UM. Autumn 2020. Homework 1 to the course «Information theory».
[should be returned by Sep 15 to be counted in *contrôle continu*]

Problem 1. Recall or learn any algorithm that sorts an array of n elements with $O(n \log n)$ comparisons (in the worst case).

Reminder : In the class we proved that no algorithm can do this faster than in $\log(n!)$ *comparisons.*

Problem 2. Prove that there exists a constant $\lambda > 0$ such that for all integer numbers $n \geq 1$

$$\log_2(n!) \geq \lambda n \log_2 n.$$

Problem 3 (Sorting algorithms)**.**
Find the number of comparisons needed in the worst case
(a) to sort an array of size 2 ;
(b) to sort an array of size 3 ;
(c) to sort an array of size 4 ;
(d) to sort an array of size 5.

Problem 4. We are given $n = 14$ coins, and one of them is fake. All genuine coins have the same weights, the fake one is heavier or lighter. We can use balance scales to compare weights of any two groups of coins. How many weighings do we need to find the fake coin (in the worst case) ?

Remark : If you cannot suggest a complete solution of this problem, please provide any lower or upper bounds.

Problem 5 (optional and difficult)**.** In what follows $\mathrm{Inf}(S)$ stands for Hartley's combinatorial information in a set S, i.e.,

$$\mathrm{Inf}(S) := \log_2 |S|.$$

Let S be a finite set in $\mathbb{Z}^3$. We denote by $\pi_{ij}[S]$ the projection of S onto the coordinates i and j (e.g., π_{13} applied to the point (x, y, z) gives (x, z)). Prove that

$$2 \cdot \mathrm{Inf}(S) \leq \mathrm{Inf}(\pi_{12}[S]) + \mathrm{Inf}(\pi_{13}[S]) + \mathrm{Inf}(\pi_{23}[S]).$$