Pomset Logic

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For Michele Abrusci on the occasion of his retirement and birthday workshop

Results on pomset logic:

1. Issued from (a remark of Girard on) **coherence semantics** which has a non commutative self dual multiplicative connective.
2. **Proof net calculus** (with cut-elimination):
   π correct ⇔ $[\pi]$ is a clique
3. **Handsome proof-nets** calculus (no links) with rewriting (as in deep inference)
4. Complete sequent calculus?
   Work by me, S. Pogodalla, L. Strassburger…Solved by Sergey Slavnov in 2019

Pomset logic	Michele's NL
1. Family: calculus of structures, deep inference	Family: Lambek, Cyclic, Abrusci, Abrusci-Ruet
2. An extension of MLL	A restriction of LL (or restriction + commutative LL)
3. Semantics of proofs (coherence semantics)	Truth value semantics (phase semantics)
4. Proof nets with cuts & coherence caraterisation)	Proof nets (cuts?)
5. Sequent calculus (Slavnonv 2019)	Perfect sequent calculus from the begining

Coherence Semantics

- Formulae: (possibly infinite) graphs
- Proofs up to normalisation: cliques
- Morphisms, linear maps:
  - F sends cliques to cliques
  - When a union is a clique:
    - Commute with union
    - Commute with intersection

Multiplicative coherence spaces Girard's remark

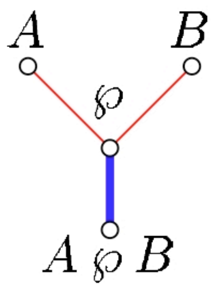
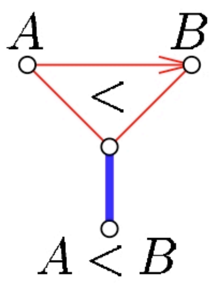
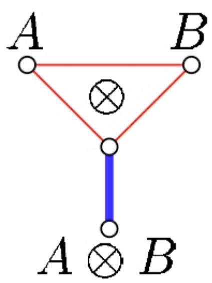
- Vertices: pairs of vertices
- Par: both $\frown$
- Times: both $\smile$
- One non commutative« < »: A: $\smile$ and B: $\frown$
- No other multiplicative.

$A \backslash B$	$\frown$	$=$	$\smile$
$\frown$	$\frown$	$\frown$	?
$=$	$\frown$	$=$	$\smile$
$\smile$	?	$\smile$	$\smile$

Before

- Written <
  - Non commutative
  - Associative
  - Self-dual $(A < B)^{\perp} \equiv (A^{\perp} < B^{\perp})$
- Girard's question:
  what syntax for this calculus?

Bicoloured proof nets

Name	*axiom*-link	*par*-link	*before*-link	*times*-link
Premises	none	A and B	A and B	A and B
R&B-graph				
Conclusions	a and $a^\perp$	$A \wp B$	$A < B$	$A \otimes B$

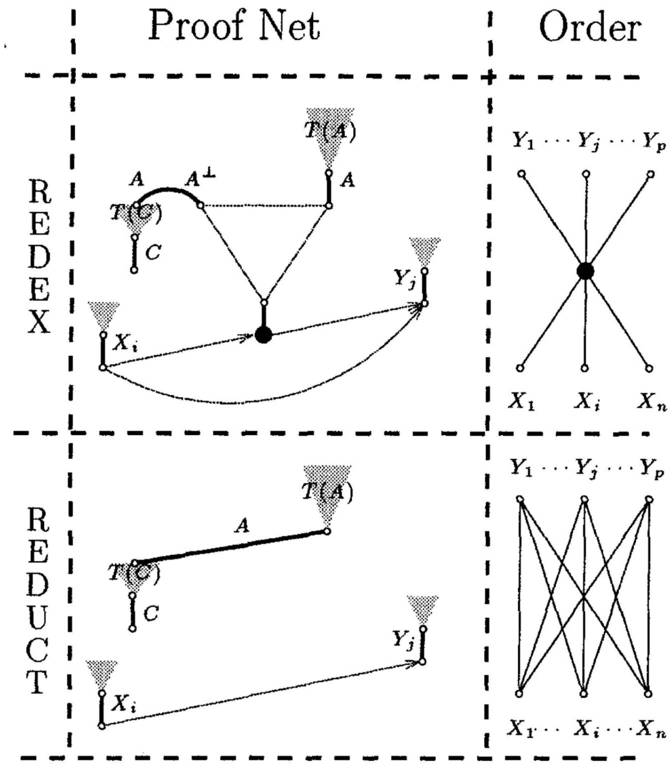
INR

Proof nets

- Extra-arc for denoting an order (preferably SP, definable) between conclusions
- Criterion no alernate elementary cycle
- Viewing cuts as $(\exists K) K \otimes K$ they take part in the order

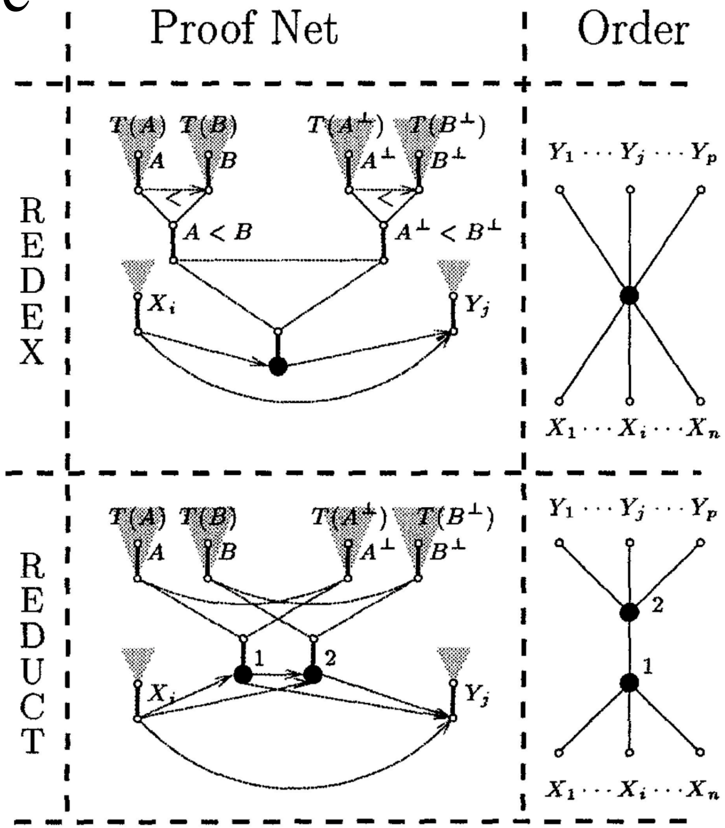
Cut elimination perserves correctness

Cut on axiom



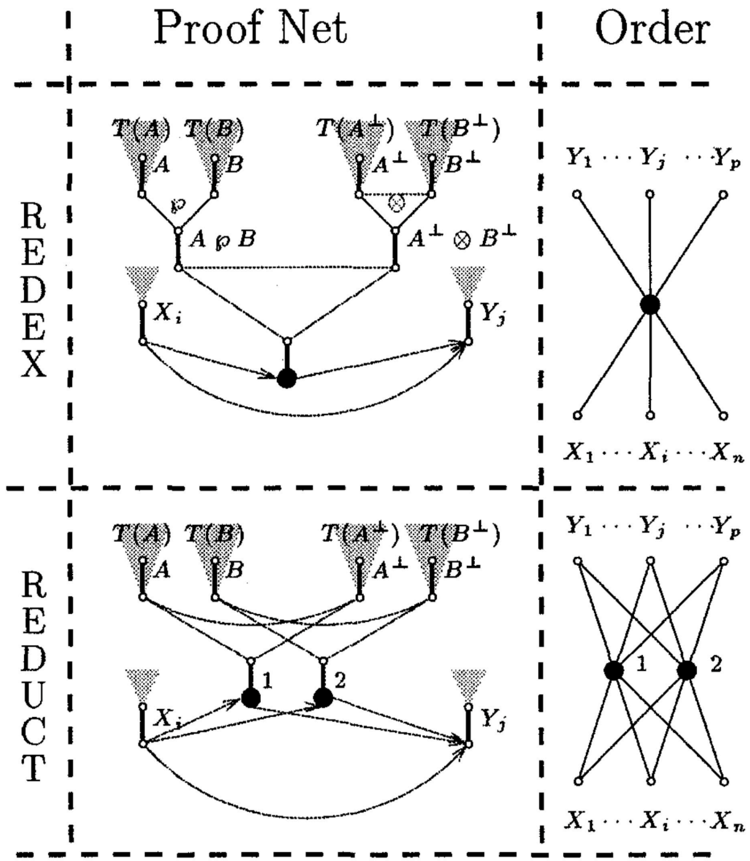
Cut elimination perserves correctness and order

Cut before/before



Cut elimination perserves correctness and order

Cut times/par



Interpreting proofs

- Choose a token for each axiom
- Collect the tuples: they are a clique of the coherence space associated with the partially ordered set of conclusions:

$$\vec{x} \smile \vec{y}[(A_i)_{i\in(I,<)}]$$

$$\Leftrightarrow$$

$$\exists i\; x_i \smile y_i \wedge (\forall j > i\; x_j = y_j)$$

Interpreting proofs: soundness and « completeness »

- Proof: would lead to an infinite alternate elementary path incoherent moving up, coherent moving down.
- Moreover the converse is true: if the proofnet is not correct, some interpretations are not cliques even in a single finite coherence space: N (isomorphic to its orthogonal Z)

Directed cographs

- Directed cographs for denoting formulae:
  - Containing the single vertex graphs
  - Closed under
    - Disjoint union
    - Undirected series composition
    - Directed series composition
    - (Hence under complementation if an undirected edge is viewed a pair of opposite directed edges)

Directed cographs

- Universal characterisation:
  - The directed part is an SP order
  - The undirected part is a cograph
  - Weak transitivity

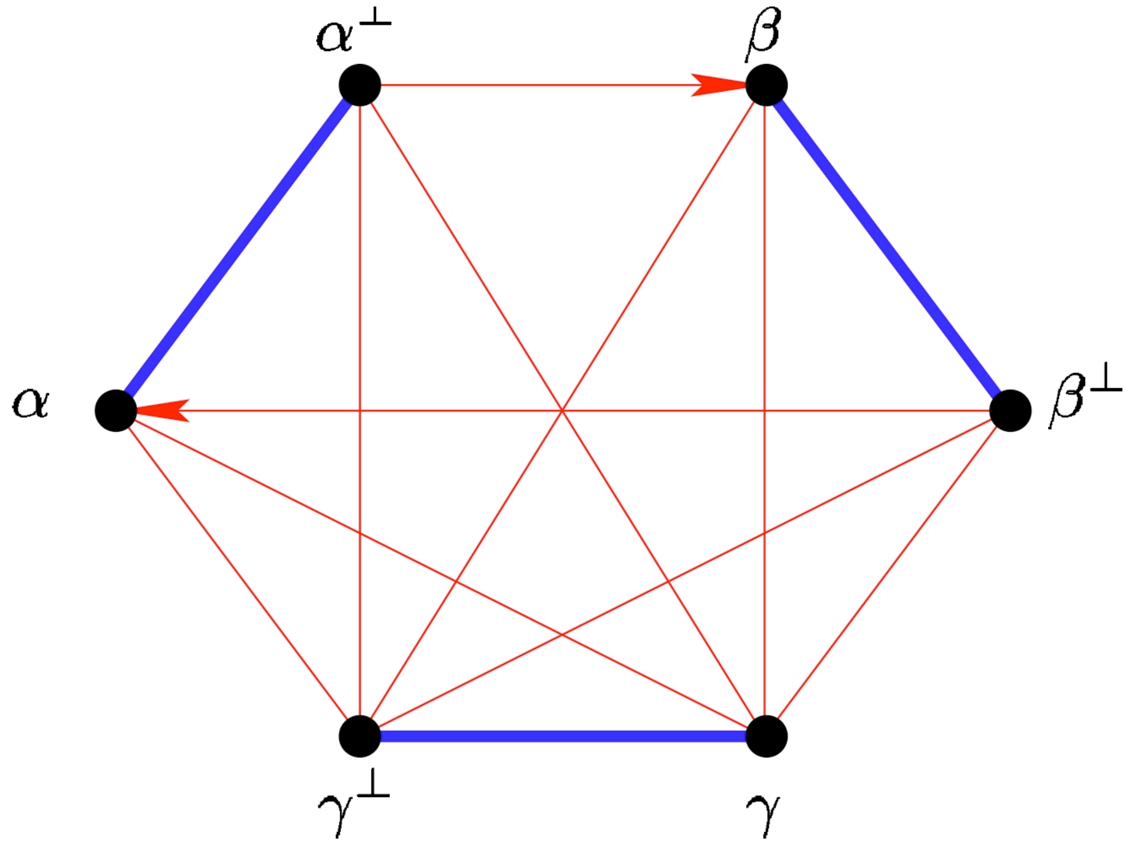
$$(x,y) \in R \wedge (y,x) \notin R \wedge (y,z) \in R \Rightarrow (x,z) \in R$$

$$(x,y) \in R \wedge (y,z) \in R \wedge (z,y) \notin R \Rightarrow (x,z) \in R$$

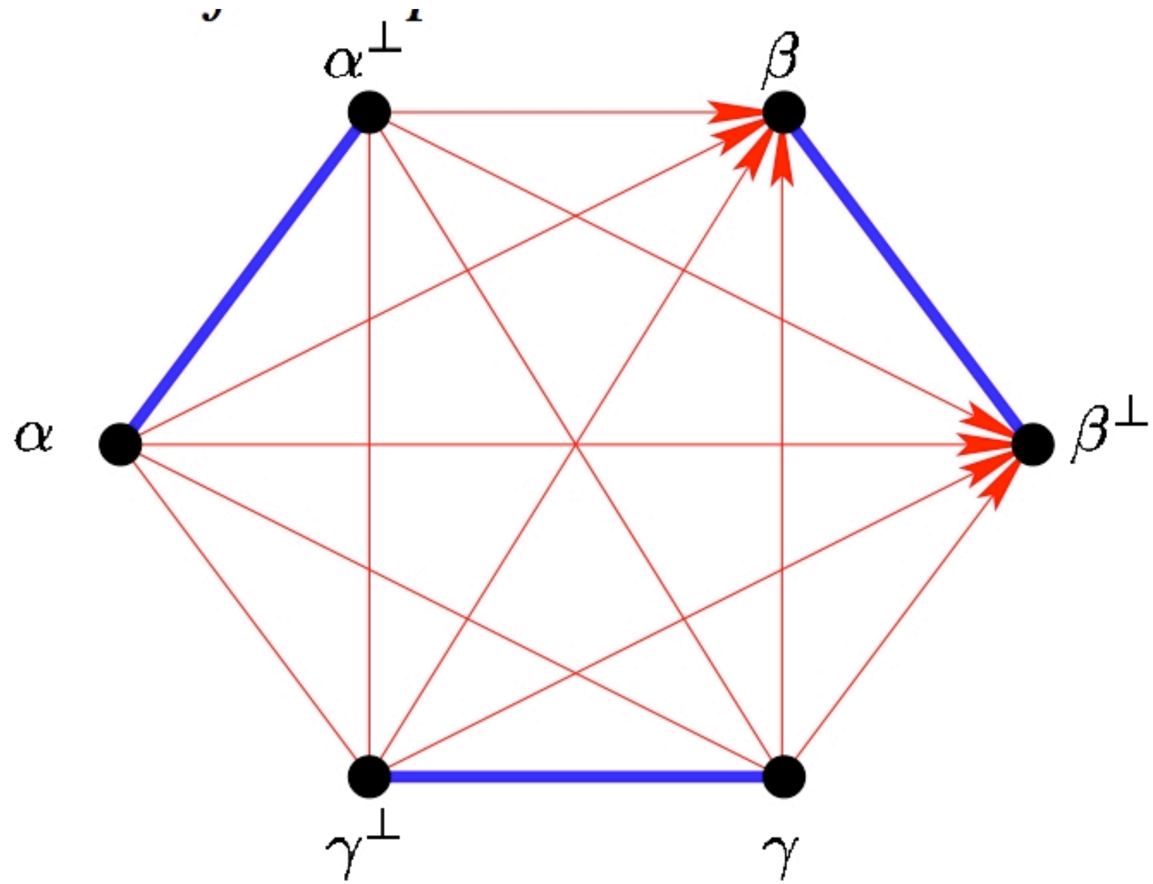
Handsome proofnets

- Vertices: propositional variables and their negations
- A directed cograph (the formula)
- Plus a perfect matching (the axioms)
- Criterion:
  - Every alternate elementary cycle contains a chord

Uncorrect

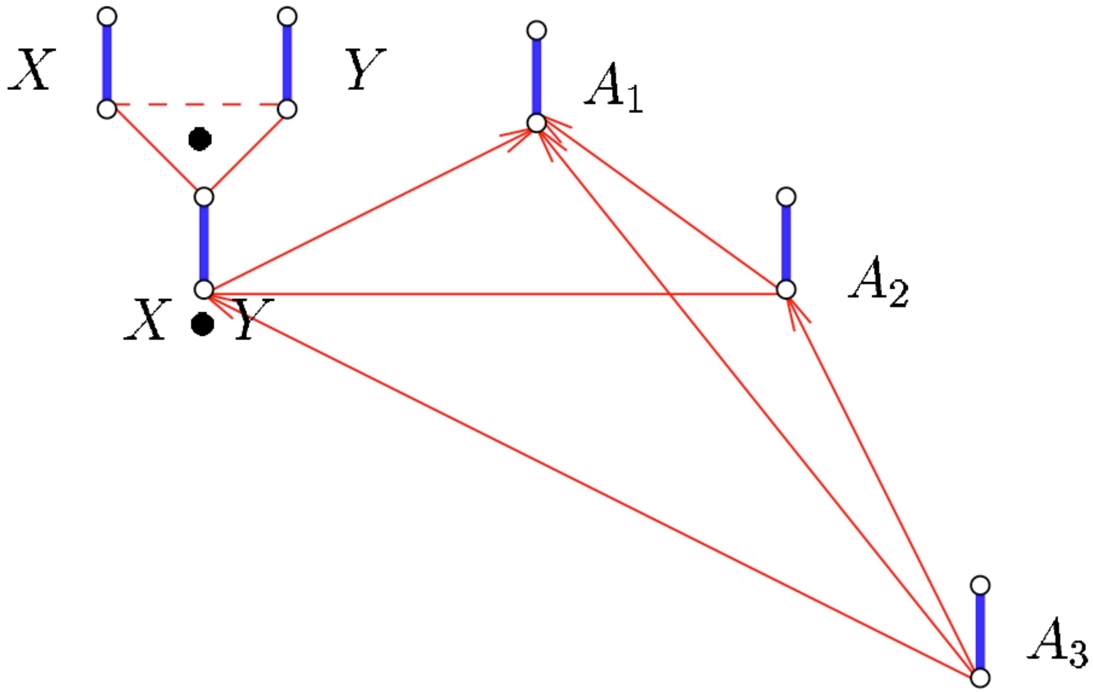


Correct

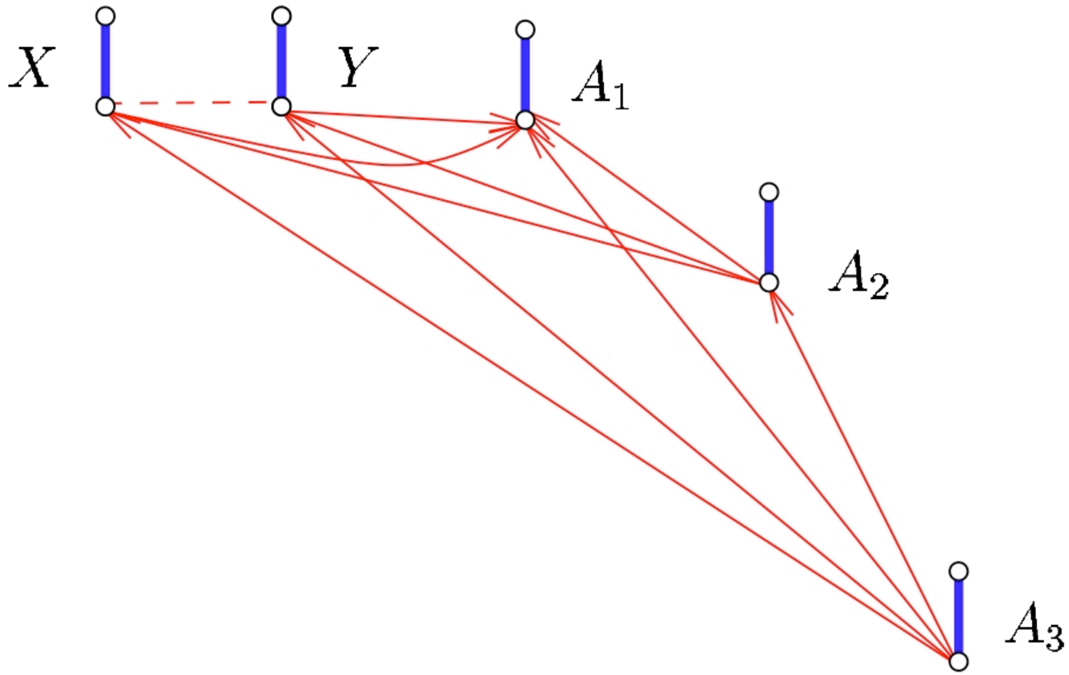


Fold

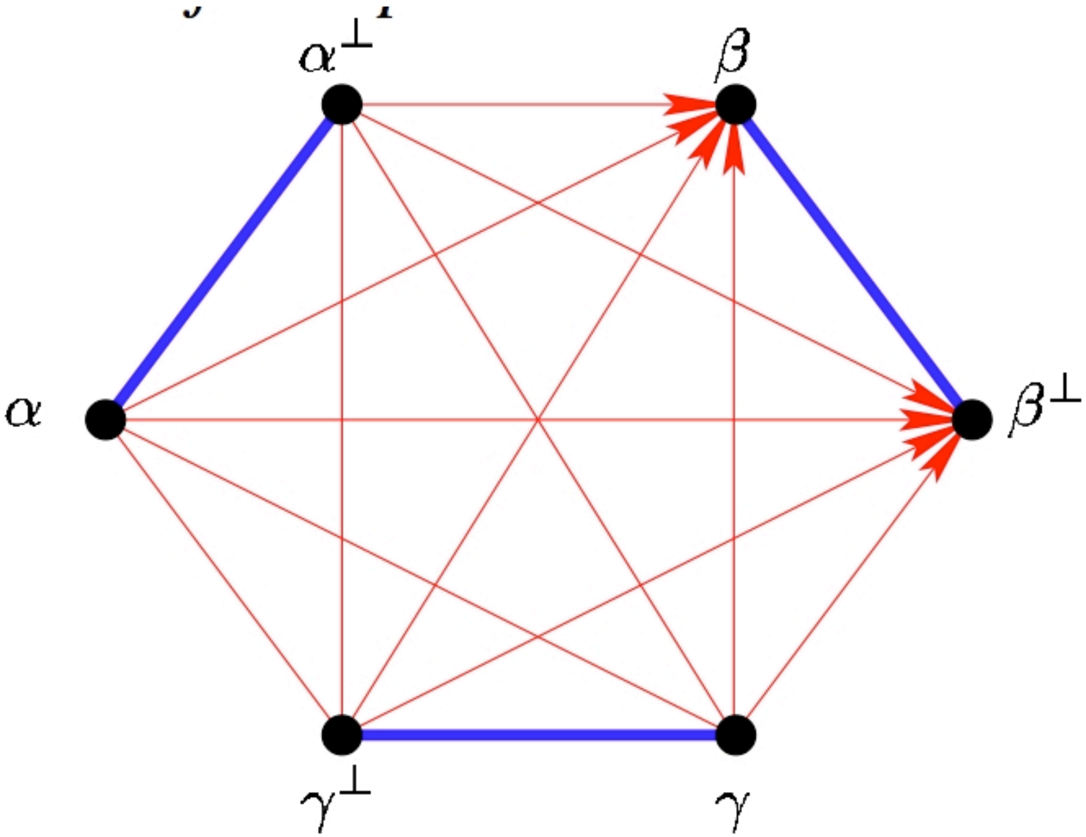
$\boxed{\Pi_\bullet}$



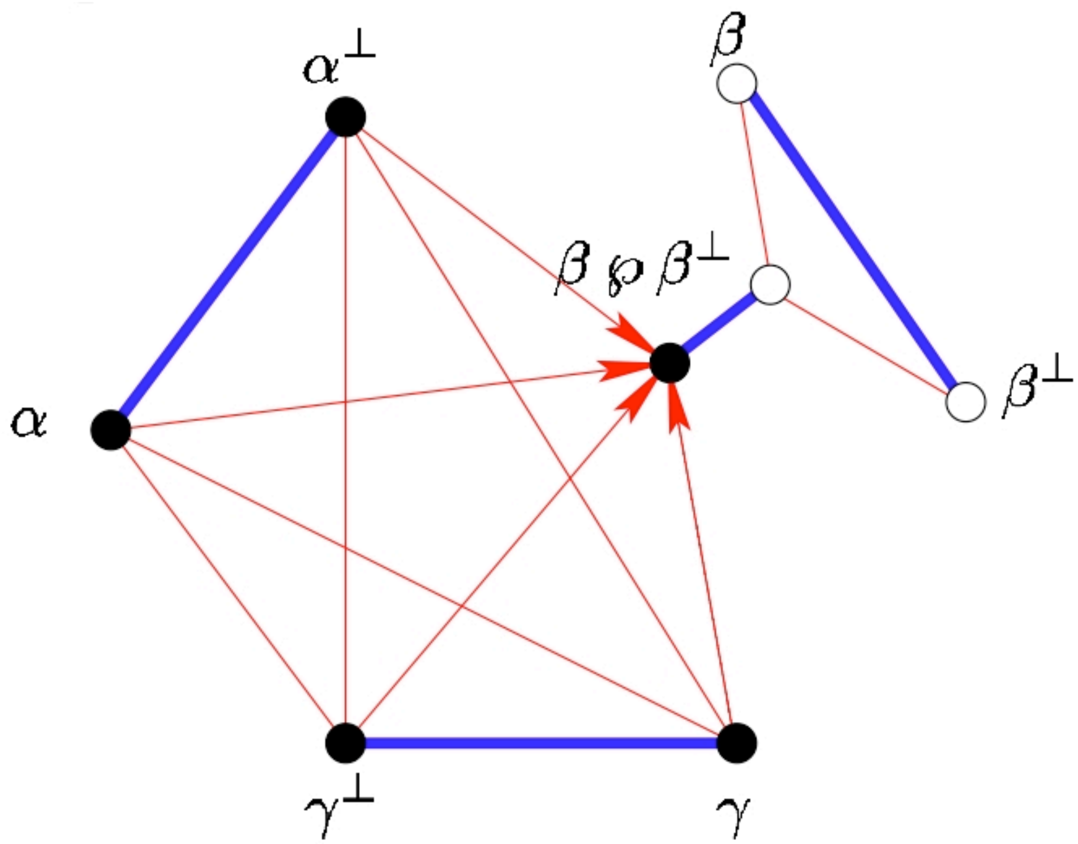
Unfold



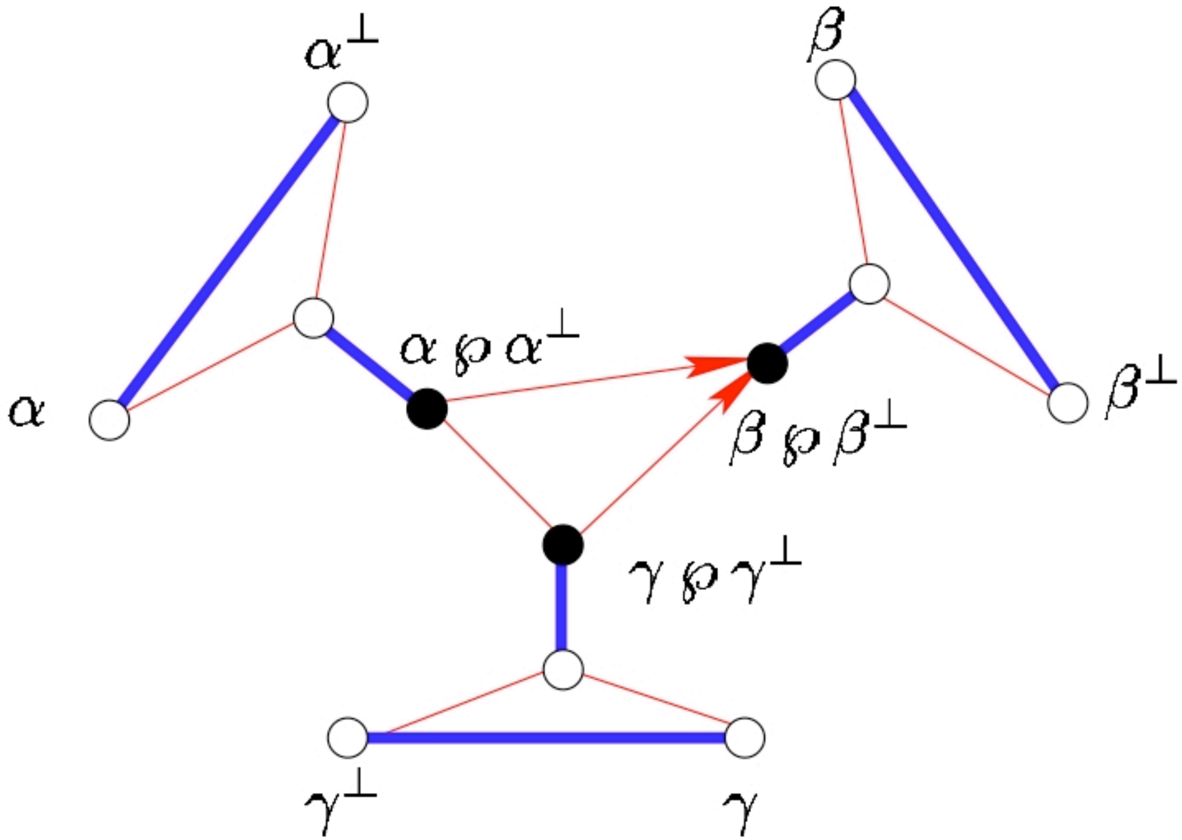
Correct



Correct with a link



Correct with three links



Property

- Fold and unfold preserve the criterion that every alternate lementary cycle contains a chord.
- Observe that when there are only links, this means that there is no alternate elementary cycle at all.

Cut-elimination

- Works directly on axioms
- Also derives from the one on proof nets with links.
- Looks like Girard's turbo cut-elimination

Rewriting (black lollipop preserves correctness)

$(\otimes\wp 4)$	$(X \mathbin{\widehat{\wp}} Y)$	$\widehat{\otimes}$	$(U \mathbin{\widehat{\wp}} V)$	$\longrightarrow$	$(X \mathbin{\widehat{\otimes}} U)$	$\widehat{\wp}$	$(Y \mathbin{\widehat{\otimes}} V)$
$(\otimes\wp 3)$	$(X \mathbin{\widehat{\wp}} Y)$	$\widehat{\otimes}$	U	$-\!\!\bullet$	$(X \mathbin{\widehat{\otimes}} U)$	$\widehat{\wp}$	Y
$(\otimes\wp 2)$	Y	$\widehat{\otimes}$	U	$-\!\!\bullet$	U	$\widehat{\wp}$	Y
$(\otimes{<}4)$	$(X \mathbin{\widehat{<}} Y)$	$\widehat{\otimes}$	$(U \mathbin{\widehat{<}} V)$	$-\!\!\bullet$	$(X \mathbin{\widehat{\otimes}} U)$	$\widehat{<}$	$(Y \mathbin{\widehat{\otimes}} V)$
$(\otimes{<}l3)$	$(X \mathbin{\widehat{<}} Y)$	$\widehat{\otimes}$	U	$-\!\!\bullet$	$(X \mathbin{\widehat{\otimes}} U)$	$\widehat{<}$	Y
$(\otimes{<}r3)$	Y	$\widehat{\otimes}$	$(U \mathbin{\widehat{<}} V)$	$-\!\!\bullet$	U	$\widehat{<}$	$(Y \mathbin{\widehat{\otimes}} V)$
$(\otimes{<}2)$	Y	$\widehat{\otimes}$	U	$-\!\!\bullet$	U	$\widehat{<}$	Y
$({<}\wp 4)$	$(X \mathbin{\widehat{\wp}} Y)$	$\widehat{<}$	$(U \mathbin{\widehat{\wp}} V)$	$-\!\!\bullet$	$(X \mathbin{\widehat{<}} U)$	$\widehat{\wp}$	$(Y \mathbin{\widehat{<}} V)$
$({<}\wp l3)$	$(X \mathbin{\widehat{\wp}} Y)$	$\widehat{<}$	U	$-\!\!\bullet$	$(X \mathbin{\widehat{<}} U)$	$\widehat{\wp}$	Y
$({<}\wp r3)$	Y	$\widehat{<}$	$(U \mathbin{\widehat{\wp}} V)$	$-\!\!\bullet$	U	$\widehat{\wp}$	$(Y \mathbin{\widehat{<}} V)$
$({<}\wp 2)$	Y	$\widehat{<}$	U	$-\!\!\bullet$	U	$\widehat{\wp}$	Y

Conjecture

- All correct handsome proofnets are obtained by the correct rewriting from

$$\bigotimes_i (a_i \parr a_i^\perp)$$

- (True for MLL)

Sequent calculus?

- Times as usual
- Par as usual
- MIX introduces the order
  the restrictions of K to G and D should be I and J

$$\frac{\vdash \Gamma[I] \qquad \vdash \Delta[J]}{\vdash \Gamma,\Delta[K]}$$

- Yields all correct proof nets?

Alternative conjecture (would directly yield sequentialisation)

- Given a correct handsome proofnet, there exists a partition A_1 A_2 of the axiom links (hence a partition V_1 V_2 of the vertices, since they are a complete matching) such that:
  - All the crossing edges are undirected and define a complete bipartite graph $K(U_1,U_2)$ with U_1 included in V_1 and U_2 included in V_2
  - All the crossing edges are directed and they all go from V_1 to V_2 or they all go from V_2 to V_1.

Old references

- 1993 Réseaux et séquents ordonnés PhD Thesis Paris 7
- 1997 Pomset logic a non commutative extension of classical linear logic. TLCA
- 1997 (with Bechet and de Groote) A complete axiomatisation of the inclusion a SP orders. RTA
- 1997 A semantic characterisation of the correctness of a proof nets. MSCS / INRIA Report
- 2003 Handsome proofnets: perfect matchings and cographs. TCS / INRIA Report

Slavnov sequent calulus (hint) 1) rules

- |- **X** (D)
- **X** multiset of formulas
- D binary relation (decoration) between pairs submultisets of **X**
  - – with the same number of elements and
  - – without common elements

  (two occurrences of the same formula are considered as distinct elements)

Slavnov sequent calulus (hint)
2) rules acting decorated sequents

- Rules are as usual as far as the multisets of formulas is concerned
- The decoration are merged in 4 various ways but preserve the relation between multisets of other formulas.
- 4 ways:
  - Par
  - Times
  - Directed par (dual of the later)
  - Directed times (dual of the former)

Slavnov sequent calulus (hint) 3) sequentialisation

- Slavnov as a proof net calculus ScMLL
- A sequentialisation theorem
  Idea: decoration correspond to disjoint directed paths from n conclusions to n conclusions in the proof net (cf. maximal alternate elementary paths in Michele's work with Elena Maringelli)
- Pomset logic is the calculus when
  - Directed par
  - Directed times

  are identified into self dual before

Conclusion and perspective

- The recent work by Sergey Slavnov, as well as ongoing research by Lutz Strassburger open new perspectives.
- Relation to deep inference and BV should be explored.
- Application to formal grammars and computational linguistics developed with Alain Lecomte will be better accepted with a seuqent calculus (not all linguists like proof nets).
- In particular, it introduces some resemblence with Michele's NL (two pairs of connectives).

➔ Merry retirement and happy new year Michele!