Edge-partitioning regular graphs for ring traffic grooming with a priori placement of the ADMs [*] [**]

Xavier Muñoz[1] and Zhentao Li[2] and Ignasi Sau[3]

[1] Graph Theory and Combinatorics group - UPC - Barcelona, Spain.
xml@ma4.upc.edu
[2] School of Computer Science - McGill University - Montreal, Canada.
zhentao.li@mail.mcgill.ca
[3] AlGCo project-team, CNRS, LIRMM, Montpellier, France.
ignasi.sau@lirmm.fr

Abstract. We study the following graph partitioning problem: Given two positive integers C and Δ, find the least integer $M(C, \Delta)$ such that the edges of any graph with maximum degree at most Δ can be partitioned into subgraphs with at most C edges and each vertex appears in at most $M(C, \Delta)$ subgraphs. This problem is naturally motivated by traffic grooming, which is a major issue in optical networks. Namely, we introduce a new pseudo-dynamic model of traffic grooming in unidirectional rings, in which the aim is to design a network being able to support any request graph with a given bounded degree. We show that optimizing the equipment cost under this model is essentially equivalent to determining the parameter $M(C, \Delta)$. We establish the value of $M(C, \Delta)$ for almost all values of C and Δ, leaving open only the case where $\Delta \geq 5$ is odd, $\Delta \pmod{2C}$ is between 3 and $C - 1$, $C \geq 4$, and the request graph does not contain a perfect matching. For these open cases, we provide upper bounds that differ from the optimal value by at most one.

Keywords: graph decomposition, edge partition, regular graph, optical networks, traffic grooming, ADM, cubic graph, perfect matching.

1 Introduction

In this article we study the following graph partitioning problem: Given two positive integers C and Δ, find the least integer $M(C, \Delta)$ such that the edges of any graph with maximum degree at most Δ can be partitioned into subgraphs with at most C edges and each vertex appears in at most $M(C, \Delta)$ subgraphs. This problem is naturally motivated by traffic grooming in optical networks. In the following we provide an introduction to traffic grooming and explain why this partitioning problem is relevant to the design of optical networks. Readers interested only in the graph-theoretic problem may safely skip to the equivalent definition of $M(C, \Delta)$ after Remark 1 in Section 2, after reading the notation paragraph.

Motivation. Traffic grooming is the generic term for packing low-rate signals into higher-speed streams in optical networks [4, 8, 14, 17]. By using traffic grooming, it is possible to bypass the electronics at the nodes which are not sources or destinations of traffic, and therefore reduce the cost of the network. Typically, in a Wavelength Division Multiplexing (WDM) network, instead of having one SONET Add Drop Multiplexer (ADM) on every wavelength at every node, it is possible to have ADMs only for the wavelengths used at that node; the other wavelengths being optically routed without electronic switching.

The so called traffic grooming problem consists of minimizing the total number of ADMs to be used, in order to reduce the overall cost of the network. The problem is easily seen to be NP-hard for an arbitrary set of requests in very simple topologies. In fact, hardness and approximation results exist for traffic grooming in ring, star, and tree networks [2, 10, 11].

Here we consider unidirectional SONET/WDM ring networks with symmetric requests. In this case, the routing is unique and to each request between two nodes, we assign a wavelength and some bandwidth on this wavelength. If the traffic is uniform and any given wavelength can carry at most C requests, we can assign at most $1/C$ of the bandwidth to each request; the integer C is known as the *grooming factor*. Furthermore, if the traffic requirement is symmetric, we may assume that symmetric requests are assigned the same wavelength, as it is easy to show (by exchanging wavelengths) that there exists an optimal solution where all symmetric requests are given the same wavelength. Then each pair of symmetric requests uses $1/C$ of the bandwidth in the whole ring. If the two end-nodes are u and v, we need one ADM at node u and one at node v. The main point is that if two requests have a common end-node, they can share an ADM if they are assigned the same wavelength.

The traffic grooming problem for a unidirectional SONET ring with n nodes, grooming ratio C, and a symmetric request graph R has been modeled as a graph partition problem as follows (see [3, 13]). Each edge of R corresponds to a pair of symmetric requests, and edges are colored by their assigned wavelength λ. All edges of color λ induce a connected subgraph B_λ of R, where each node corresponds to an ADM. The grooming constraint, i.e., the fact that a wavelength can carry at most C requests, translates to an upper bound C on the number of edges in each B_λ. The cost corresponds to the total number of vertices used in the subgraphs, and the objective is therefore to minimize $\sum_\lambda |V(B_\lambda)|$.

While most of previous work has focused on the case where the requests are given as input [2–4, 7, 8, 10, 11, 13, 14], we consider the case where only the network topology is given, together with a bound Δ on the maximum degree of request graph. We would like to place, for each value of the grooming factor C, a minimum number of ADMs at each node in such a way that they could support *any* traffic pattern where each node is the end-node of at most Δ requests. This model is interesting because the network can support dynamic traffic without replacement of the ADMs; the existing theoretical models in the literature are much more rigid, and do not allow such adaptability.

From a practical point of view, it is interesting to design a network being able to support any request graph with maximum degree not exceeding a given constant. This situation is usual in real optical networks, since due to technology constraints the number of allowed communications for each node is usually bounded. This flexibility can also be thought from another point of view: given a fixed number of ADMs and a grooming factor, it is interesting to ask which is the maximum degree of a request graph that the network can support. Equivalently, given a maximum degree and a number of available ADMs, it is useful to know which values of the grooming factor the network will support.

The aim of this article is to provide a theoretical framework to design such networks with dynamically changing traffic. We study the case where the physical network is given by an unidirectional ring, which is a widely used topology (for instance, SONET rings). The formal definition of the problem is provided below. We first define the notation used throughout the article.

Notation. The (multi)graphs considered in this paper are finite and without self-loops. Edges are denoted $\{u, v\}$. The *degree* of a vertex v, denoted by $\deg(v)$, is the number of edges containing v as an end-point. When we speak about a *subgraph H* of a graph G defined by a subset of edges $F \subseteq E(G)$, we assume that $V(H)$ is the set of vertices spanned by the edges in F. For a positive integer C, a partition of a (multi)graph G into subgraphs with at most C edges is called a *C-edge-partition* of G. The *maximum degree* of a (multi)graph is the maximum degree over all its vertices. A *Δ-graph* is a (multi)graph with maximum degree at most Δ. $\mathcal{G}_\Delta$ denotes the class of all Δ-graphs. A *Δ-regular* (multi)graph is a graph in which all vertices have degree Δ. An *almost Δ-regular* (multi)graph is a (multi)graph in which all vertices have degree Δ except possibly one which has degree $\Delta - 1$. A *bridge* in a connected (multi)graph G is an edge whose removal disconnects G. A *matching* in a (multi)graph $G = (V, E)$ is a subset $M \subseteq E$ which contains each vertex at most once. A *perfect matching* is a matching containing all vertices. A *digon* is a cycle of length 2. A *trail* in a (multi)graph is a sequence $\{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{k-1}, x_k\}\}$ of distinct edges in which the second end of an edge is the first end of the next edge (the same pair of vertices may appear more than once if there is more than one edge between them). The vertices $x_2, x_3, \ldots, x_{k-1}$ of a trail are called *midpoints*. The *length* of a trail is the number of edges in it. Given a (multi)graph $G = (V, E)$ and a subset of vertices $V' \subseteq V$, we denote by $G - V'$ the (multi)graph obtained from G by removing the vertices in V', the edges incident with vertices in V', and isolated vertices (if any). Similarly, given a subset of edges $E' \subseteq E$, we denote by $G - E'$ the (multi)graph obtained from G by removing the edges in E' and isolated vertices (if any).

Statement of the problem. The problem we study can be formulated as a graph partitioning problem as follows.

Δ-DEGREE-BOUNDED TRAFFIC GROOMING IN UNIDIRECTIONAL RINGS

Input: Three positive integers n, C, and Δ.

Output: An assignment $A : \{1, \ldots, n\} \to \mathbb{N}$ such that for every n-vertex Δ-graph G with $V(G) = \{v_1, ..., v_n\}$, there exists a C-edge-partition of $E(G)$ such that for every $i \in \{1, \ldots, n\}$, vertex v_i occurs in at most $A(i)$ subgraphs of the partition.

Objective: Minimize $\sum_{i=1}^{n} A(i)$. The optimum is denoted by $A(n, C, \Delta)$.

This definition indeed corresponds to the grooming problem we discussed above, where n is the size of the ring, C is the grooming factor, Δ is the maximum degree of the request graph, $A(i)$ is the number of ADMs assigned to vertex v_i, and G is the request graph.

The function $A(n, C, \Delta)$ satisfies some straightforward properties.

Lemma 1. *The following statements hold:*

(i) $A(n, C, 1) = n$ *for every* $n, C \geq 1$.
(ii) $A(n, 1, \Delta) = \Delta \cdot n$ *for every* $\Delta \geq 1$.
(iii) *If* $C' \geq C$, *then* $A(n, C', \Delta) \leq A(n, C, \Delta)$.
(iv) *If* $\Delta' \geq \Delta$, *then* $A(n, C, \Delta') \geq A(n, C, \Delta)$.
(v) $A(n, C, \Delta) \geq n$ *for every* $\Delta \geq 1$.
(vi) *If* $C \geq \frac{n\Delta}{2}$, $A(n, C, \Delta) = n$.

Proof:

(i) For each vertex v_i, consider a graph G consisting only of an edge containing v_i. Therefore, it holds $A(i) \geq 1$ for every $i \in \{1, \ldots, n\}$. On the other hand, G must be a matching, so for every vertex v_i we may assume that $A(i) \leq 1$. Therefore, $A(n, C, 1) = n$ for every $n, C \geq 1$.
(ii) A Δ-graph can be partitioned into $\frac{n\Delta}{2}$ disjoint edges, and the bound is tight for Δ-regular graphs.

(iii) Any solution for C is also a solution for C'.
(iv) If $\Delta' \geq \Delta$, the graphs on n vertices with maximum degree at most Δ form a subclass of the class of graphs with maximum degree at most Δ'.
(v) Combine *(i)* and *(iv)*.
(vi) In this case all the edges of any Δ-graph fit into one subgraph.   □

Organization of the paper. In Section 2 we show that the Δ-Degree-Bounded Traffic Grooming in Unidirectional Rings problem is essentially equivalent to establishing the value of the parameter $M(C, \Delta)$ (see Definition 1) for each value of C and Δ. We solve the cases where $\Delta \geq 2$ is even in Section 3. In Section 4 we focus on the cases where $\Delta \geq 3$ is odd, leaving open only the cases where $\Delta \geq 5$ is odd, $\Delta \pmod{2C}$ is between 3 and $C - 1$, $C \geq 4$, and the graph does not contain a perfect matching (see Table 1). In Section 4.6 we present an attempt to solve these remaining cases, that may lead to an eventual proof. Finally, Section 5 concludes the article.

2   The parameter $M(C, \Delta)$

The following definition will play a fundamental role in the remainder of the article.

Definition 1. $M(C, \Delta)$ *is the least number M such that $A(n, C, \Delta) \leq M \cdot n$ for all n.*

Lemma 2. $M(C, \Delta)$ *is a natural number.*

Proof: For every $C \geq 1$, we know by Lemma 1 that $n \leq A(n, C, \Delta) \leq A(n, 1, \Delta) = \Delta \cdot n$. Suppose that M is not a natural number. That is, suppose that $r < M < r + 1$ for some positive integer r. Therefore, there must be at least $(r + 1 - M) \cdot n$ vertices such that each of them occurs at most r times in any C-edge-partition. For each n, let $V_{n,r}$ be the subset of vertices of the request graph with at most r occurrences. Then, since $r + 1 - M > 0$, we have $\lim_{n \to \infty} |V_{n,r}| = \infty$. In other words, there is an arbitrarily big subset of vertices with at most r occurrences per vertex. But we can consider a Δ-graph on (possibly a subset of) the set of vertices $V_{n,r}$, and this means that with r occurrences per vertex we can construct a C-edge-partition, a contradiction with the optimality of M.   □

If the request graph is further restricted to belong to a subclass of graphs $\mathcal{C} \subseteq \mathcal{G}_\Delta$, then the corresponding parameter is denoted by $M(C, \Delta, \mathcal{C})$. Note that, as long as the class $\mathcal{C}$ contains infinitely many graphs, the proof of Lemma 2 shows that $M(C, \Delta, \mathcal{C})$ is a positive integer. This is the case, for instance, of the class of regular graphs with a perfect matching.

By the discussion above, $A(n, C, \Delta)$ is of the form $A(n, C, \Delta) = M(C, \Delta) \cdot n - \alpha(C, \Delta)$, where $M(C, \Delta)$ and $\alpha(C, \Delta)$ are integers depending only on C and Δ. Suppose that a Δ-graph H requires at least $M(C, \Delta) + 1$ occurrences of some vertex. Since any Δ-graph must admit a C-edge-partition with at most the same number of occurrences, by relabeling the vertices of H we could force at least $M(C, \Delta) + 1$ occurrences in $\Omega(n)$ vertices of the graph. This would contradict the definition of $M(C, \Delta)$. Therefore, each vertex can appear in at most $M(C, \Delta)$ subgraphs. So we may conclude the following.

Remark 1 *For each value of C and Δ, the Δ-Degree-Bounded Traffic Grooming in Unidirectional Rings problem reduces to finding the least integer $M(C, \Delta)$ such that any Δ-graph admits a C-edge-partition with each vertex appearing in at most $M(C, \Delta)$ subgraphs.*

This allows us to give an equivalent definition of $M(C, \Delta)$. Let $G \in \mathcal{G}_\Delta$ and let $\mathcal{P}_C(G)$ be the set of C-edge-partitions of G. For $P \in \mathcal{P}_C(G)$ with $P = \{B_\lambda\}_{1 \leq \lambda \leq \Lambda}$, let $\mathrm{occ}(P)$ be the maximum number of occurrences of a vertex in the partition, that is,

$$\mathrm{occ}(P) = \max_{v \in V(G)} |\{B_\lambda \in P : v \in B_\lambda\}|, \quad \text{and then} \quad M(C, \Delta) = \max_{G \in \mathcal{G}_\Delta} \left(\min_{P \in \mathcal{P}_C(G)} \mathrm{occ}(P) \right).$$

In the remainder of this paper, we use Remark 1 and focus on determining $M(C, \Delta)$ for each value of C and Δ. Observe also that any Δ-graph H is a subgraph of some Δ-regular graph G (with possibly more vertices). Note also that if we restrict a partition of G to the vertices of H, the number of occurrences of the vertices cannot increase. Therefore,

Remark 2 $M(C, \Delta) = M(C, \Delta, \mathcal{C})$, *where* $\mathcal{C}$ *is the class of* Δ-*regular graphs.*

The following lemma will be used throughout the article.

Lemma 3. *The following statements hold trivially from Lemma 1:*

(i) $M(C, 1) = 1$ *for all* $C \geq 1$.
(ii) $M(1, \Delta) = \Delta$ *for all* $\Delta \geq 1$.
(iii) *If* $C' \geq C$, *then* $M(C', \Delta) \leq M(C, \Delta)$.
(iv) *If* $\Delta' \geq \Delta$, *then* $M(C, \Delta') \geq M(C, \Delta)$.
(v) $M(C, \Delta) \leq \Delta$ *for all* $C, \Delta \geq 1$.

The following proposition establishes a general lower bound on $M(C, \Delta)$, that will allow us to prove in many cases the optimality of the constructions of the next sections.

Proposition 1. $M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ *for every* $C, \Delta \geq 1$.

Proof: Given $C, \Delta \geq 1$, let G be a Δ-regular graph with girth at least $C + 1$, which exists by the seminal result of Erdös and Sachs [9], and let $n = |V(G)|$. Let P be a C-edge-partition of G. All the subgraphs involved in P are trees, since each such subgraph has at most C edges and the girth of G is larger than C. Let p be the number of subgraphs in P, so $p \geq \frac{n\Delta}{2C}$. Let $n_1, \ldots, n_p$ be the orders of the subgraphs in P. Since each of them is a tree, $\sum_{i=1}^{p} (n_i - 1) = \frac{n\Delta}{2}$, so $\sum_{i=1}^{p} n_i = \frac{n\Delta}{2} + p \geq \frac{\Delta(C+1)}{2C} \cdot n$. Therefore, there exists a vertex spanned by at least $\frac{\Delta(C+1)}{2C}$ subgraphs. By the definition of $M(C, \Delta)$, the lower bound follows. $\square$

3   Case $\Delta \geq 2$ even

In this section we establish the value of $M(C, \Delta)$ for an even $\Delta \geq 2$ and any value of C.

Theorem 3. *Let* $\Delta \geq 2$ *be even. Then for any* $C \geq 1$, $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Proof: The lower bound follows from Proposition 1. Let us give an explicit construction for any Δ-regular graph $G = (V, E)$. Orient the edges of G in an Eulerian tour, and assign to each vertex $v \in V$ its $\Delta/2$ out-edges, namely E_v^+. For each $v \in V$, partition E_v^+ into $\left\lceil \frac{\Delta}{2C} \right\rceil$ stars with C edges centered at v (except, possibly, one star with fewer edges). Each vertex v appears as a leaf in stars centered at other vertices exactly $\Delta - \Delta/2 = \Delta/2$ times. Therefore, the number of occurrences of each vertex in this partition is

$$\left\lceil \frac{\Delta}{2C} \right\rceil + \frac{\Delta}{2} = \left\lceil \frac{\Delta}{2} \left(1 + \frac{1}{C} \right) \right\rceil = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil.$$

$\square$

Note that for the special case $\Delta = 2$, Theorem 3 implies that $M(C, 2) = 2$ for all $C \geq 1$. In fact, for $\Delta = 2$ it is possible to give the exact expression of the cost function $A(n, C, 2)$. Indeed, it is easy to see that given a set of disjoint cycles, we can always find a C-edge-partition such that $C-1$ prescribed (arbitrary) vertices appear in only one subgraph. On the other hand, if we pretend that at least C vertices appear in at most one subgraph, we can consider as Δ-graph a cycle of length at least $C+1$ containing the prescribed C vertices, and then necessarily one of those vertices appears in at least two subgraphs of any C-edge-partition, a contradiction. Summarizing,

Proposition 2. $A(n, C, 2) = 2n - (C - 1)$.

4   Case $\Delta \geq 3$ odd

The cases where Δ is odd turn out to be inherently much more complicated than the cases where Δ is even. In Section 4.1, we present a general construction which differs from the lower bound of Proposition 1 by at most 1, and we determine when this construction is optimal. In Section 4.2 we provide an improved lower bound when $\Delta \equiv C \pmod{2C}$, which meets our upper bound. In Section 4.4 we solve the case $\Delta = 3$ and $C = 4$, which is the only case for $\Delta = 3$ that does not follow from the results presented so far. In Section 4.5 we present an optimal construction for graphs with a perfect matching, after proving that the lower bound of Proposition 1 still holds when the request graph is restricted to have a perfect matching. We then discuss in Section 4.3 the relation of the parameter $M(C, \Delta)$ with the *linear C-arboricity* [1, 5, 16]. Finally, we describe in Section 4.6 an attempt to solve the remaining cases where $\Delta \geq 5$ is odd, using the ideas developed in the previous sections.

4.1   General upper bound

The following proposition provides a general upper bound, which differs from the lower bound of Proposition 1 by at most 1.

Proposition 3. *Let $\Delta \geq 3$ be odd. Then for any $C \geq 1$, $M(C, \Delta) \leq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.*

Proof: Let G be a Δ-regular graph. Since Δ is odd, $|V(G)|$ is even. Add a perfect matching M to G to obtain a $(\Delta + 1)$-regular multigraph G'. Orient the edges of G' in an Eulerian tour, and assign to each vertex $v \in V(G')$ its $(\Delta+1)/2$ out-edges E_v^+. Remove the edges of M and, as in the case Δ even, partition E_v^+ into stars with at most C edges. To count the number of occurrences of each vertex, we distinguish two cases. If an edge of M is in E_v^+, then v appears as center in $\left\lceil \frac{\Delta - 1}{2C} \right\rceil$ stars and as a leaf in $\Delta - \frac{\Delta - 1}{2}$ stars. Summing both terms yields

$$\left\lceil \frac{\Delta - 1}{2C} \right\rceil + \Delta - \frac{\Delta - 1}{2} = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil.$$

Otherwise, if no edge of M is in E_v^+, the number of occurrences of v is

$$\left\lceil \frac{\Delta + 1}{2C} \right\rceil + \Delta - \frac{\Delta + 1}{2} = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} + \frac{1 - C}{2C} \right\rceil \leq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil. \qquad \square$$

The upper bound of Proposition 3 and the lower bound of Proposition 1 are equal for, roughly speaking, half of the pairs C, Δ, as shown in the following corollary.

Corollary 1. *Let $\Delta \geq 3$ be odd. If $\Delta \pmod{2C} = 1$ or $\Delta \pmod{2C} \geq C + 1$, then $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.*

Proof: Let $\Delta = \lambda \cdot 2C + h$, with h odd, $1 \leq h \leq 2C - 1$. Writing $k := \lambda(C + 1) + \frac{h-1}{2}$, the lower bound of Proposition 1 equals $k + \left\lceil \frac{1}{2} + \frac{h}{2C} \right\rceil$, and the upper bound of Proposition 3 equals $k + \left\lceil 1 + \frac{h-1}{2C} \right\rceil$. If $h = 1$ both bounds equal $k + 1$, and if $h \geq C + 1$ both bounds equal $k + 2$. $\quad\square$

In particular, when $C = 2$ and Δ is odd, $\Delta \pmod{2C}$ is either 1 or 3, and then by Corollary 1 the lower bound is attained, as stated in the following corollary.

Corollary 2 (Case $C = 2$). *For any $\Delta \geq 3$ odd, $M(2, \Delta) = \left\lceil \frac{3\Delta}{4} \right\rceil$.*

For all the cases we solved so far, the value of $M(C, \Delta)$ equals the lower bound of Proposition 1. It seems natural to think that the value $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ may be always attained. We shall see in the next section that this is not true. Namely, we prove in Theorem 4 that if $\Delta \equiv C \pmod{2C}$, then $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.

4.2   Improved lower bound

In this section we prove a new lower bound which strictly improves on Proposition 1 when $\Delta \equiv C$ (mod $2C$).

Theorem 4. *Let $\Delta \geq 3$ be odd and let $\Delta \equiv C$ (mod $2C$). Then $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.*

Proof: We prove that if $\Delta = kC$ with k odd, then $M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$ and thus, by Proposition 3, $M(C, \Delta)$ is equal to $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$. Since both Δ and k are odd, so is C, and therefore $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil = k \cdot \frac{C+1}{2}$.

We proceed to build a Δ-regular graph G with no C-edge-partition where each vertex is incident to at most $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ subgraphs, hence implying that $M(C, \Delta) > \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$. First, we construct a graph H where all vertices have degree Δ except one which has degree $\Delta - 1$. Furthermore, we build H so that it has girth strictly greater than C. H exists by [6, 9]. Make Δ copies of H and add a cut-vertex v joined to all vertices of degree $\Delta - 1$ to make our Δ-regular graph G (see Fig. 1 for an example of the construction of such a graph for $\Delta = C = 3$).
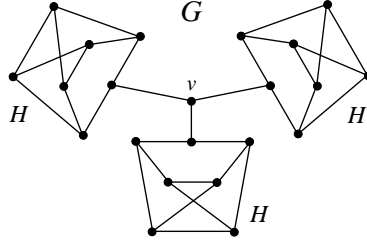
Fig. 1. 3-regular graph with girth 4, which is a counterexample showing that $M(3, 3) = 3$.

Now suppose for the sake of contradiction that there is a C-edge-partition $\mathcal{B}$ of G where each vertex is incident to at most $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ subgraphs. Since the girth of G is greater than C, all the subgraphs in $\mathcal{B}$ are trees. Since $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil < \Delta$, v must have degree at least 2 in some subgraph $T' \in \mathcal{B}$. Since $|E(T')| \leq C$, the tree T' contains at most $\left\lfloor \frac{C-2}{2} \right\rfloor = \frac{C-3}{2}$ edges of a copy H' of H intersecting T'. Now we only work in H'. Let $\alpha = |E(T' \cap H')| \leq \frac{C-3}{2}$ (note that $\alpha = 0$ for $C = \Delta = 3$).

Let $\mathcal{B}' = \{B \cap H'\}_{B \in (\mathcal{B} - \{T'\})}$, with the empty subgraphs removed. That is, $\mathcal{B}'$ contains the subgraphs in $\mathcal{B}$ that partition the edges in H' that are not in T'. Let $n = |V(H')|$, which is odd as in H' there is one vertex of degree $\Delta - 1$ and all the others have degree Δ. Therefore, the total number of edges of the trees in $\mathcal{B}'$ is

$$\sum_{T \in \mathcal{B}'} |E(T)| \;=\; |E(H')| - \alpha \;=\; \frac{n\Delta - 1}{2} - \alpha \;=\; \frac{nkC - 1}{2} - \alpha. \tag{1}$$

As $\alpha \leq \frac{C-3}{2}$, from Equation (1) we get

$$\sum_{T \in \mathcal{B}'} |E(T)| \;\geq\; \frac{nkC - 1}{2} - \frac{C - 3}{2} \;=\; \left(\frac{nk - 1}{2} \right) \cdot C + 1. \tag{2}$$

As each tree in $\mathcal{B}'$ has at most C edges, from Equation (2) we get that $|\mathcal{B}'|$, the number of trees in $\mathcal{B}'$, satisfies

$$|\mathcal{B}'| \;\geq\; \left\lceil \frac{nk - 1}{2} + \frac{1}{C} \right\rceil \;=\; \frac{nk - 1}{2} + \left\lceil \frac{1}{C} \right\rceil \;=\; \frac{nk - 1}{2} + 1. \tag{3}$$

Clearly, the total number of vertices in the trees in $\mathcal{B}'$ is exactly the total number of edges in the trees in $\mathcal{B}'$ plus the number of trees in B', that is, $\sum_{T \in \mathcal{B}'} |V(T)| = \sum_{T \in \mathcal{B}'} |E(T)| + |\mathcal{B}'|$. On the

other hand, the tree T' contains $\alpha + 1$ vertices of H', that is, $|V(T' \cap H')| = \alpha + 1$. Therefore, using Equations (1) and (3), we get that the total number of occurrences of the vertices in H' in some tree of $\mathcal{B}$ is

$$\sum_{v \in V(H')} |\{T \in \mathcal{B} : v \in T\}| = \sum_{T \in \mathcal{B}'} |V(T)| + |V(T' \cap H')| = \sum_{T \in \mathcal{B}'} |E(T)| + |\mathcal{B}'| + \alpha + 1$$

$$= \frac{nkC - 1}{2} - \alpha + |\mathcal{B}'| + \alpha + 1 \geq \frac{nkC - 1}{2} + \frac{nk - 1}{2} + 1 + 1$$

$$= nk \cdot \frac{C+1}{2} + 1 = n \cdot \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1 ,$$

which implies that at least one vertex of H' appears in at least $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$ subgraphs, which is a contradiction to $\mathcal{B}$ being a C-edge-partition of G in which each vertex appears in at most $\left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ subgraphs. The theorem follows. $\qquad \square$

It turns out that Theorem 4 allows us to find the value of $M(3, \Delta)$ for any $\Delta \geq 3$ odd.

Corollary 3 (Case $C = 3$). *For any $\Delta \geq 3$ odd, $M(3, \Delta) = \left\lceil \frac{2\Delta + 1}{3} \right\rceil$.*

Proof: If $\Delta \equiv 1 \pmod 6$ or $\Delta \equiv 5 \pmod 6$, then by Corollary 1, $M(3, \Delta) = \left\lceil \frac{2\Delta}{3} \right\rceil = \left\lceil \frac{2\Delta + 1}{3} \right\rceil$. Otherwise, if $\Delta \equiv 3 \pmod 6$, then by Theorem 4, $M(3, \Delta) = \left\lceil \frac{2\Delta}{3} \right\rceil + 1 = \left\lceil \frac{2\Delta + 1}{3} \right\rceil$. $\qquad \square$

4.3   Relation of $M(C, \Delta)$ with the linear C-arboricity

A result of Thomassen [16], which settled a conjecture of Bermond *et al.* [5], states that the edges of a cubic graph can be 2-colored such that each monochromatic component is a path of length at most 5. That is, in such a coloring (that can be seen as a partition into paths) each vertex appears in exactly 2 paths with at most 5 edges each. Therefore, combining this result with *(iii)* of Lemma 3 we deduce that $M(C, 3) = 2$ for any $C \geq 5$.

Let us now discuss how these ideas can be extended to other values of C and Δ. A *linear C-forest* in a graph is a forest consisting of paths of length at most C. The *linear C-arboricity* of a graph G is the minimum number of linear C-forests required to partition $E(G)$, and is denoted by $\mathrm{la}_C(G)$ [5]. Let $\mathrm{la}_C(\Delta) = \max_{G \in \mathcal{G}_\Delta} \mathrm{la}_C(G)$. Clearly $M(C, \Delta) \leq \mathrm{la}_C(\Delta)$ for all C, Δ, since the paths in a *linear C-forest* are graphs with at most C edges. Therefore, the following upper bound given by Alon *et al.* [1] also applies to $M(C, \Delta)$.

Theorem 5 (Alon *et al.* [1]). *There is an absolute constant $\beta > 0$ such that for $\sqrt{\Delta} > C \geq 2$,*

$$\mathrm{la}_C(\Delta) \leq \frac{C+1}{C} \frac{\Delta}{2} + \beta \sqrt{C\Delta \log \Delta}. \tag{4}$$

It turns out that the first addend of the right-hand side of Equation (4) is equal to the lower bound of Proposition 1, so Theorem 5 provides an additive $\mathcal{O}(\sqrt{C\Delta \log \Delta})$-approximation of $M(C, \Delta)$ for $\sqrt{\Delta} > C \geq 2$. Although we have improved this bound for $M(C, \Delta)$ in Sections 3 and 4.1, the relation between $M(C, \Delta)$ and $\mathrm{la}_C(\Delta)$ is of theoretical interest of its own.

4.4   Case $\Delta = 3$, $C = 4$

As discussed in Section 4.3, $M(C, 3) = 2$ for $C \geq 5$. On the other hand, Theorem 4 implies that $M(3, 3) = 3$, so by *(ii)* and *(iii)* of Lemma 3 we have that $M(C, 3) = 3$ for $C \leq 3$. Therefore, the interesting question is whether $M(4, 3)$ equals 2 or 3. The remainder of this section is devoted to prove that $M(4, 3) = 2$ (see Corollary 5). First we need a classical result concerning cubic graphs and an easy extension to cubic multigraphs.

Theorem 6 (Petersen [15]). *Any cubic bridgeless graph has a perfect matching.*

Corollary 4. *Any cubic bridgeless multigraph without self-loops has a perfect matching.*

Proof: Let G be a cubic multigraph without self-loops. We can assume that G has no triple edges, otherwise G has only 2 vertices and any of the 3 edges is a perfect matching. Consider the simple graph G' built from G as follows: for each digon $\{\{u,v\},\{u,v\}\}$, add 2 new vertices s_{uv} and t_{uv}, and replace the digon with the edges $\{u,s_{uv}\},\{u,t_{uv}\},\{v,s_{uv}\},\{v,t_{uv}\}$, and $\{s_{uv},t_{uv}\}$. By Theorem 6, G' has a perfect matching M'. We now construct a perfect matching M of G from M'. For each edge $e \in M'$ such that e was also an edge of G, put e in M. For each digon $\{\{u,v\},\{u,v\}\}$ of G, if any of the pairs $\{\{u,s_{uv}\},\{v,t_{uv}\}\}$ or $\{\{u,t_{uv}\},\{v,s_{uv}\}\}$ is in M', put one of the copies of $\{u,v\}$ in M. Otherwise, $\{s_{uv},t_{uv}\}$ belongs to M' and we do nothing. It is easy to check that M is a perfect matching of G. $\qquad\square$

We are ready to prove the main result of this section.

Theorem 7. *The edges of every almost 3-regular multigraph G without self-loops can be partitioned into a set $\mathcal{W} = \{W_1, W_2, \ldots, W_k\}$ of trails of length at most 4 such that each vertex appears as the midpoint of a trail.*

Proof: Suppose the theorem is false and let G be a counterexample with the minimum number of vertices. G is connected as otherwise, we can take the union of the partitions of its connected components, which exist by minimality of G.

Suppose first that G contains a bridge $e = \{u,v\}$. Then $G - \{e\}$ has exactly two components: U containing u and V containing v. Without loss of generality, we may choose U to be the component with no degree 2 vertex in G and e is chosen so that U is maximal with this property. Thus this component U of $G - \{e\}$ is almost 3-regular (only u has degree 2). By minimality of G, U can be partitioned into a set $\mathcal{W}^u$ of trails as in the statement of the theorem.

If v has degree 2 in G then $V - \{v\}$ is almost 3-regular. By minimality of G, $V - \{v\}$ can be partitioned into a set $\mathcal{W}^v$ of trails as in the theorem. Now the only edges of G not in any trail in $\mathcal{W}^u \cup \mathcal{W}^v$ are those incident to v. Thus taking $\mathcal{W}^u \cup \mathcal{W}^v$ together with a trail consisting of the 2 edges incident to v (which has v as a midpoint) yields the required partition of the edges of G into trails. This contradicts the fact that G is a counterexample.

If v has degree 3 in G, let x, y be the neighbors of v in V (see Fig. 2(a)). We can assume $x \neq y$ (i.e., $\{v,x\}$ and $\{v,y\}$ are not parallel edges) since otherwise, the third edge incident to $x = y$ is a cut edge whose choice (instead of e) would increase the size of U. Let H be the graph obtained from $V - \{v\}$ by adding an edge $f = \{x,y\}$ (see Fig. 2(b)). By minimality of G, H can be partitioned into a set $\mathcal{W}^v$ of trails. We now attempt to transform $\mathcal{W}^u \cup \mathcal{W}^v$ into a partition of G into trails.
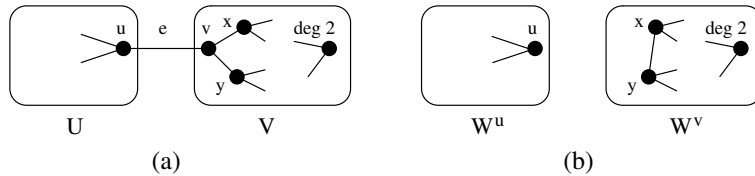
Fig. 2. (a) A bridge $e = \{u,v\}$ in an almost 3-regular graph G with components U and V of $G - \{e\}$. (b) Graphs smaller than G from which we obtain a partition into trails W^u and W^v.

The edge f appears in some trail $\{W_1, \{x,y\}, W_2\}$ of $\mathcal{W}^v$, where W_1 is a (possibly empty) trail ending at x and W_2 is a (possibly empty) trail starting at y. At least one of the subtrails
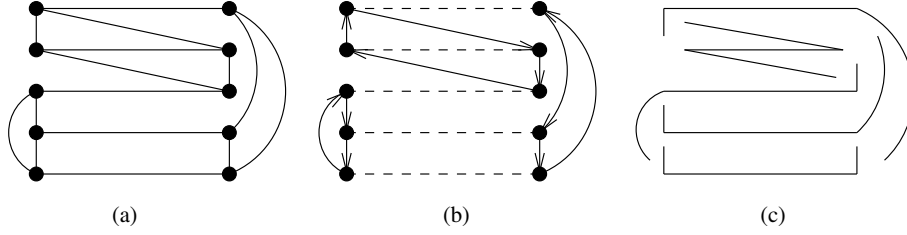
Fig. 3. (a) A 3-regular graph G' with no bridges. (b) A matching M of G' (shown in dashed lines) and an orientation of the cycles of $G' - M$. (c) A partition of the edges of G' into trails of length 3 using M and the orientation of the cycle of $G' - M$ in (b).

$\{W_1, \{x, y\}\}$ or $\{\{x, y\}, W_2\}$ has fewer than 3 edges. Without loss of generality, it is $\{W_1, \{x, y\}\}$. Replace this trail with $\{W_1, \{x, v\}, \{v, u\}\}$ which has length at most 4, and $\{\{v, y\}, W_2\}$ which has length less than or equal to $\{W_1, \{x, y\}, W_2\}$. Note that x and v are midpoints of the first trail and y is the midpoint of the second trail. Furthermore, any other vertex which was a midpoint in $\{W_1, \{x, y\}, W_2\}$ is still a midpoint (since W_1 and W_2 appear as subtrails).

Thus the union of $\mathcal{W}^u$ and $\mathcal{W}^v$ with the above replacement yields a partition of G into trails of length at most 4 with the desired property, which is a contradiction.

We may now assume that G contains no bridges. If G is 3-regular, let $G' = G$. Otherwise, let G' be the graph obtained from G by replacing the vertex of degree 2 with an edge between its neighbors. Note that G' is 3-regular and contains no bridges. Therefore, by Corollary 4, G' contains a perfect matching $M \subseteq E(G')$.

Since G' is 3-regular, $G' - M$ is 2-regular. Thus, $G' - M$ is a union of disjoint cycles. We can orient the cycles of $G' - M$ so that each vertex v has exactly one edge e_v pointing towards v. For each edge $\{u, v\} \in M$, $W_{uv} = \{e_u, \{u, v\}, e_v\}$ is a trail of length 3 (see Fig. 3). Note that $\mathcal{W} = \{W_{uv} \mid \{u, v\} \in M\}$ is a partition of the edges of G' into trails of length 3. Furthermore, every vertex u in the matching appears as the midpoint of the trail corresponding to the edge of the matching in which u appears. Since M is a perfect matching, every vertex appears as the midpoint of some trail in $\mathcal{W}$. Thus $G' \neq G$ as otherwise, we have constructed a partition as required by the theorem. So G has a vertex v of degree 2 which we replaced with an edge $e = \{x, y\}$ to obtain G'. Let $W = \{W_1, \{x, y\}, W_2\}$ be the trail in $\mathcal{W}$ containing e, and recall that W has length 3. Replacing W with $\{W_1, \{x, v\}, \{v, y\}, W_2\}$ in $\mathcal{W}$ yields a partition of $E(G)$ into trails of length at most 4, which is a contradiction. □

Note that the simple trees with some vertex of degree 3 and the digon with a pendant edge at each side are *not* allowed in the partition stated in Theorem 7, since these graphs cannot be thought of as trails. The following corollary settles the value of $M(4, 3)$.

Corollary 5. $M(4, 3) = 2$.

Proof: By Remark 2, we may restrict ourselves to 3-regular graphs. Thus, a 3-regular graph G is almost 3-regular and we may apply Theorem 7 to obtain a partition $\mathcal{W}$. Let $\mathcal{B} = \{E(W)\}_{W \in \mathcal{W}}$. Each vertex of G appears in at most two elements of $\mathcal{B}$, as G is 3-regular and each vertex appears as the midpoint of some trail in $\mathcal{W}$. □

To conclude this section, we would like to mention that Fouquet and Vanherpe study in [12] *normal* partitions of cubic graphs, which are defined as a partition of the edges of a cubic graph into trails such that each vertex is the end-point of exactly one trail of the partition. The *length* of a normal partition is the length (in terms of number of edges) of the longest trail in it. Following this notation, Theorem 7 implies that any cubic multigraph admits a normal partition of length at most 4.

4.5   Optimal construction for graphs with a perfect matching

In this section we focus on the case where the Δ-regular graphs are further restricted to contain a perfect matching. First observe that the proof of the general lower bound provided in Proposition 1 does not imply that the same lower bound carries over to Δ-regular graphs with a perfect matching. Indeed, the proof of Proposition 1 uses the existence of Δ-regular graphs with girth at least $C+1$, but those graphs may not necessarily contain a perfect matching. Fortunately, we can prove that the lower bound does not decrease when we assume that the graph contains a perfect matching.

Proposition 4. *Let $\Delta \geq 3$ be odd and let $\mathcal{C}$ be the class of Δ-regular graphs that contain a perfect matching. Then $M(C, \Delta, \mathcal{C}) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ for all $C \geq 1$.*

Proof: We shall construct a Δ-regular graph G with a perfect matching and girth at least $C+1$, and then the proof of Proposition 1 applied to G yields the desired bound. The details follow.

For any two positive integers Δ and C, Chandran provided in [6, Section 2.1] an explicit and simple construction of a graph H such that

- H has girth strictly greater than C;
- H contains a perfect matching (in fact, H is obtained from a perfect matching by adding the appropriate edges); and
- The degree of a vertex of H is either $\Delta - 2$, $\Delta - 1$, or Δ.

We will construct from H our Δ-regular graph G. Let $v_1, \ldots, v_{k_1}$ be the vertices of degree $\Delta - 1$ in H, and let $v_{k_1+1}, \ldots, v_{k_1+k_2}$ be the vertices of degree $\Delta - 2$ in H. Let F be a $(k_1 + 2k_2)$-regular graph with girth at least $C+1$ (which exists by the result of Erdös and Sachs [9]), and let $f = |V(F)|$. Let the vertices of F be $u^1, \ldots, u^f$. To construct G, first make f copies of H, and let v_i^j be the copy of vertex v_i in the j-th copy of H, for $i = 1, \ldots, k_1 + k_2$ and $j = 1, \ldots, f$. Intuitively, each copy of H corresponds to a vertex of F. We now add $|E(F)|$ edges among the f copies of H as follows. We assign labels from 1 to $k_1 + 2k_2$ to the vertices of H with degree less than Δ in the following way: for $i = 1, \ldots, k_1$, vertex v_i gets label i, and for $i = k_1 + 1, \ldots, k_1 + k_2$, vertex v_i gets labels i and $k_2 + i$. For each vertex of F, we label arbitrarily the edges incident to it with distinct integers from 1 to $k_1 + 2k_2$ (recall that F is $(k_1 + 2k_2)$-regular). This way, each edge of F gets two labels, one from each end-vertex. Then, for each edge $\{u^{j_1}, u^{j_2}\} \in E(F)$ with labels (ℓ_1, ℓ_2), we add an edge between the vertices labeled ℓ_1 and ℓ_2 in the j_1-th and j_2-th copies of H, respectively.

This completes the construction of G. Note that the copies of the vertices that had degree Δ in H have also degree Δ in G. Since one (resp. two) edges have been added to each vertex of degree $\Delta - 1$ (resp. $\Delta - 2$), it is clear that G is Δ-regular. Since each copy of H had a perfect matching and no edge of any copy of H has been removed, G has also a perfect matching. Finally, the girth of G is at least $C+1$. Indeed, the girth of each copy of H is at least $C+1$ by [6]. Therefore, each cycle c of length at most C in G should visit strictly more than one copy of H. By the construction of G, such a cycle c in G would induce a cycle of length at most C in F among the vertices corresponding to the copies of H visited by c. But this is impossible as the girth of F is at least $C+1$. $\square$

We are now ready to provide an optimal construction for all $\Delta \geq 3$ odd and $C \geq 1$ when the request graph is restricted to have a perfect matching.

Proposition 5. *Let $\Delta \geq 3$ be odd and let $\mathcal{C}$ be the class of Δ-regular graphs that contain a perfect matching. Then $M(C, \Delta, \mathcal{C}) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ for all $C \geq 1$.*

Proof: The lower bound follows from Proposition 4. To prove the upper bound, let G be a Δ-regular with a perfect matching M. Then $G - M$ is $(\Delta - 1)$-regular, with $\Delta - 1$ even. We orient the edges of $G - M$ in an Eulerian tour, and assign to each vertex $v \in V(G)$ its $\frac{\Delta - 1}{2}$ out-edges E_v^+. We distinguish three cases.

(1) $\Delta < C$. For each edge $\{u, v\} \in M$, build a tree with Δ edges consisting of $\{u, v\}$, $\frac{\Delta-1}{2}$ edges from E_u^+, and $\frac{\Delta-1}{2}$ edges from E_v^+. The number of occurrences of each vertex is $1 + \Delta - \frac{\Delta+1}{2} = \frac{\Delta+1}{2}$. The lower bound equals $\lceil \frac{C+1}{C} \frac{\Delta}{2} \rceil = \frac{\Delta-1}{2} + \lceil \frac{1}{2} + \frac{\Delta}{2C} \rceil$, which equals $\frac{\Delta+1}{2}$ as $\Delta < C$.

(2) $\Delta \geq C$ and $C \geq 3$ is odd (the case $C = 1$ is trivial by Lemma 3). For each edge $\{u, v\} \in M$, build a tree with C edges consisting of $\{u, v\}$, $\frac{C-1}{2}$ edges from E_u^+, and $\frac{C-1}{2}$ edges from E_v^+. Partition the remaining $\frac{\Delta-1}{2} - \frac{C-1}{2} = \frac{\Delta-C}{2}$ edges assigned to each vertex into $\lceil \frac{\Delta-C}{2C} \rceil$ stars with at most C edges. The number of occurrences of each vertex is

$$1 + \left\lceil \frac{\Delta - C}{2C} \right\rceil + \Delta - \frac{\Delta + 1}{2} = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil.$$

(3) $\Delta \geq C$ and $C \geq 4$ is even (the case $C = 2$ is solved by Corollary 2). Build a tree with $C - 1$ edges consisting of $\{u, v\}$, $\frac{C-2}{2}$ edges from E_u^+, and $\frac{C-2}{2}$ edges from E_v^+. Partition the remaining $\frac{\Delta-1}{2} - \frac{C-2}{2} = \frac{\Delta-C+1}{2}$ edges assigned to each vertex into stars with at most C edges. The number of occurrences of each vertex is

$$1 + \left\lceil \frac{\Delta - C + 1}{2C} \right\rceil + \frac{\Delta - 1}{2} = \left\lceil \frac{\Delta(C+1) + 1}{2C} \right\rceil = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil,$$

where the last equality holds because both Δ and $(C + 1)$ are odd. □

4.6 Towards a proof for the remaining cases

In this section, we describe an attempt to prove that the lower bound $\lceil \frac{C+1}{C} \frac{\Delta}{2} \rceil$ of Proposition 1 is attained in the remaining cases where $\Delta \geq 5$ is odd. We attempt to resolve the remaining cases by using induction and Tutte's matching theorem. We may assume that Δ is odd by Theorem 3.

The idea is to use the construction from Proposition 5 of Section 4.5, which solves the case where the graph contains a perfect matching, as a base case for a proof by induction. Then, if the graph does not contain a perfect matching, an easy consequence of Tutte's matching theorem shows that it contains an edge-cut of size at most $\Delta - 1$. We would then like to recurse on each side of the cut as we did in the proof of Theorem 7 and combine the edge-partitions of each side into a partition of the whole graph. However, as opposed to Theorem 7, it is more difficult to deal with the edges across the cut in this case.

We may orient each edge $e = \{u, v\}$ across the cut from u to v and let the side containing u decide which partition will contain e. To guarantee that v is not incident to too many subgraphs at the end, we can simply force v to be incident to one fewer subgraph in the edge-partition of the side containing v.

We note that, since the cuts have size less than Δ, it is possible to recursively orient the edges of cuts so that no side has more than $\Delta - 1$ edges pointing towards it (including edges from previous steps of the recursion).

However, it seems difficult to control the distribution of the edges pointing towards a side. If, for example, a single vertex v had $\Delta - 1$ edges pointing towards it, then it is clearly impossible to obtain the desired edge-partition, as v would need to be in a negative number of parts. On the other hand, if it were possible to control the distribution of the edges pointing towards a side, the following strengthening of Proposition 5 would be sufficient to prove the base case of the induction.

Definition 2. *G is* near-Δ-regular *if the vertices of G have degrees between $\frac{\Delta}{2}$ and Δ and $|E(G)| \geq \frac{\Delta}{2}(|V(G)| - 1) - 1$ (i.e., the total degree is off by at most $\Delta - 1$).*

Lemma 4. *Let $LB(C, \Delta) = \lceil \frac{C+1}{C} \frac{\Delta}{2} \rceil$, the lower bound of Proposition 1. Let C, Δ be positive integers with Δ odd and $(\Delta - 1)/2$ not a multiple of C. Let G be a near-Δ-regular graph with girth at least 5 and a perfect matching. Then G has an edge-partition where each vertex v is incident to at most $LB(C, \Delta) - (\Delta - \deg(v))$ subgraphs of the partition.*

We note that it may be possible to first recursively find all the cuts and then orient the edges so that no vertex has more than $(\Delta - 1)/2$ edges pointing towards it. We also note that the above lemma is not in its strongest form (e.g., we could have the total degree differ by more the one stated in the lemma) but we clearly cannot relax the condition that every vertex v has degree greater than $\Delta/2$ (otherwise, v is contained in too many subgraphs even if we use stars of size C centered at v). We now prove Lemma 4.

Proof: Let M be a perfect matching in G. Since at most Δ vertices of G have degree not equal to Δ, at most Δ edges in M connect an odd degree vertex to an even degree vertex. Let G' be the graph obtained from G by removing edges of M matching odd degree vertices of G and adding (at most $\Delta/2$) edges to pair up the remaining odd degree vertices of G. Thus, G' is an even graph and we may obtain an Eulerian orientation O' of G'.

O' induces an orientation of some of the edges of G. We orient the remaining edges of G "both ways" and count half towards the in-degree and half towards the out-degree of the vertex. Let S be the set of vertices of G with degree less than Δ. We reverse some of the arcs of O so that all vertices in S have out-degree at least $\Delta/2$. This can be done greedily since G has no C_4 and we are only off by $\Delta - 1$ from the total degree. We call this new orientation O.

Let S' be the set of vertices with out-degree less than $\Delta/2$ in O. Note that the vertices of S' have degree Δ and out-degree at least $(\Delta - 3)/2$. Let $N^-(v)$ denote the set of vertices with an arc to v in O. Note that for two distinct vertices u and v in G, their neighborhood intersects in at most one vertex (since G has girth at least 5). Therefore $N^-(u) \cap N^-(v)$ also contains at most one vertex.

Therefore, we may find a subgraph H of the graph induced by the edges $N^-(v)$ to v for all v in S' with the following properties:

○ Each vertex in $N^-(S')$ has degree at most 1.
○ Each vertex in $s \in S'$ has degree at least $\delta^-(s) - (\Delta - 1)/2 \geq \delta^-(s) - (2C - 1)$, where $\delta^-(s)$ is the in-degree of s.

We say that a star or double-star is *full* if it contains exactly C edges.

Now, we can find a set $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ of edge disjoint subgraphs of G such that

○ $\mathcal{S}_1$ is a set of full double stars centered at the endpoints of unoriented edges in O,
○ $\mathcal{S}_2$ is a set of full stars,
○ $\mathcal{S}_3 = \{S_v\}_{v \in V(G)}$, where S_v is a star centered at v of size at most $C - 1$, and
○ only stars in $\mathcal{S}_3$ contain edges of H.

These edge disjoint subgraphs can be found greedily by first finding $\mathcal{S}_1$ and then partitioning the out-edges of each vertex into sets of size C and a remainder set of edges of size $\leq C$.

Now, for each $s \in S'$, remove all but two stars centered at s in $\mathcal{S}_2$ and remove one star centered at s in $\mathcal{S}_3$. Let R be the set of edges removed in this way. For each edge $e = \{u, v\} \in E(H)$, add an out-edge of v in R if there is any left (and remove this edge from R) to the star containing e. By the properties of H, no edges of R are left in the end.

We claim that this new set of subgraphs form a C-edge-partition where each vertex v is incident to at most $LB(C, \Delta) - (\Delta - \deg(v))$ partitions.

Indeed, the elements of $\mathcal{S}'$ are edge disjoint and have size at most C (since every star in $\mathcal{S}_3$ has size at most $C - 1$). The vertices $v \in V - S''$ are incident to

$$\begin{aligned}
\frac{\Delta + 1}{2C} + \deg(v) - \frac{\Delta + 1}{2} \;&=\; \frac{\Delta + 1}{2C} + \frac{\Delta - 1}{2} - (\Delta - \deg(v)) \\
&=\; \frac{\Delta(C + 1) + 1 - C}{2C} - (\Delta - \deg(v)) \\
&\leq\; LB(C, \Delta) - (\Delta - \deg(v))
\end{aligned}$$

subgraphs if v is not incident to an unoriented edge, and

$$
\begin{aligned}
1 + \left\lceil \frac{\Delta - C}{2C} \right\rceil + \deg(v) - \frac{\Delta + 1}{2} \;&=\; 1 + \left\lceil \frac{\Delta - C}{2C} \right\rceil + \frac{\Delta - 1}{2} - (\Delta - \deg(v)) \\
&=\; \left\lceil \frac{\Delta + C + C(\Delta - 1)}{2C} \right\rceil - (\Delta - \deg(v)) \\
&=\; \left\lceil \frac{(C + 1)\Delta}{2C} \right\rceil - (\Delta - \deg(v)) \\
&=\; LB(C, \Delta) - (\Delta - \deg(v))
\end{aligned}
$$

subgraphs if v is incident to an unoriented edge. This satisfies the conditions in the lemma.

Recall that vertices in S' have out-degree at least $(\Delta - 3)/(2C)$. If $v \in S'$ and v is incident to an unoriented edge, v appears in

$$
\begin{aligned}
\frac{\Delta - 3 - (2C - 2)}{2C} + \deg(v) - \frac{\Delta - 3}{2} \;&=\; \frac{\Delta - 1 - 2C}{2C} + \deg(v) - \frac{\Delta + 1}{2} + \frac{4}{2} \\
&=\; \frac{\Delta - 1}{2C} - 2 + \deg(v) - \frac{\Delta + 1}{2} + 2 \\
&\leq\; \frac{\Delta + 1}{2C} + \deg(v) - \frac{\Delta + 1}{2} \\
&\leq\; LB(C, \Delta) - (\Delta - \deg(v))
\end{aligned}
$$

subgraphs. If $v \in S'$ and and v is not incident to an unoriented edge, v appears in

$$
\begin{aligned}
1 + \left\lceil \frac{\Delta - 3 - C - (2C - 2)}{2C} \right\rceil + \deg(v) - \frac{\Delta - 3}{2} \;&=\; 1 + \left\lceil \frac{\Delta - 1 - C}{2C} \right\rceil - 2 + \deg(v) - \frac{\Delta + 1}{2} + 2 \\
&\leq\; 1 + \left\lceil \frac{\Delta - C}{2C} \right\rceil + \deg(v) - \frac{\Delta + 1}{2} \\
&=\; LB(C, \Delta) - (\Delta - \deg(v))
\end{aligned}
$$

subgraphs. Again, the conditions in the lemma are satisfied as required. $\qquad\square$

5   Conclusions

In this article we introduced the traffic grooming problem in unidirectional WDM rings when the request graph belongs to the class of graphs with maximum degree Δ. Such a model allows the network to support dynamic traffic without reconfiguring the electronic equipment. We showed that this problem is essentially equivalent to finding the least integer $M(C, \Delta)$ such that the edges of any graph with maximum degree at most Δ can be partitioned into subgraphs with at most C edges and each vertex appears in at most $M(C, \Delta)$ subgraphs. We established the value of $M(C, \Delta)$ for many cases, leaving open only the case where $\Delta \geq 5$ is odd, $\Delta \pmod{2C}$ is between 3 and $C - 1$, $C \geq 4$, and the graph does not contain a perfect matching. Table 1 summarizes what is known about $M(C, \Delta)$, including the case where the graph has a perfect matching. For the remaining cases, we hope to either extend the counterexample given in Section 4.2 or to complete the partial proof given in Section 4.6, which can be seen as a strengthening of Proposition 5.

Considering bounded-degree request graphs is natural from a networking perspective. It would be also interesting to consider as input other families of request graphs that make sense from a telecommunications point of view, like circulant graphs or graphs of bounded diameter.

$C\|\Delta$	1	2	3	4	5	6	7	8	9	...	Δ even	Δ odd
1	1	2	3	4	5	6	7	8	9	...	Δ	Δ
2	1	2	3	3	4	5	6	6	7	...	$\lceil\frac{3\Delta}{4}\rceil$	$\lceil\frac{3\Delta}{4}\rceil$
3	1	2	3 (2)	3	4	5 (4)	5	6	7 (6)	...	$\lceil\frac{2\Delta}{3}\rceil$	$\lceil\frac{2\Delta+1}{3}\rceil$ ($\lceil\frac{2\Delta}{3}\rceil$)
4	1	2	2	3	4	4	5	5	6	...	$\lceil\frac{5\Delta}{8}\rceil$	$\geq \lceil\frac{\mathbf{5\Delta}}{\mathbf{8}}\rceil$ (=)
5	1	2	2	3	4 (3)	4	5	5	6	...	$\lceil\frac{3\Delta}{5}\rceil$	$\geq \lceil\frac{\mathbf{3\Delta}}{\mathbf{5}}\rceil$ (=)
6	1	2	2	3	$\geq \mathbf{3}$ (=)	4	5	5	6	...	$\lceil\frac{7\Delta}{12}\rceil$	$\geq \lceil\frac{\mathbf{7\Delta}}{\mathbf{12}}\rceil$ (=)
7	1	2	2	3	$\geq \mathbf{3}$ (=)	4	5 (4)	5	6	...	$\lceil\frac{4\Delta}{7}\rceil$	$\geq \lceil\frac{\mathbf{4\Delta}}{\mathbf{7}}\rceil$ (=)
8	1	2	2	3	$\geq \mathbf{3}$ (=)	4	$\geq \mathbf{4}$ (=)	5	6	...	$\lceil\frac{9\Delta}{16}\rceil$	$\geq \lceil\frac{\mathbf{9\Delta}}{\mathbf{16}}\rceil$ (=)
9	1	2	2	3	$\geq \mathbf{3}$ (=)	4	$\geq \mathbf{4}$ (=)	5	6 (5)	...	$\lceil\frac{5\Delta}{9}\rceil$	$\geq \lceil\frac{\mathbf{5\Delta}}{\mathbf{9}}\rceil$ (=)
...	...	...	...	...	...	...	...	...	...	...	...	...
C	1	2	2	3	$\geq \mathbf{3}$ (=)	4	$\geq \mathbf{4}$ (=)	5	$\geq \mathbf{5}$ (=)	...	$\lceil\frac{\mathbf{C+1}}{\mathbf{C}}\frac{\Delta}{\mathbf{2}}\rceil$	$\geq \lceil\frac{\mathbf{C+1}}{\mathbf{C}}\frac{\mathbf{\Delta}}{\mathbf{2}}\rceil$ (=)

Table 1. Known values of $M(C,\Delta)$. The **bold** cases with "$\geq$" remain open. The cases in brackets only hold if the graph has a perfect matching; "(=)" means that the corresponding lower bound is attained.

References

1. N. Alon, V. Teague, and N. C. Wormald. Linear Arboricity and Linear k-Arboricity of Regular Graphs. *Graphs and Combinatorics*, 17(1):11–16, 2001.
2. O. Amini, S. Pérennes, and I. Sau. Hardness and Approximation of Traffic Grooming. *Theoretical Computer Science*, 410(38-40):3751–3760, 2009.
3. J.-C. Bermond and D. Coudert. Traffic Grooming in Unidirectional WDM Ring Networks using Design Theory. In *Proceedings of IEEE International Conference on Communications (ICC)*, volume 2, pages 1402–1406, 2003.
4. J.-C. Bermond and D. Coudert. *The CRC Handbook of Combinatorial Designs (2nd edition)*, volume 42 of *Discrete Mathematics and its Applications*, chapter VI.27, Grooming, pages 493–496. CRC Press, C.J. Colbourn and J.H. Dinitz edition, 2006.
5. J.-C. Bermond, J.-L. Fouquet, M. Habib, and B. Péroche. On linear k-arboricity. *Discrete Mathematics*, 52(2-3):123–132, 1984.
6. L. S. Chandran. A high girth graph construction. *SIAM Journal on Discrete Mathematics*, 16(3):366–370, 2003.
7. C. J. Colbourn, A. C. H. Ling, G. Quattrocchi, and V. R. Syrotiuk. Grooming traffic to minimize load. To appear in *Discrete Mathematics*, `doi:10.1016/j.disc.2011.03.016`, 2011.
8. R. Dutta and N. Rouskas. Traffic grooming in WDM networks: Past and future. *IEEE Network*, 16(6):46–56, 2002.
9. P. Erdös and H. Sachs. Reguläre graphe gegebener taillenweite mit minimaler knotenzahl. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe*, 12:251–257, 1963.
10. M. Flammini, G. Monaco, L. Moscardelli, M. Shalom, and S. Zaks. Approximating the traffic grooming problem in tree and star networks. *Journal of Parallel and Distributed Computing*, 68(7):939–948, 2008.
11. M. Flammini, L. Moscardelli, M. Shalom, and S. Zaks. Approximating the traffic grooming problem. *Journal of Discrete Algorithms*, 6(3):472–479, 2008.
12. J.-L. Fouquet and J. Vanherpe. On normal partitions in cubic graphs. *Discussiones Mathematicae Graph Theory*, 29(2):293–312, 2009.
13. O. Goldschmidt, D. Hochbaum, A. Levin, and E. Olinick. The SONET edge-partition problem. *Networks*, 41(1):13–23, 2003.

14. E. Modiano and P. Lin. Traffic grooming in WDM networks. *IEEE Communications Magazine*, 39(7):124–129, 2001.
15. J. P. Petersen. Die Theorie der Regulären Graphs. *Acta Mathematica*, 15:193–220, 1891.
16. C. Thomassen. Two-coloring the edges of a cubic graph such that each monochromatic component is a path of length at most 5. *Journal of Combinatorial Theory, Series B*, 75(1):100–109, 1999.
17. K. Zhu and B. Mukherjee. A review of traffic grooming in WDM optical networks: Architectures and challenges. *Optical Networks Magazine*, 4(2):55–64, 2003.