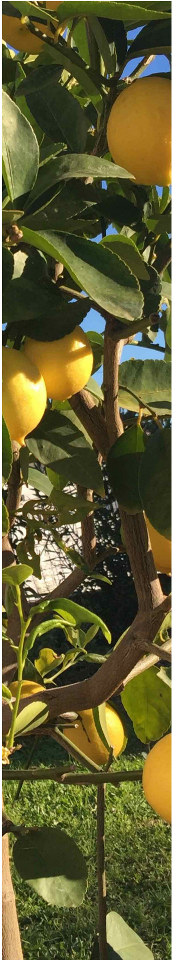
F.3. Completeness for presheaf semantics

If every presheaf models satisfies φ
then φ is provable in **intuitionistic** logic.

Usually established by:

- equivalence with $\Omega-$models;
- construction of a canonical Kripke model.

Here: canonical model (separated presheaf) in which *F* valid means *F* intuitionistically provable.



F.4. Canonical model construction: the underlying site

For completeness with a theory Th add things in GREEN.

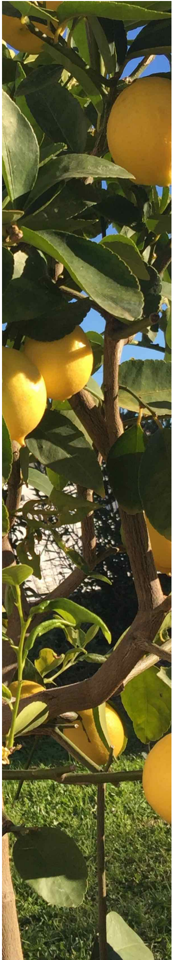
Direct proof: a canonical ”syntactic” model.

Canonical site:

- **Category:** we take the Lindenbaum-Tarski algebra $\overline{\mathscr{L}}$
  - Objects: classes of provably equivalent formulas $\overline{\varphi}$.
    $\overline{\varphi}$ contains formulas φ' such that $\varphi \vdash \varphi'$ and $\varphi' \vdash \varphi$
    Using Th: $Th, \varphi \vdash \varphi'$ and $Th, \varphi' \vdash \varphi$
  - Arrows: $\overline{\varphi} < \overline{\psi} \iff \varphi \vdash \psi$ (using Th: $\varphi, Th \vdash \psi$)
- **Grothendieck topology:** $\overline{\varphi} \lhd \{\psi_i\}_{i \in I}$ whenever

$$\forall \chi \left[\quad \varphi \vdash \chi \quad \textit{iff} \quad (\forall i \in I \quad \psi_i \vdash \chi) \right]$$

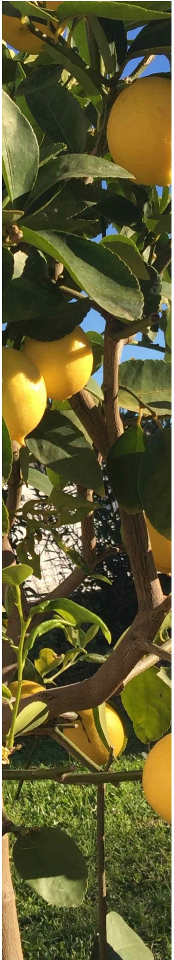
Think of the last line as $\varphi = \bigvee_i \psi_i$
(incorrect, because FOL formulae are finite!)



F.5. Properties of this site

The proposed site is actually a site
i.e. it enjoys the following three properties
which generalise coverings (Grothendieck pretopology)

1. $\varphi \lhd \{\varphi\}$;
2. if $\psi \vdash \varphi$ (i.e. $\psi < \varphi$)and $\varphi \lhd \{\varphi_i \,|\, i \in \mathscr{I}\}$
   then $\psi \lhd \{\psi \wedge \varphi_i \,|\, i \in \mathscr{I}\}$;
3. if $\varphi \lhd \{\varphi_i \,|\, i \in \mathscr{I}\}$
   and if for each $i \in \mathscr{I}$, $\varphi_i \lhd \{\psi_{i,k} \,|\, k \in \mathscr{K}_i\}$,
   then $\varphi \lhd \{\psi_{i,k} \,|\, i \in \mathscr{I}, k \in \mathscr{K}_i\}$.

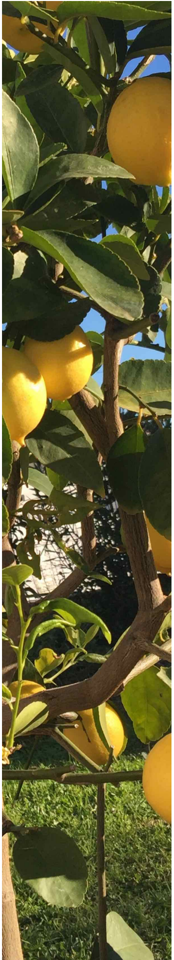


F.6. Canonical model construction: the presheaf

- Put $t \equiv_\varphi t'$ in case $\varphi \vdash t = t'$.
- Denote by t^φ the equivalence class of t modulo $\equiv_\varphi$.

Canonical presheaf:

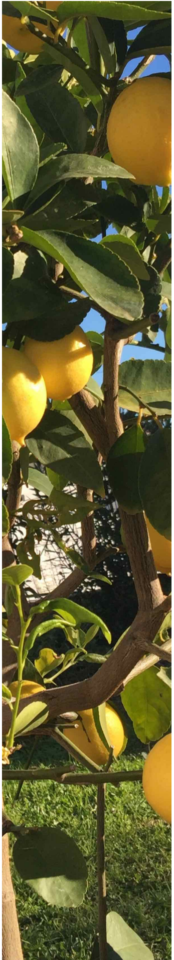
- Model $M_{\overline{\varphi}}$:
  1. Universe $|M_{\overline{\varphi}}|$:
     set of equivalence classes t^φ of closed terms;
  2. Function symbols: $f_{\overline{\varphi}}(\vec{t}^\varphi) = f(\vec{t})^\varphi$;
  3. Relation symbols: $\vec{t}^\varphi \in R_{\overline{\varphi}} \iff \varphi \vdash R(\vec{t})$.
- Restriction. If $t^\psi \in M_{\overline{\psi}}$ and $\overline{\varphi} \leq \overline{\psi}$, put $t^\psi \upharpoonright_{\overline{\varphi}} = t^\varphi$.



F.7. The canonical presheaf is well defined

The canonical presheaf is separated: when two elements coincidate on each part of a cover, then they are equal.

Interpretation of atomic formulas is local: if an atomic formula holds on each part of a cover of U then it holds on U.



F.8. Method for the proof of completeness

$$\forall\psi\Big[\,\forall\varphi\ [\ \textit{if}\quad \overline{\varphi}\Vdash\psi\quad \textit{then}\quad \varphi\vdash\psi]\,\Big]$$

or without much additional effort

$$\forall\psi\Big[\,\forall\varphi[\overline{\varphi}\Vdash\psi\ \textit{iff}\ \varphi\vdash\psi]\,\Big]$$

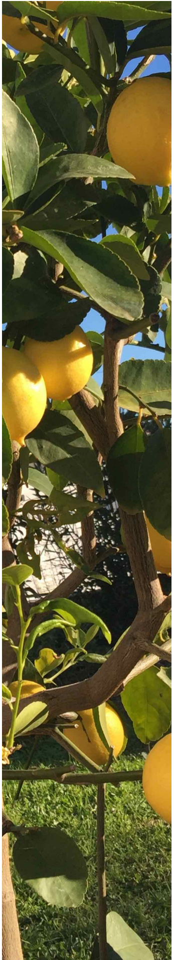
By induction on the formula ψ.

What is fun is that soundness mainly uses introduction rules while completeness mainly uses elimination rules.

The quotient on formulas is not really needed.

Having equality is not mandatory but pleasant.

With a theory Th (strong completeness):
if in every interpretation (for any u, $u\Vdash Th$ *entails* $u\Vdash X$)
then (iff) $Th\vdash X$.



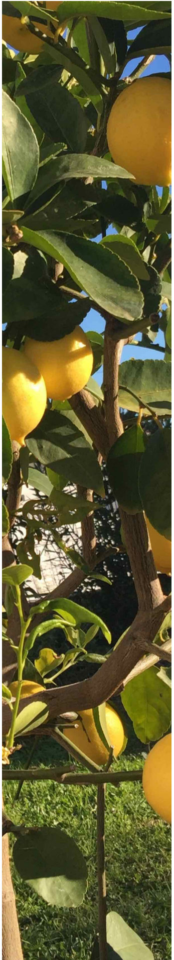
F.9. Sketch of completeness proof

Truth Lemma 1. For any formula φ and sentence ψ,

$$\overline{\varphi} \Vdash \psi \iff \varphi \vdash \psi$$

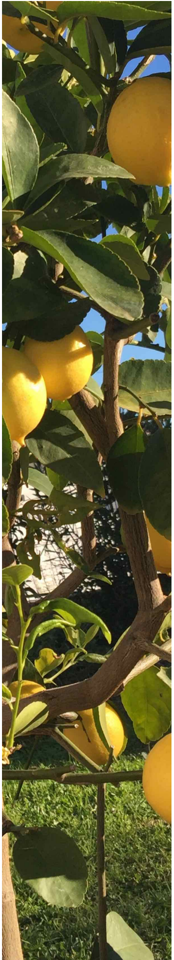
Proof By induction on ψ. The two directions of each inductive step amount to the introduction and elimination rules for the given logical constant.

Let us look at the case of the existential quantifier.



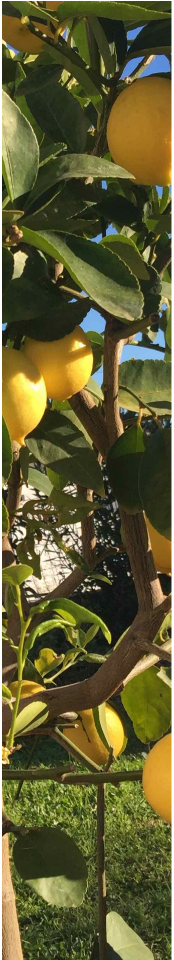
F.10. Completeness $\exists$ direction $\Rightarrow$

- Suppose $\overline{\varphi} \Vdash \exists x\, \psi(x)$.
- There is a family $\{\overline{\varphi_i} \mid i \in \mathscr{I}\}$ and elements $t_i^{\varphi_i} \in M_{\overline{\varphi_i}}$ such that $\overline{\varphi_i} \vdash_{[x \mapsto t_i^{\varphi_i}]} \psi(x)$ for all $i \in \mathscr{I}$.
- Since $[t] = t^{\varphi_i}$ for closed t at $\overline{\varphi_i}$, this is $\overline{\varphi_i} \Vdash \psi(t_i)$.
- By induction hypothesis amounts to $\varphi_i \vdash \psi(t_i)$.
- By rule $(\exists i)$, for any $i \in \mathscr{I}$ we have $\varphi_i \vdash \exists x\, \psi(x)$.
- Since $\overline{\varphi} \lhd \{\overline{\varphi_i} \mid, i \in \mathscr{I}\}$, by the meaning of $\lhd$ we have $\varphi \vdash \exists x\, \psi(x)$.



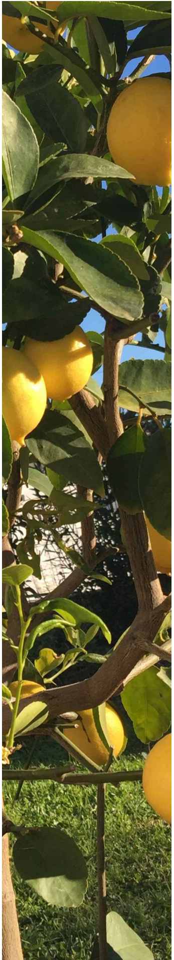
F.11. Completeness $\exists$ direction $\Leftarrow$

- Suppose $\varphi \vdash \exists x \, \psi(x)$.
- We must provide a covering of $\overline{\varphi}$ and local witnesses.
- For any constant c, define $\varphi_c = \varphi \wedge \psi(c)$.
- Since $\varphi_c \vdash \psi(c)$, by induction hypothesis $\overline{\varphi_c} \Vdash \psi(c)$.
- Since $[c] = c^{\varphi_c}$ at $\overline{\varphi_c}$, also $\overline{\varphi_c} \vdash_{[x \mapsto c^{\varphi_c}]} \psi(x)$, i.e. the element c^{φ_c} is a witness for the existential at $\overline{\varphi_c}$.
- It remains to be seen that $\overline{\varphi} \lhd \{\overline{\varphi_c} \mid c \text{ a constant}\}$.



F.12. Completeness $\exists$ direction $\Leftarrow$, continued

- Suppose ξ is derivable from $\varphi \wedge \psi(c)$ for any constant c.
- Let c^* be a constant that occurs neither in φ nor in ξ.
- In particular, $\varphi \wedge \psi(c^*) \vdash \xi$, that is, $\varphi, \psi(c^*) \vdash \xi$.
- But since c^* occurs neither in φ nor in ξ, by the rule ($\exists e$) we have $\varphi, \exists x\, \psi(x) \vdash \xi$.
- Thus by the assumption $\varphi \vdash \exists x\, \psi(x)$ we also have $\varphi \vdash \xi$.
- This shows that $\overline{\varphi} \lhd \{\overline{\varphi_c} \mid c \text{ a constant}\}$.
- Hence we conclude $\overline{\varphi} \Vdash \exists x\, \psi(x)$.



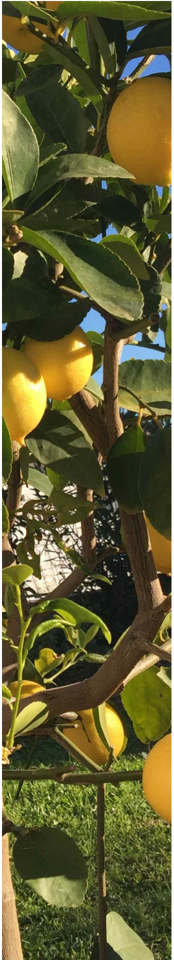
F.13. State of the art: hard to tell

Before 1995 : see survey by Makkay and Reyes, 1995.

After 1995, other work in particular by Awodey.

Direct completeness via canonical presheaf: Ivano Ciardelli. A Canonical Model for Presheaf Semantics. Talk at Topology, Algebra and Categories in Logic (TACL) 2011, Jul 2011, Marseille, France. 2011.HAL Id: inria-00618862 https://hal.inria.fr/inria-00618862

Ongoing work with David Théret in Montpellier (proof with sheaves by completion of separated pre sheaves).



F.14. Future work

Connection to Ω sets of Dana Scott (roughly speaking, if i understand properly: one classical model, but the truth value of $P(a)$ varies in a (complete) Heyting algebra, like Boolean valued models) ?

Can we construct a canonical sheaf and not just separated presheaf? e.g. with the sheaf completion method that basically simply formally adds the missing global sections? Or by imposing some additional locality condition on terms and equality?

(Pre)sheaves are particular kinds of Kripke models, conversely can any Kripke model be viewed as a pre(sheaf) with the order topology?

Is it possible to prove completeness with a standard topology (instead of a pretopology / Grothendieck topology)?

Does it applies to first order S4?

Thank you for your attention.