UM. Autumn 2019. Homework 3 to the course «Information theory».
[should be returned by Oct 1 to be counted in *contrôle continu*]

Problem 1. We are given a heap of n stones (for simplicity we assume that n is even), and we can use balance scales to compare weights of any two stones. We want to find in the heap two stones : the one with the *maximal weight* and the one with the *minimal weight*. Prove that these two stones can be found in at most $\frac{3n}{2} - 2$ weighings. Prove that this number of operations is optimal : any algorithm that solves this problem needs at least $\frac{3n}{2} - 2$ operations in the worst case.

Hint : At each stage of the search, we define the following four sets :

- `Potential-Max-or-Min` : the stones that have not yet participated in any comparison.
- `Potential-Max-not-Min` : the stones that have won at least one comparison but have not lost any comparison.
- `Potential-Min-not-Max` : the stones that have lost at least one comparison but have not won any comparison.
- `Not-Max-nor-Min` : the stones that have won at least one comparison and have lost at least one comparison.

Describe these sets at the beginning of the process (before the very first weighing) and at the very end of the process (when the maximal and the minimal stones are already found). Analyze how the stones travel between these sets after each comparison. Then propose an "adversarial strategy" that maximize the number of operations in the search.

Problem 2 (optional ; no need to bring the solution for correction)**.** Write a program that computes Shannon's entropy for a distribution with given probabilities $(p_1, \dots, p_n)$. [You will need this program for the next homework.]

Problem 3. (a) Let $S \subset \{a, b, c\}^n$ be the set of all strings with 50% of letters a, 25% of letters b, and 25% of letters c. Prove that there exists an injective mapping (a "text compressor")

$$\mathrm{Comp} : S \to \{0,1\}^{3n/2}$$

that assigns to each word from S a string of $3n/2$ bits.

(b) Prove that there is no injective mapping

$$\mathrm{Comp}_U : \{a, b, c\}^n \to \{0,1\}^{3n/2}$$

(a "text compressor" with this property does not exist).