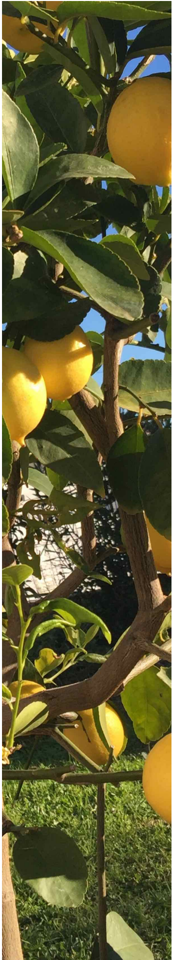

Coherence Semantics
for Pomset Logic
and a Self-Dual Modality

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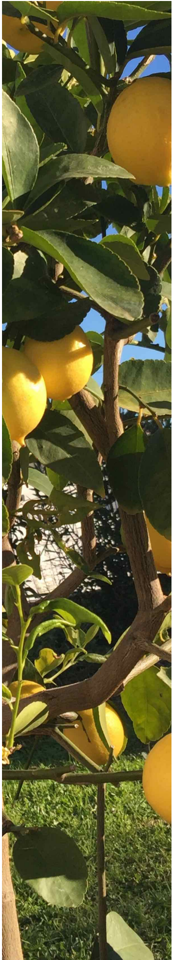
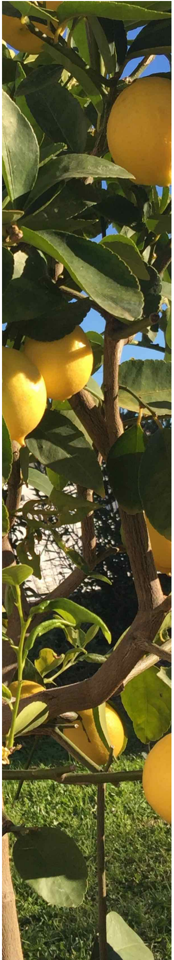
LIPN, 20 mars 2024



1. Initial Motivation

Girard (93): — Would you be able to find with ”your before connective” a self dual modality answering this?

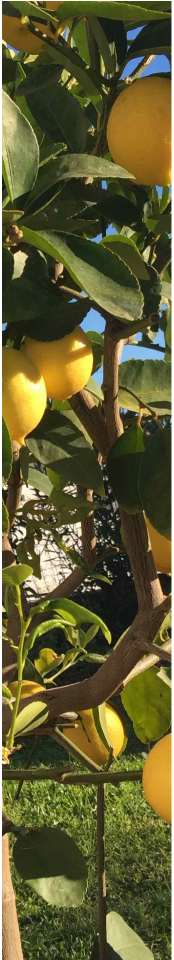
“The obvious candidate for a classical semantics was of course coherence spaces which had already given birth to linear logic; the main reason for choosing them was the presence of the involutive linear negation. However the difficulty with classical logic is to accommodate structural rules (weakening and contraction); in linear logic, this is possible by considering coherent spaces $?X$. But since classical logic allows contraction and weakening both on a formula and its negation, the solution seemed to require the linear negation of $?X$ to be of the form $?Y$, which is a nonsense (the negation of $?X$ is $!X^\perp$ which is by no means of isomorphic to a space$?Y$). Attempts to find a self-dual variant $\S Y$ of $?Y$ (enjoying $(\S Y)^\perp = \S Y^\perp$) systematically failed. The semantical study of classical logic stumbled on this problem of self-duality for years.” (J.-Y. Girard A new constructive logic classical logic, MSCS, 1991)



2. Today’s Motivation

Renewed interest in Pomset logic and on the related developments by Guglielmi and Straßburger Calculus of Structure (SBV) and Deep Inference, a complete sequent calculus for pomset logic published by Slavnov.

Further more, e.g. for process calculi it makes sense to repeat a sequence of operations.



3. The category COH : the privileged categorical interpetation of linear logic

Categorical interpretation:

Formula/type : object

proof $\pi : A \vdash B$: morphism $[\![\pi]\!] : A \mapsto B$

whenever $\pi \rightsquigarrow \pi'$: $[\![\pi]\!] = [\![\pi']\!]$.

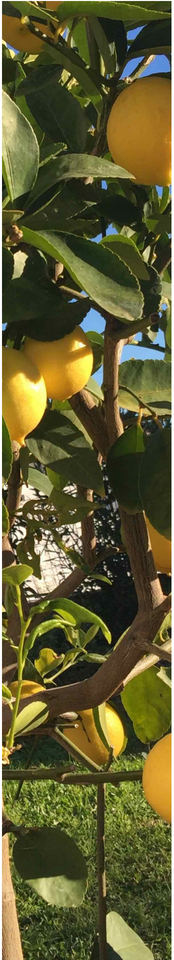
$Hom(A,B)$ corresponds to an *object* B^A.

CCC intuitionistic logic

COHerence spaces: intially introduced to interpret second order intutionistic logic because the endofunctor $X \longrightarrow T[X]$ can be represented as a coherence space.

Linear logic is issued from coherence spaces:

$$A \to B = (!A) \multimap B$$



4. The category COH. Objects: coherence spaces

A coherence space $A = (|A|, \frown_A)$ is an undirected simple graph, without loops nor multiple edges.

vertices are called tokens and they constitute the web $|A|$

$\frown_A$ is a binary symmetric and irreflexive relation on $|A|$ called **strict coherence**.

Given $\alpha, \alpha' \in |A|$

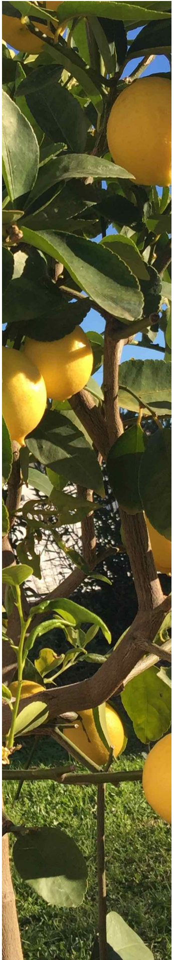
$\alpha \frown \alpha'[A]$ stands for $\alpha \frown_A \alpha'$

$\alpha \stackrel{\frown}{\smile} \alpha'[A]$ stands for $\alpha \frown \alpha'[A]$ or $\alpha = \alpha'$

$\alpha \asymp \alpha'[A]$ stands for $\alpha \not\frown \alpha'[A]$ (and holds whenever $\alpha = \alpha'$)

$\alpha \smile \alpha'[A]$ stands for $\alpha \not\stackrel{\frown}{\smile} \alpha'[A]$ (so $\alpha \neq \alpha'$).

The objects under consideration are the **cliques** of this graph, i.e. the sets of pairwise related tokens. Cliques interpret proofs of A up to cut-elimination / normalisation.



5. Involutive negation

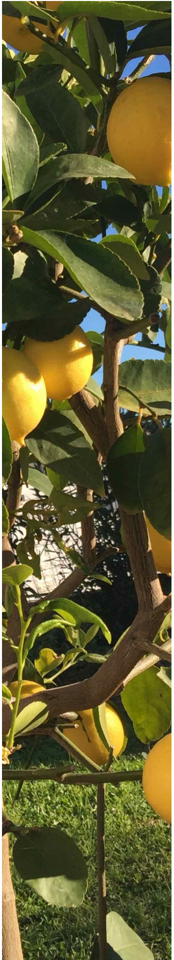
There is a natural involutive negation: the complement graph:

If $A = (|A|, \frown_A)$ then $A^{\perp} = (|A|, \frown_A^{\perp})$ with $\alpha \frown \alpha'[A^{\perp}]$ iff $\alpha \not\frown \alpha'[A]$

Given $\alpha, \alpha' \in |A|$ exactly one of the 3 relations below holds:

$\alpha \smile \alpha'[A]$ or $\alpha = \alpha'$ or $\alpha \frown \alpha'[A]$

$\alpha \frown \alpha'[A^{\perp}]$ or $\alpha = \alpha'$ or $\alpha \frown \alpha'[A]$



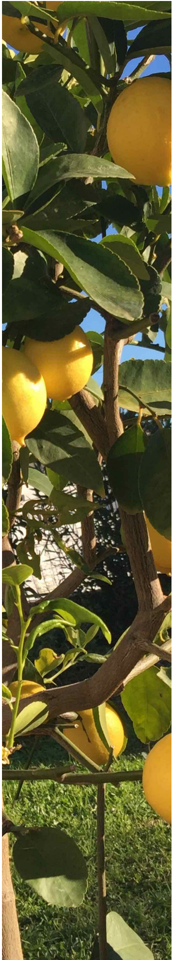
6. The category COH. Arrows: linear maps

A linear morphism F from A to B is a morphism mapping cliques of A to cliques of B such that:

- For all $x \in A$ if $(x' \subset x)$ then $F(x') \subset F(x)$
- For every family $(x_i)_{i \in I}$ of pairwise compatible cliques of A — that is to say $(x_i \cup x_j) \in A$ holds for all $i, j \in I$ — $F(\cup_{i \in I} x_i) = \cup_{i \in I} F(x_i)$.
- For all $x, x' \in A$ if $(x \cup x') \in A$ then $F(x \cap x') = F(x) \cap F(x')$.

Linear functions from A to B can be viewed as

cliques in $A^{\perp} \mathbin{⅋} B = A \multimap B$ (cf. later on).



7. Commutative Multiplicative Connectives

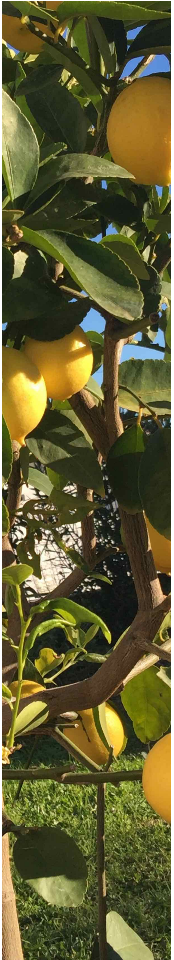
Multiplicative connectives $A * B$: $|A * B| = |A| \times |B|$. Unit $= \mathbb{1} = \{*\}$.

We may assume they are covariant in both their arguments.

Commutative multiplicative (binary) connectives, just two of them:

$A \parr B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

$A \otimes B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\smile$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\smile$	$\frown$	$\frown$



8. The category COH. Arrows as cliques of the linear function space

A linear map F corresponds to

$$\{(\alpha,\beta) \mid \alpha \in |A| \beta \in |B| \ \beta \in F(\{\alpha\})\}$$

clique of $A^{\perp} \parr B = A \multimap B$.

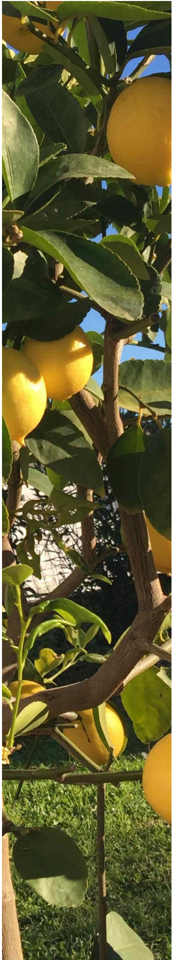
Linearity → for any clique x of A and any $\beta \in F(x)$ there is a unique $\alpha \in x$ such that $\beta \in F(\{\alpha\})$.

Conversely, given a clique f of $A^{\perp} \parr B$ a linear function can be defined by

$$F(x) = \{\beta \in |B| | \exists \alpha \in x (\alpha,\beta) \in f\}$$

Strict coherence in $A^{\perp} \parr B = A \multimap B$ is characterised as follows:

$(\alpha,\beta) \frown (\alpha',\beta')[A \multimap B]$ whenever $\alpha \frown \alpha'[A]$ entails $\beta \frown \beta'[B]$.



9. Before (pomset logic)= Seq (deep inference)

But, there is another (non commutative) multiplicative connective:

$A \lhd B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\smile$	$\frown$	$\frown$

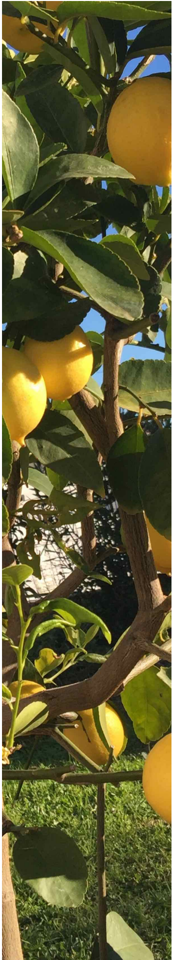
$A \rhd B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\smile$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

$$(\alpha,\beta) \frown (\alpha',\beta')[A \lhd B] \text{ whenever } \begin{cases} \alpha \frown \alpha'[A] \\ \text{or} \\ \alpha = \alpha' \text{ and } \beta \frown \beta'[A] \end{cases}$$

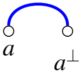
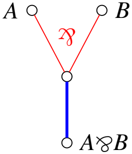
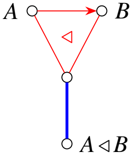
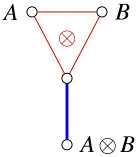
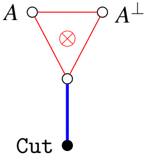
Associative, self dual $(A \lhd B)^\perp = A^\perp \lhd B^\perp$ **(no swap!)**

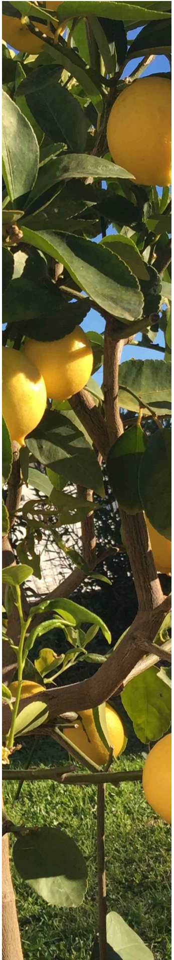
Generalisation: $\prec$ finite (partial) order over $I = \{1,\ldots,n\}$, $\Pi^I A_i$:

- web: $|A_1| \times \cdots \times |A_n|$
- strict coherence: $(\alpha_1,\ldots,\alpha_n) \frown (\alpha'_1,\ldots,\alpha'_n)$
  when there exists i s.t. $\alpha_i \frown \alpha'_i$ and $\alpha_j = \alpha'_j$ for all $j \prec i$.



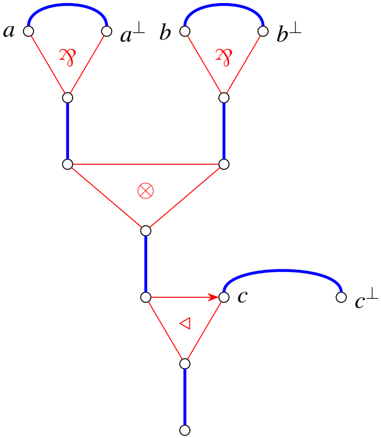
10. Pomset logic: proof net syntax (with links)

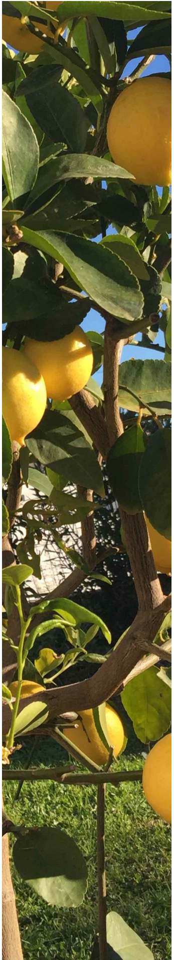
	Axiom	Par $\parr$	Before $\lhd$	Times $\otimes$	Cut
Premisses	None	A and B	A and B	A and B	K and $K^\perp$
RnB link					
Conclusion(s)	a and $a^\perp$	$A \parr B$	$A \lhd B$	$A \otimes B$	None



11. Correctness criterion

No alternate elementary circuit (directed cycle).





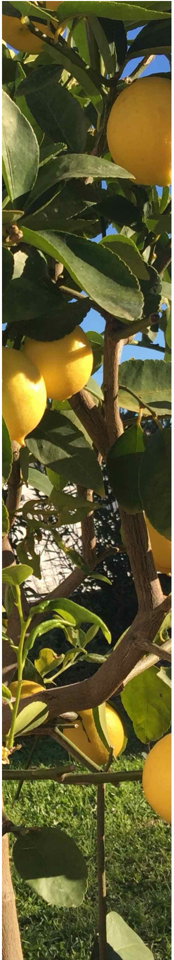
12. Cut elimination

Cut elimination preserves the criterion.

a cut may be viewed as $K \otimes K^\perp$

two cuts in parallel (⅋ of the cuts)

two cuts one before the other (< of the cuts)

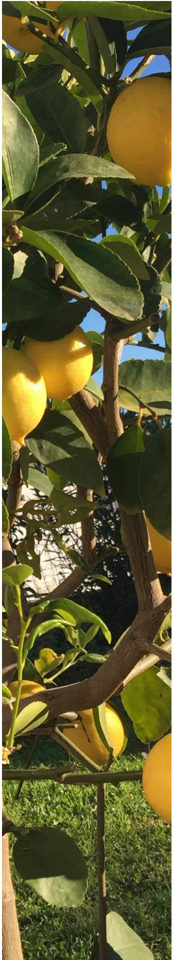


13. Semantics

A proof structure is interpreted as a set of tokens in the corresponding coherence space (experiment method).

Theorem: a proof structure is correct if and only if its interpretation if a clique of the corresponding coherence space.

Intepretation is preserved by cut-elimination.



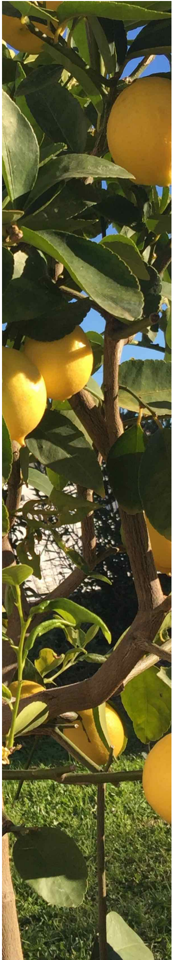
14. Sequent calculus?

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \langle \Gamma ; \Delta \rangle}\, \mathit{dimix} \qquad\qquad \frac{\vdash \Gamma}{\vdash \Gamma'}\, \text{entropy}(\Gamma' \text{ sub sp order of } \Gamma)$$

$$\frac{}{\vdash \{a, a^{\perp}\}}$$

$$\frac{\vdash \{A, \Gamma\} \qquad \vdash \{B, \Delta\}}{\vdash \{\Gamma, (A \otimes B), \Delta\}}\, \otimes \text{ / cut when } A = B^{\perp}$$

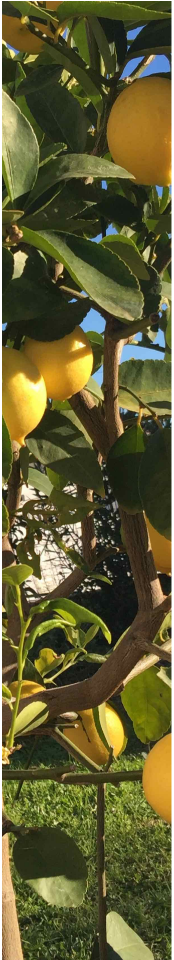
$$\frac{\vdash \Gamma[\{A, B\}]}{\vdash \Gamma[A \parr B]}\, \parr(A \overset{..}{\sim} B) \qquad\qquad \frac{\vdash \Gamma[\langle A ; B \rangle]}{\vdash \Gamma[A \triangleleft B]}\, \triangleleft(A \overset{\rightarrow}{\sim} B)$$



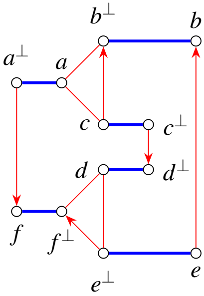
15. Sequent proof   proof net example

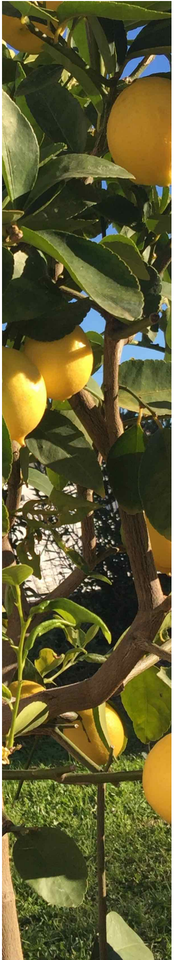
$$\cfrac{\cfrac{\cfrac{\cfrac{\vdash \{a,a^{\perp}\}}{\vdash a \parr a^{\perp}} \qquad \cfrac{\vdash \{b,b^{\perp}\}}{\vdash b \parr b^{\perp}}}{\vdash (a \parr a^{\perp}) \otimes (b \parr b^{\perp})} \qquad\qquad \vdash c,c^{\perp}}{\vdash \langle (a \parr a^{\perp}) \otimes (b \parr b^{\perp}); \{c,c^{\perp}\}\rangle}\ \textit{dimix}}{\vdash \{\langle (a \parr a^{\perp}) \otimes (b \parr b^{\perp}); c\rangle, c^{\perp}\}}\ \textit{entropy}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdash \{a,a^{\perp}\}}{\vdash a \parr a^{\perp}} \qquad \cfrac{\vdash \{b,b^{\perp}\}}{\vdash b \parr b^{\perp}}}{\vdash (a \parr a^{\perp}) \otimes (b \parr b^{\perp})} \qquad\qquad \vdash c,c^{\perp}}{\vdash \langle (a \parr a^{\perp}) \otimes (b \parr b^{\perp}); \{c,c^{\perp}\}\rangle}\ \textit{dimix}}{\vdash \{\langle (a \parr a^{\perp}) \otimes (b \parr b^{\perp}); c\rangle, c^{\perp}\}}\ \textit{entropy}$$

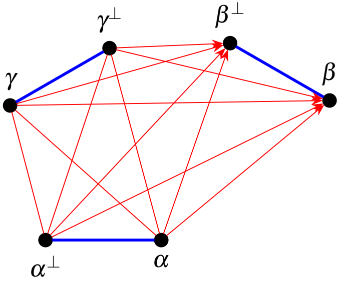


16. Not derivable in sequent calculus

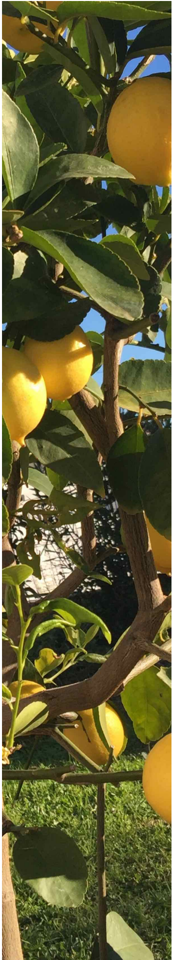




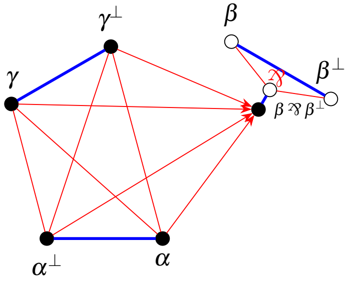
17. Folding/unfolding 1



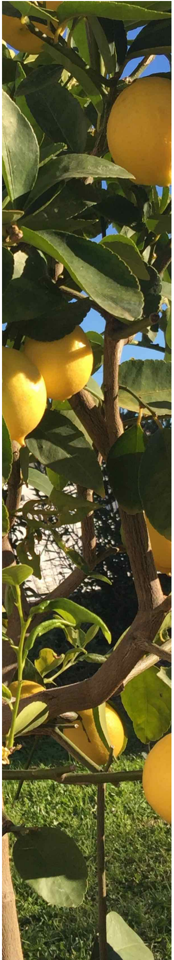
(a) π_1



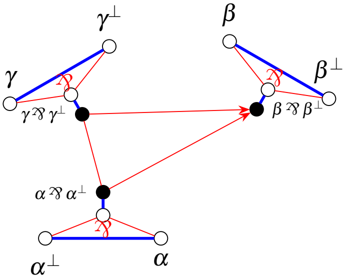
18. Folding/unfolding 2



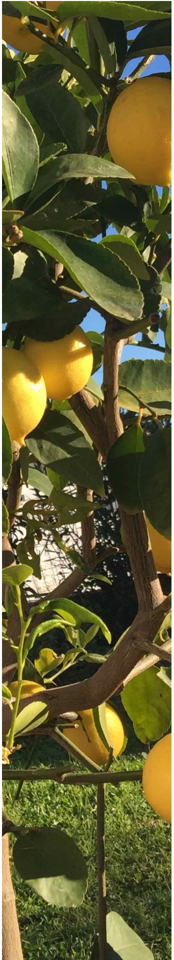
(b) π_2



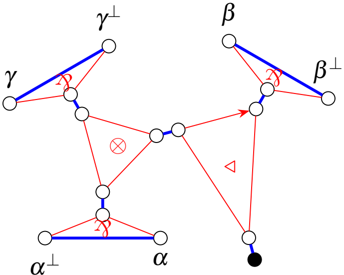
19. Folding/unfolding 3



(c) π_3

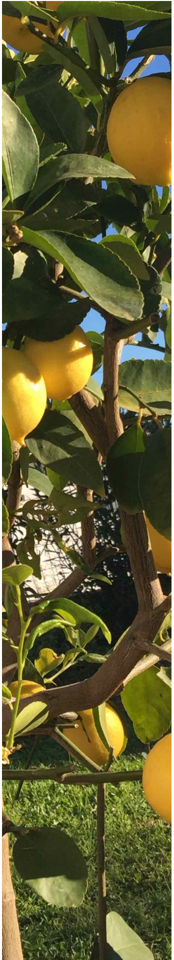


20. Folding/unfolding 4



$((\alpha ⅋ \alpha^\perp) \otimes (\gamma ⅋ \gamma^\perp)) \triangleleft (\beta ⅋ \beta^\perp)$

(d) π_4



21. Handsome proof nets

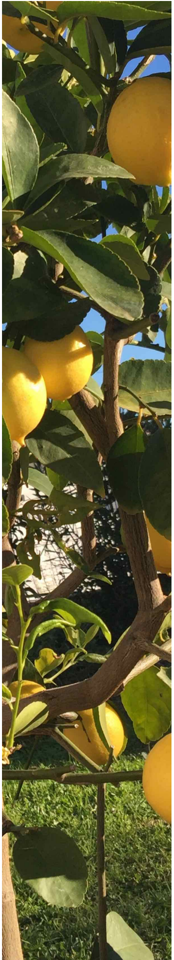
Proof net

vertices atoms a $a^{\perp}$

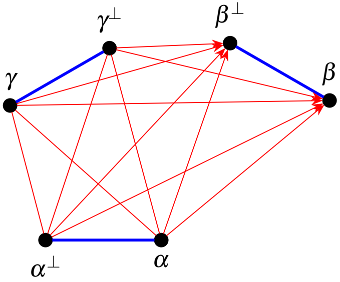
B edges axioms, perfect matching

R directed cograph (directed part: series parallel partial order; symmetric part: cograph; weak transitivity between both)

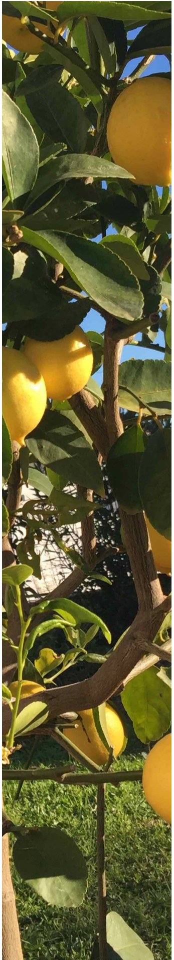
Criterion: every alternate elementary circuit contains a chord (an edge or an arc not in the circuit but between two vertices of the circuit)



22. A handsome proof net

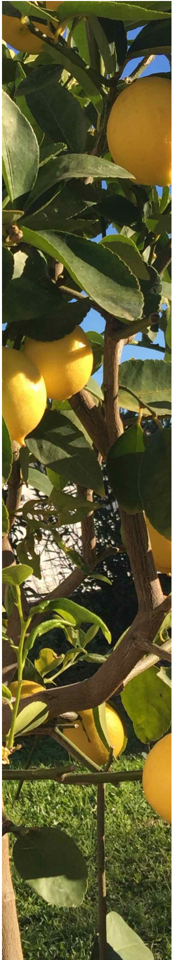


(a) π_1



23. Rewriting

rule name	*dicograph*	$\rightsquigarrow$	*dicograph*$'$
☠ $\widehat{\otimes}\widehat{\parr}4$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\otimes}} (U \mathbin{\widehat{\parr}} V)$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\parr}} (Y \mathbin{\widehat{\otimes}} V)$
$\otimes\parr 3$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\parr}} Y$
$\otimes\parr 2$	$Y \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$U \mathbin{\widehat{\parr}} Y$
$\otimes\triangleleft 4$	$(X \mathbin{\widehat{\triangleleft}} Y) \mathbin{\widehat{\otimes}} (U \mathbin{\widehat{\triangleleft}} V)$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\triangleleft}} (Y \mathbin{\widehat{\otimes}} V)$
$\otimes\triangleleft 3l$	$(X \mathbin{\widehat{\triangleleft}} Y) \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\triangleleft}} Y$
$\otimes\triangleleft 3r$	$Y \mathbin{\widehat{\otimes}} (U \mathbin{\widehat{\triangleleft}} V)$	$\rightsquigarrow$	$U \mathbin{\widehat{\triangleleft}} (Y \mathbin{\widehat{\otimes}} V)$
$\otimes\triangleleft 2$	$Y \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$U \mathbin{\widehat{\triangleleft}} Y$
$\triangleleft\parr 4$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\triangleleft}} (U \mathbin{\widehat{\parr}} V)$	$\rightsquigarrow$	$(X \mathbin{\widehat{\triangleleft}} U) \mathbin{\widehat{\parr}} (Y \mathbin{\widehat{\triangleleft}} V)$
$\triangleleft\parr 3l$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\triangleleft}} U$	$\rightsquigarrow$	$(X \mathbin{\widehat{\triangleleft}} U) \mathbin{\widehat{\parr}} Y$
$\triangleleft\parr 3r$	$Y \mathbin{\widehat{\triangleleft}} (U \mathbin{\widehat{\parr}} V)$	$\rightsquigarrow$	$U \mathbin{\widehat{\parr}} (Y \mathbin{\widehat{\triangleleft}} V)$
$\triangleleft\parr 2$	$Y \mathbin{\widehat{\triangleleft}} U$	$\rightsquigarrow$	$U \mathbin{\widehat{\parr}} Y$

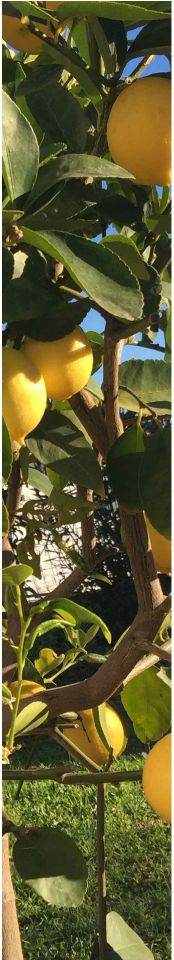


24. Relation to Deep Inference

Starting with $\otimes_i(a_i \parr a_i^\perp)$

some rules handling 1 the common unit of $\otimes, \lhd, \parr$.

Equivalent to SBV and you easily get SBV ”cut” elimination (removal of $a \uparrow 1$) when $c \otimes c^\perp$ vanishes.



25. Not derivable in Deep Inference

Tito N'Guyen results (partly with Lutz Strassburger)

Conclusion **Joyeux anniversaire Christian !**

Retoré's *Pomset Logic* (PL) and Guglielmi's *BV*: 2 logics over the same formulas, from the 1990s, conservatively extending Multiplicative Linear Logic with Mix

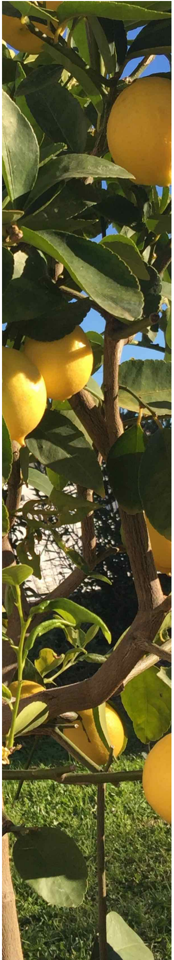
Our result [N. & Straßburger]: refuting Guglielmi's two-decades-old conjecture

- *There is some formula A such that* BV $\nvdash A$ *but* PL $\vdash A$.

$$A = ((a \triangleleft b) \otimes (c \triangleleft d)) \parr ((e \triangleleft f) \otimes (g \triangleleft h)) \parr (a^\perp \triangleleft h^\perp) \parr (e^\perp \triangleleft b^\perp) \parr (g^\perp \triangleleft d^\perp) \parr (c^\perp \triangleleft f^\perp)$$

  Causally meaningful variant (K.–S.): $(((p^1)^\perp \triangleleft q^1) \otimes ((r^1)^\perp \triangleleft s^1)) \parr (((q^1)^\perp \triangleleft r^1) \otimes ((s^1)^\perp \triangleleft p^1))$

- Moreover, "BV $\vdash A$?" is NP-complete while "PL $\vdash A$?" is Σ_2^{p}-complete.



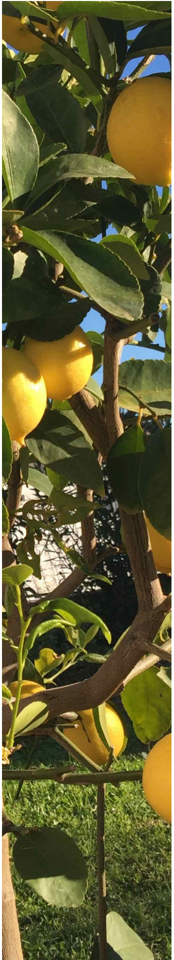
26. Slavnov's — complete but *ad hoc* — sequent calculus

Very complex: if n conclusions, pairs of tuples of length k for all $k \leqslant n/2$.

Only unary rules but mix.

Intuition: dependent alternate elmentary paths between k conclusions and k other conclusions are known.

One very itneresting idea: usual commutatives $\parr, \otimes$ plus a pair of dual non commutative connective: $\vec{\parr}$ and $\vec{\otimes}$, and $\lhd$ is a degenerate case, when both are equal.



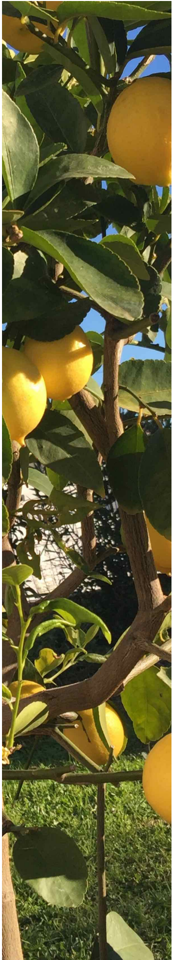
27. Conclusion on sequent calculus

Not yet!

But some new ideas (like Slavnov $\vec{\parr}$ and $\vec{\otimes}$),

and some graph theoretical ideas as well.

However from 1991, there are moments when think i can solve this problem...



28. A selfdual modality for contraction/duplication with ”$<$”: What we are looking for?

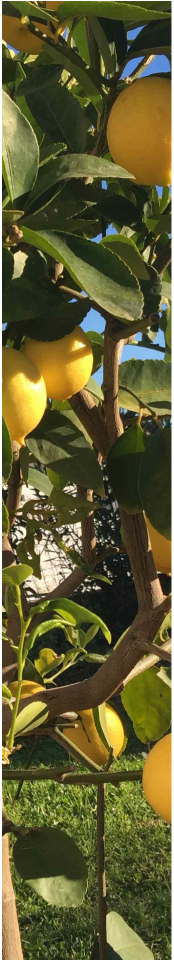
Usual modalities:

$$\begin{tikzcd} !A \arrow[r, -o] \arrow[d, -o] \arrow[dr, -o] & (!A \otimes !A) \\ 1 & A \end{tikzcd} \qquad \begin{tikzcd} ?A & ?A \parr ?A \arrow[l, -o] \\ 1 \arrow[u, -o] & A \arrow[ul, -o] \end{tikzcd}$$

Self dual contraction/duplication Flag:

$$\lhd\!\!\mid A \multimapinv \text{linear iso} \multimap (\lhd\!\!\mid A \lhd \lhd\!\!\mid A)$$

Of course there is no relation between $\lhd\!\!\mid A$ and 1, otherwise, with a self dual modality, the system would collapse.



29. Continuous functions from Cantor space to a discrete topological space

$\mathbb{2}^\omega$, infinite words on $\mathbb{2}$, with standard order and topology:

- usual total lexicographical order:

$$w_1 < w_2 \text{ iff } \exists m \in \mathbb{2}^* \ \exists w'_1, w'_2 \in \mathbb{2}^\omega \ w_1 = m0w'_1 \ and \ w_2 = m1w'_2$$

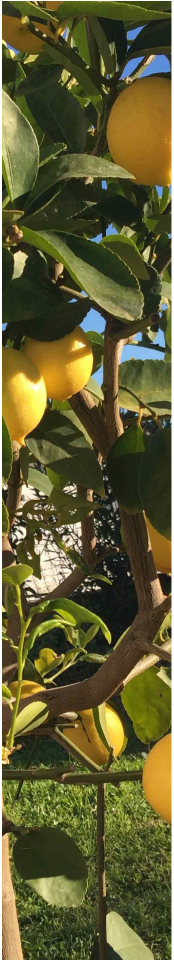
- usual product topology generated by clopen sets $(U_m)_{m \in \mathbb{2}^*}$

$$U_m = \{w \in \mathbb{2}^\omega | \exists w' \in \mathbb{2}^\omega \ w = mw'\}$$

Continuous function from $\mathbb{2}^\omega$ to a set M (discrete topology) = finite binary tree with M-labelled leaves
without two sister leaves with the same M-label.

gt_M generic trees over M = binary tree representing continuous functions $\mathbb{2}^\omega \mapsto M$.

Let $f \in \mathrm{gt}_M$ for $w \in \mathbb{2}^\omega$ there is a unique prefix of w that is a root-to-leaf path of f. If the M-label is a then $f(w) = a$.



30. Justification

$2^{\omega} = \cup_{\alpha \in M} f^{-1}(\{\alpha\})$

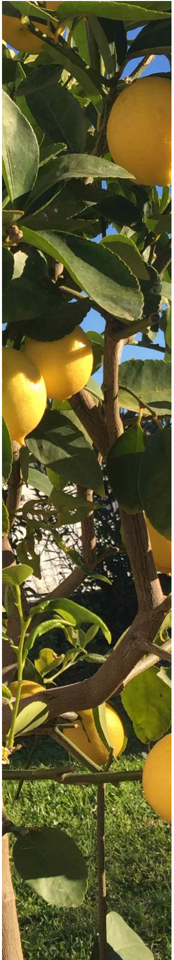
$\{\alpha\}$ are clopen sets and so is $f^{-1}(\{\alpha\})$.

Hence one can extract a finite covering of 2^{ω} from the $f^{-1}(\{\alpha\})$ (compacity of 2^{ω}).

So the function has finitely many values.

Each of these $f^{-1}(\{\alpha\})$ can be written as a finite union of base clopen sets and a finite union of finite union is a finite union, and this gives the tree structures.

Observe that two base clopen sets never have a non trivial intersection: their intersection is either empty or one contains the other.



31. A generic tree, i.e. continuous function from 2^ω to a set M

A binary tree with labels in M without sisters leaves with a common label, like:

$$[[a,b],[b,[a,c]]]$$

corresponds to a unique continuous function from 2^ω to M:

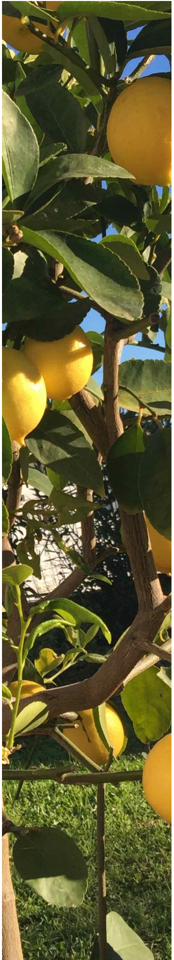
$f(00w) = a$

$f(01w) = b$

$f(10w) = b$

$f(110w) = a$

$f(111w) = c$



32. A remark on continuous functions from the Cantor space to a discrete topological space

Let $f,g \in \mathrm{gt}_M$.
If $f \neq g$, then there exists $w \in \mathbb{2}^{\omega}$ such that

$$f(w) \neq g(w) \text{ and } \forall w' < w \quad f(w') = g(w')$$

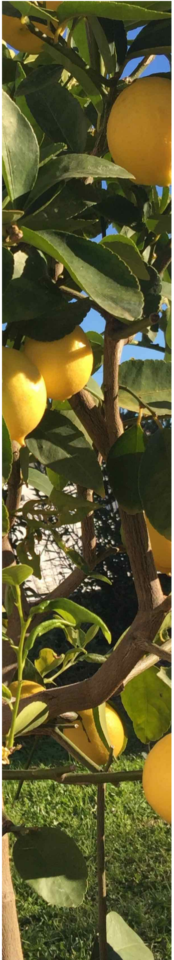
Consider the two continuous maps:

$$\begin{array}{rrcll} (f,g): & \mathbb{2}^{\omega} & \mapsto & M \times M & \text{(product of discrete topology)} \\ & w & \mapsto & (f(w),g(w)) & \end{array}$$

$$\begin{array}{rrcll} \Delta: & M \times M & \mapsto & \mathbb{2} & \text{(discrete topology)} \\ & (a,b) & \mapsto & 1 \; si \; a = b & \\ & & & 0 \; si \; a \neq b & \end{array}$$

$\Delta \circ (f,g)$ is continuous, hence $(\Delta \circ (f,g))^{-1}(0)$ is a clopen set.

It has a lowest element $w_0 = u(0)^{\omega}$ with
$f(w_0) \neq g(w_0)$ and $f(w') = g(w')$ for all $w' \prec w_0$.



33. The flag modality

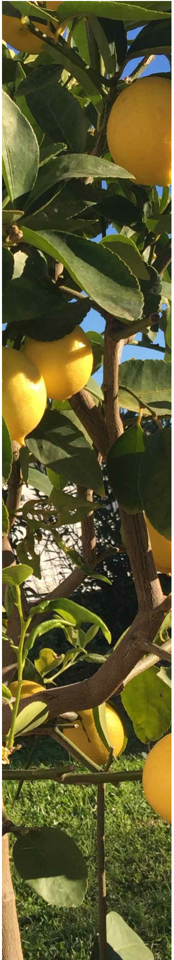
Web of $\triangleleft A$: $\mathrm{gt}_{|A|}$

the continuous functions from 2^ω to $|A|$ the web of A.

Observe that if $|A|$ is countable so is $\mathrm{gt}_{|A|}$.

Coherence $f \frown g[\triangleleft A]$ with $f, g \in |\triangleleft A = \mathrm{gt}_{|A|}$ whenever

$$\exists w \in 2^\omega \begin{cases} f(w) \frown g(w)[A] \\ \text{and} \\ \forall w' < w \quad f(w') = g(w') \end{cases}$$



34. Flag is self dual

The modality $\triangleleft\!\!\!|$ is self-dual, i.e. $(\triangleleft\!\!\!|A)^{\perp} \equiv \triangleleft\!\!\!|(A^{\perp})$

Those two coherence spaces obviously have the same web.

Let $f \neq g$ be two distinct continuous functions from 2^{ω} to $|A|$.

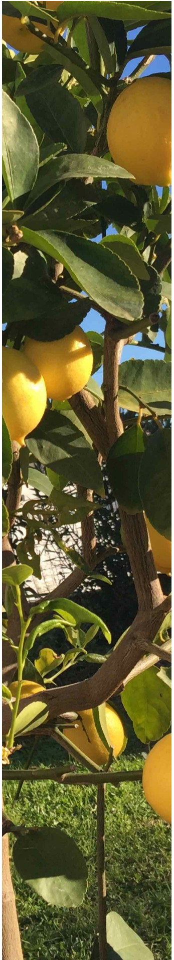
Let $w \in 2^{\omega}$ satisfying $f(w) \neq g(w)$ and $\forall w' < w.\ f(w') = g(w')$.

Either $f(w) \frown g(w)[A]$ and therefore $f \frown g[\triangleleft\!\!\!|A]$

or $f(w) \frown g(w)[A^{\perp}]$ and therefore $f \frown g[\triangleleft\!\!\!|(A^{\perp})]$

Hence when $f \neq g$ either $f \frown g[\triangleleft\!\!\!|A]$ or $f \frown g[\triangleleft\!\!\!|(A^{\perp})]$.

So $\triangleleft\!\!\!|A^{\perp} = (\triangleleft\!\!\!|A)^{\perp}$.



35. Linear iso $\triangleleft\!\!\mid\! A \;\circ\!\!-\!-\!\!\circ\; (\triangleleft\!\!\mid\! A \triangleleft \triangleleft\!\!\mid\! A)$

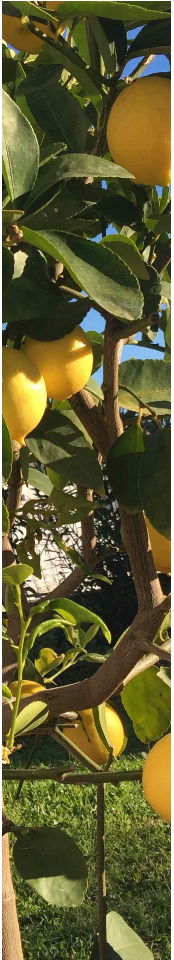
$$C = \{(h,(h_0,h_1)) \,|\, \forall w \in 2^\omega \; h(0w) = h_0(w) \text{ and } h(1w) = h_1(w)\}$$

defines a linear isomorphism between $\triangleleft\!\!\mid\! A$ and $\triangleleft\!\!\mid\! A < \triangleleft\!\!\mid\! A$.

bijection between the webs, i.e. between

pairs of continuous functions from 2^ω to $|A|$

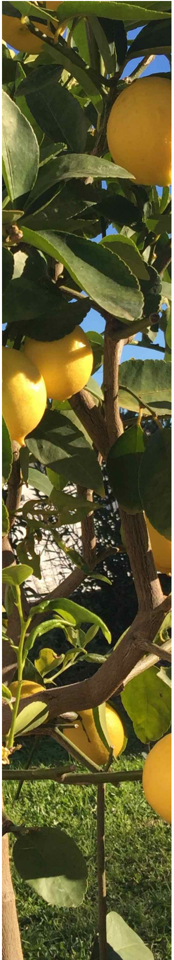
continuous functions from 2^ω to $|A|$.



36. Linear iso $\lhd\!\!|A \circ\!\!-\!\!-\!\!\circ (\lhd\!\!|A \lhd \lhd\!\!|A)$

Given $(h,(h_0,h_1))$ and $(g,(g_0,g_1))$, both in C we have to prove that:

$$(1):h \frown g[\lhd\!\!|A] \iff (h_0,h_1) \frown (g_0,g_1)[\lhd\!\!|A \lhd \lhd\!\!|A]:(2)$$



37. Linear iso ${\triangleleft\!\!\mid}A \ \circ\!\!-\!\!-\!\!\circ\ ({\triangleleft\!\!\mid}A \triangleleft {\triangleleft\!\!\mid}A)$

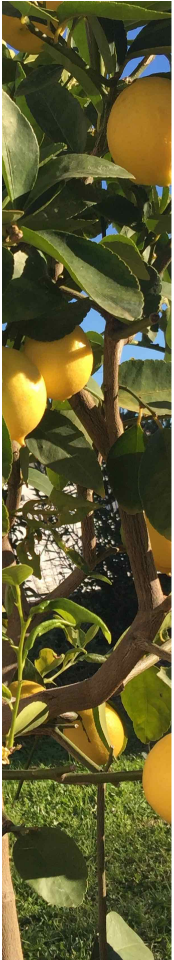
$(1) \Longrightarrow (2)$ We assume that $h \frown g[{\triangleleft\!\!\mid}A]$,

i.e. that $\exists w \in \mathbb{2}^\omega\ h(w) \frown g(w)$ and $\forall v{<}w \quad h(v) = g(v)$.

Either $w = 0w'$ or $w = 1w'$.

In both cases $(h_0, h_1) \frown (g_0, g_1)[{\triangleleft\!\!\mid}A \triangleleft {\triangleleft\!\!\mid}A]$

0. If $w = 0w'$ we have $h_0 \frown g_0[{\triangleleft\!\!\mid}A]$:
   - $h_0(w') \frown g_0(w')$ since $h_0(w') = h(0w') = h(w)$, $g(w) = g(0w') = g_0(w')$ and $h(w) \frown g(w)$.
   - $h_0(v') = g_0(v')$ for all $v'{<}w'$; indeed, $0v' < 0w' = w$ hence $h_0(v') = h(0v') = g(0v') = g_0(v')$.

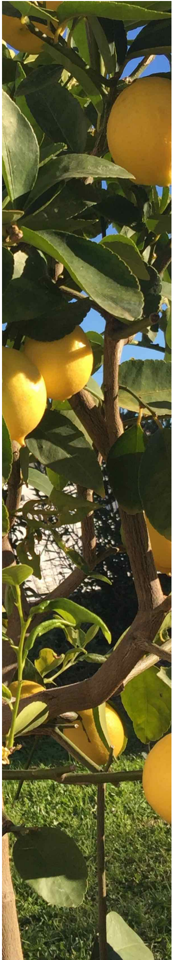


38. Linear iso $\triangleleft\!\!\mid A \;\circ\!\!-\!-\!\!\circ\; (\triangleleft\!\!\mid A \lhd \triangleleft\!\!\mid A)$

$(1) \Longrightarrow (2)$ We assume that $h \frown g[\triangleleft\!\!\mid A]$,

i.e. that $\exists w \in 2^\omega \; h(w) \frown g(w)$ and $\forall v{<}w \quad h(v) = g(v)$.

1. If $w = 1w'$ then $h_1 \frown g_1[\triangleleft\!\!\mid A]$ and $h_0 = g_0$:
   - $h_1 \frown g_1[\triangleleft\!\!\mid A]$
     - $h_1(w') \frown g_1(w')$ since $h_1(w') = h(1w') = h(w)$, $g_1(w') = g(1w') = g(w)$, $h(w) \frown g(w)$.
     - $h_1(v') = g_1(v')$ for all $v'{<}w'$; indeed, $h_1(v') = h(1v') = g(1v') = g_1(v')$ since $1v'{<}1w' = w$.
   - $h_0 = g_0$ since $h_0(u) = h(0u) = g(0u) = g_0(u)$ because $0u < 1w' = w$.

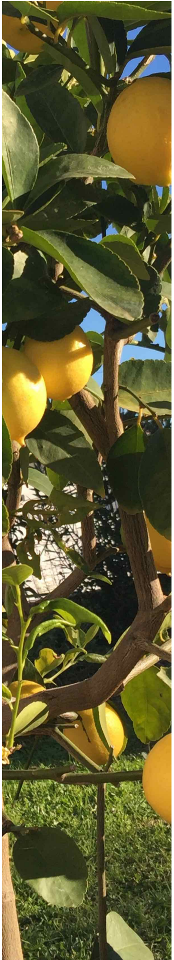


39. Linear iso $\triangleleft\!\!\mid A \;\circ\!\!-\!\!-\!\!\circ\; (\triangleleft\!\!\mid A \lhd \triangleleft\!\!\mid A)$

$(2) \Longrightarrow (1)$ We assume that $(h_0, h_1) \frown (g_0, g_1)[\triangleleft\!\!\mid A \lhd \triangleleft\!\!\mid A]$ i.e. that either 0. $h_0 \frown g_0[\triangleleft\!\!\mid A]$ or 1. ($h_0 = g_0$ and $h_1 \frown g_1[\triangleleft\!\!\mid A]$).

We first show $h \frown g[\triangleleft\!\!\mid A]$ in case 0.

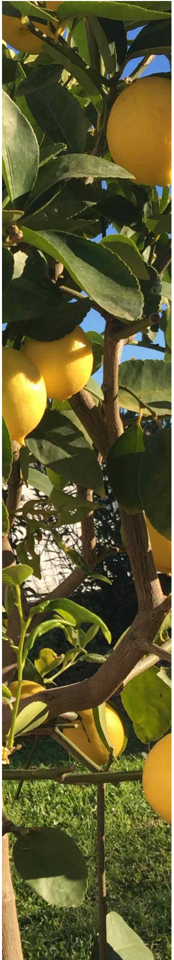
0. If $h_0 \frown g_0[\triangleleft\!\!\mid A]$ then there exists w' such that $h_0(w') \frown g_0(w')$ and $h_0(v') = g_0(v')$ for all $v' < w'$ and $h \frown g[\triangleleft\!\!\mid A]$. Indeed:
   - $h(0w') \frown g(0w')$ because $h(0w') = h_0(w')$, $g(0w') = g_0(w')$ and $h_0(w') \frown g_0(w')$.
   - for all $u < 0w'$, one has $h(u) = g(u)$; indeed, if $u < 0w'$ then $u = 0u'$ with $u' < w'$, so $h(u) = h(0u') = h_0(u')$, $g(u) = g(0u') = g_0(u')$ and $h_0(u') = g_0(u')$ because $u' < w'$.



40. Linear iso $\triangleleft\!\!\!\mid A \;\circ\!-\!-\!\circ\; (\triangleleft\!\!\!\mid A \triangleleft \triangleleft\!\!\!\mid A)$

$(2) \Longrightarrow (1)$ We assume that $(h_0, h_1) \frown (g_0, g_1)[\triangleleft\!\!\!\mid A \triangleleft \triangleleft\!\!\!\mid A]$ i.e. that either 0. $h_0 \frown g_0[\triangleleft\!\!\!\mid A]$ or 1. $(h_0 = g_0$ and $h_1 \frown g_1[\triangleleft\!\!\!\mid A])$.
We now show that $h \frown g[\triangleleft\!\!\!\mid A]$ in case 1.

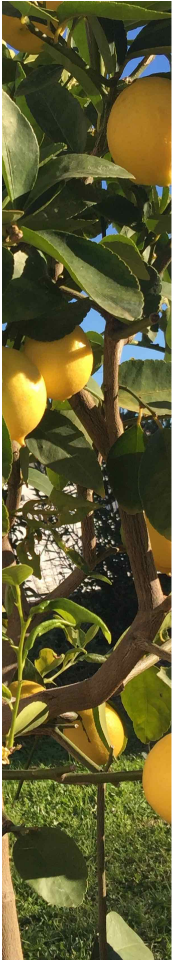
1. If $h_1 \frown g_1[\triangleleft\!\!\!\mid A]$ and $h_0 = g_0$ then there exists w' such that $h_1(w') \frown g_1(w')$ and $h_1(v') = g_1(v')$ for all $v' < w'$, and for all u, $h_0(u) = g_0(u)$. We have $h \frown g[\triangleleft\!\!\!\mid A]$:
    - $h(1w') \frown g(1w')[A]$; indeed $h(1w') = h_1(w')$ and $g(1w') = g_1(w')$ and $g_1(w') \frown h_1(w')$.
    - for all $v < 1w'$ one has $h(v) = g(v)$ since $v = 0u$ or $v = 1u'$ and
        - if $v = 0u$ then $h(v) = h(0u) = h_0(u) = g_0(u) = g(0u) = g(v)$.
        - if $v = 1u'$ then $u' < w'$ and therefore $h(v) = h(1u') = h_1(u')$, $h_1(u') = g_1(u')$ (because $u' < w'$), and $g(v) = g(1u') = g_1(u')$.



41. Explanation

This is because, so to speak, f is $f_0 \triangleleft f_1$.

I initially defined the web of $\triangleleft\!\!\mid A$ as $\triangleleft_{i \in Q} A$ (Q copies of $|A|$, a token was a function from Q to A) and Achim young suggested to use 2^ω to get a finite representation of the tokens.



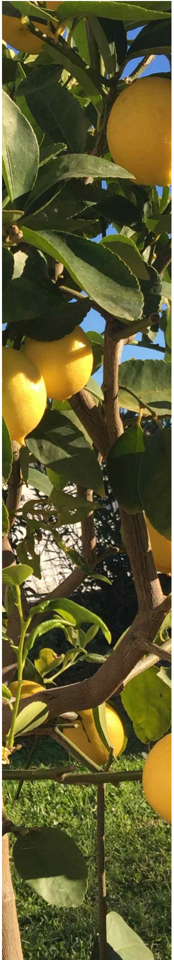
42. A is a retract of Flag A

Let $\rho = \{\alpha, \underline{\alpha})|\alpha \in |A|\}$ where $\underline{\alpha}$ is the constant continuous function from 2^{ω} to $|A|$ mapping every infinite word to $\alpha \in |A|$.

ρ is linear. It dual $\sigma = \{(\underline{\alpha}, \alpha)|\alpha \in |A|\}$ is linear as well.

$\rho \circ \sigma$ is Id_A,

while $\sigma \circ \rho \subsetneq Id_{\triangleleft A}$ (identity, butonly for constant functions).



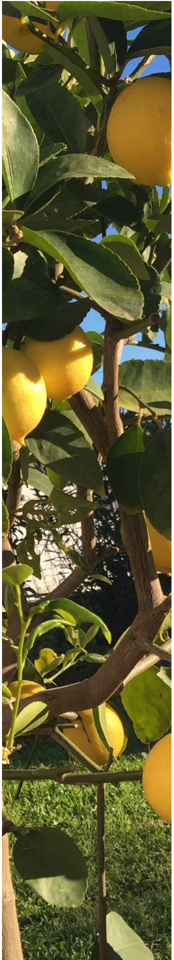
43. Flag is functorial

Given $\ell : A \to B$ defines $\triangleleft\ell : \triangleleft A \to \triangleleft B$ by the linear map:

$$\triangleleft\ell = \{(f,g)/\forall w \in 2^{\omega}(f(w),g(w)) \in \ell\}$$

This makes $\triangleleft$ an endo-functor of COH with linear maps.

This is not difficult but a bit tedious to prove.



44. Concluding question: syntax?

Pomset logic is better defined with (handsome) proof nets,

or as a rewriting system like Deep Inference.

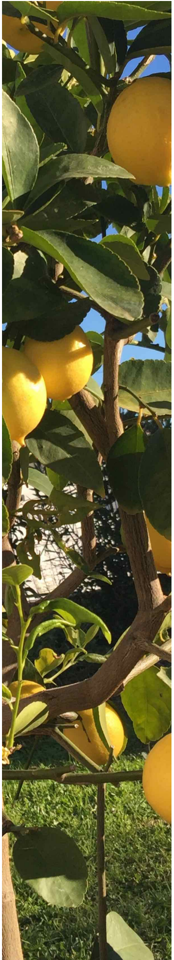
The design of a self dual modality should perhaps proceed with handsome proof nets

whose correction is equivalent to their interpretability in coherence spaces.

However modalities are complicated in the the proof net framework

an exception being the essential nets of Lamarche for intuitionistic logic.

Guglielmi proposed in the last years several versions of a self dual modality with deep inference coherence semantics should be a guideline to find the right one, if any.



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