Variational models in color science

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Summary

1. Part **one**: presentation of perceptually inspired color enhancement models;
2. Part **two**: introduction to variational principles in imaging, with the noticeable example of histogram equalization;
3. Part **three**: variational framework for perceptual models.

Part one: presentation of perceptually inspired color enhancement models

Scope of these models

- The human visual system (HVS) has some powerful properties that we will examine. Computational models of color perception are interesting for two main reasons (cross-fertilization):

    1. Being able to reproduce/imitate these features can help enhancing cameras hardware and software, image quality, algorithms for tracking in large camera networks, and so on;
    2. Modeling the computation performed by the HVS can guide or at least give some hints to neuroscientists, biologists and psychologists for their experiments.

Anticipation of a result:

Figure: *Left*: Original image. *Right*: Image filtered by a PICE algorithm.

A plethora of PICE models

- The job performed by the HVS is very complex;
- In literature, one can find *tens of models* that try to **directly implement** some of the HVS features in order to perform perceptual color correction;
- These models are, in general, very difficult to compare, so we have a plethora of algorithms that generates confusion.

Why a variational framework for perceptual models of color images?

- Job performed in collaboration with (in alphabetical order):
    - Marcelo Bertalmío;
    - Vicent Caselles;
    - Rodrigo Palma-Amestoy.

- **Motivation**: building a general 'house' for perceptual color enhancement models where existing algorithms can be analyzed in terms of important image features and compared.

Two (non mutually exclusive) choices: phenomenology or neuroscience?

- When we deal with HVS properties we can choose to study them through...
  1. Neurophysiological ('microscopical') properties;
  2. Phenomenological ('macroscopical') properties (the global result of neurodynamics);
- We chose to consider **phenomenological features**.

4 fundamental phenomenological features of the HVS

- There is a common agreement on the fact that the 4 most important phenomenological features of the HVS are:

    1. Adaptation to the average luminance level;
    2. Local contrast enhancement;
    3. Color constancy;
    4. Weber-Fechner's law.

Summary of the most important phenomenological features of color vision (1)

- **Adaptation to the average luminance level**: the radiance of a natural visual scene can span up to 10 orders of magnitude, but neurons can only deal with signals which span up to **2 orders of magnitude around the average luminance level of the scene**;

- The dynamic range shrinking is already performed before light hits the retina by:
  - Cornea;
  - Crystalline lens;
  - Aqueous and Vitreous Humor;
  - Macula.

Summary of the most important phenomenological features of color vision (1)

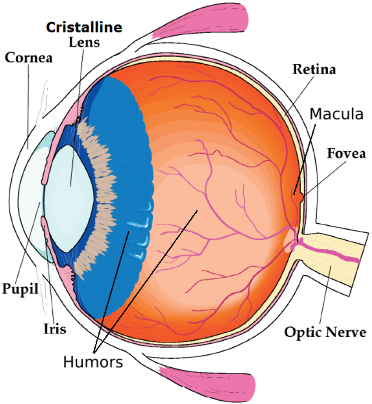


Figure: A simplified representation of the human eye.

Summary of the most important phenomenological features of color vision (1)

- When photons hit the retina, they activate **rods** and **cones** that **transduce** electromagnetic energy into electric current by changing their electric potential according to the 'Michaelis-Menten's formula'.

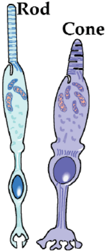


Figure: Prototypical shape of rods (left) and cones (right).

Summary of the most important phenomenological features of color vision (1)

- Michaelis-Menten's formula:

$$r(I) = \frac{\Delta V}{\Delta V_{\max}} = \frac{I^\gamma}{I^\gamma + I_S^\gamma} \ll I, \tag{1}$$

  where

  - $\Delta V_{\max}$: highest difference of potential that can be generated by the cell;
  - γ: constant (measured as 0.74 for the rhesus monkey);
  - I_S: luminous intensity at which the photoreceptor response is half maximal, *semisaturation level* or **adaptation level**.

Summary of the most important phenomenological features of color vision (2)

- **Local contrast enhancement**: to (partially) remedy the loss of information, we extracted local infos through **saccadic movements**;

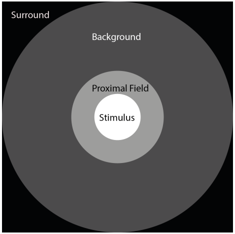


Figure: Locality of vision.

- **Vision is a local process**: the color of a certain point is determined not only by its absolute intensity, but also from the intensity distribution of the surrounding points (**induction**).

Summary of the most important phenomenological features of color vision (2)

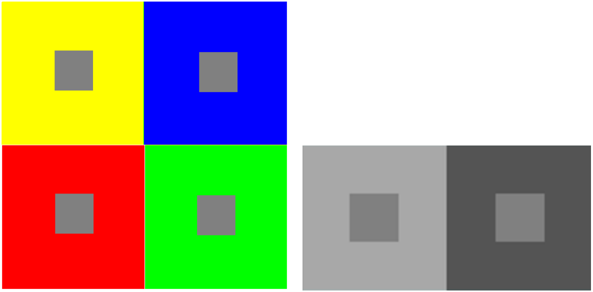


Figure: *Top*: Colored simultaneous contrast. *Bottom*: Grayscale simultaneous contrast. In both pictures, the inner gray squares have exactly the same physical luminance, however, their perceived luminance is very different.

Summary of the most important phenomenological features of color vision (2)

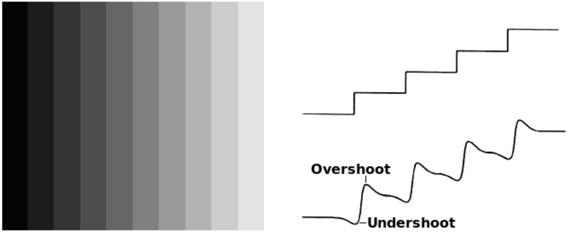


Figure: *Left*: Mach bands effect. *Right*: real and appearent luminance pattern.

Summary of the most important phenomenological features of color vision (2)

- Post-retinal cells and neurons are responsible for local contrast enhancement.

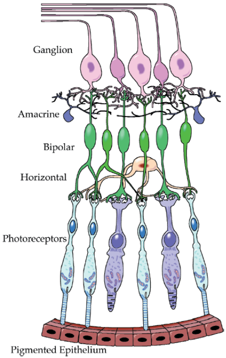


Figure: Composition of retinal layers.

Measuring (a)chromatic induction: Wallach's experiment

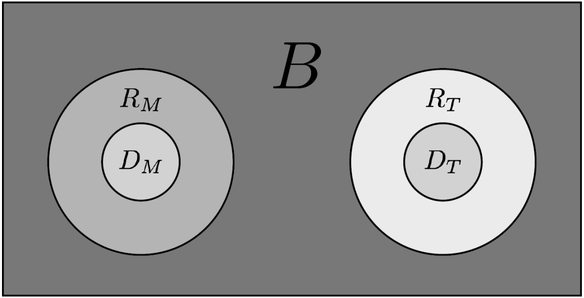


Figure: **Wallach**'s classical experiment (1948): R_T is changed from one value to another and the observer is asked to tune the luminance of D_M to have perceptual match with D_T.

Measuring (a)chromatic induction: Rudd-Zemach's model

- Rudd-Zemach's model (2005): perceptual match when

$$w_1 \log \frac{D_M}{R_M} + w_2 \log \frac{R_M}{B} = w_1 \log \frac{D_T}{R_T} + w_2 \log \frac{R_T}{B},$$

  w_1: **induction weight** between the barycenters of the inner circle and the ring
  w_2: **induction weight** between the barycenters of inner circle and the background

- Solving w.r.t. $\log D_M$ we have

$$\log D_M = \log D_T + \left(1 - \frac{w_2}{w_1}\right) \log R_M - \left(1 - \frac{w_2}{w_1}\right) \log R_T.$$

- Linear relationship in logarithmic scale, interpolating the slope from experiments they measured the induction weights: as expected: $w_2 < w_1$, i.e. **induction decreases with distance**.

Summary of the most important phenomenological features of color vision (3)

- **Color constancy**: our perception of colors is very stable with respect to change of illuminant conditions, **we use this feature all the time**;
- Color constancy works when we are **embedded in a visual scene** and **adapted to the illumination conditions**, it doesn't work when we're looking at a picture on a digital screen;
- So, we can't eliminate the color cast of a digital picture just by looking at it on a digital screen!

Summary of the most important phenomenological features of color vision (3)

Figure: *Left*: Image with color cast. *Right*: output of a perceptually-inspired color correction algorithm.

- This is one of the reasons why perceptually-inspired color correction algorithms are useful.

Summary of the most important phenomenological features of color vision (3)

- The **simplest image formation model** is:

$$I(x) = \lambda \cdot \rho(x) \qquad \text{(in each chromatic channel RGB)}$$

- λ: **illuminant**, $\rho(x)$: **reflectance** of the point x;
- Perfect color constancy implies that:

$$\lambda_1 \cdot \rho(x) \sim \lambda_2 \cdot \rho(x)$$

where $\sim$ means 'perceptually indistinguishable'.

Summary of the most important phenomenological features of color vision (3)

- Human color constancy is not perfect, and λ has a certain influence on us;
- In perfect color constancy models, the most important thing is to be able to compute λ to retreive $\rho(x)$, the 'intrinsic color' (not correct, there is no such thing, color depends on context!);
- Searching for λ starting from $I(x) = \lambda \cdot \rho(x)$ is an ill-posed problem that has no solution, so we must break the ambiguity by adding some hypothesis, e.g. white patch and gray world.

White patch hypothesis

- White Patch (WP) hypothesis: there is a 'white patch' WP in the visual scene with perfect reflectance, i.e. $\rho(x) = 1$, then

$$\forall x \in WP : \quad I(x) = \lambda = I_{\max}$$

- Thus, if the WP hypothesis holds true: $\rho(x) = \frac{I(x)}{I_{\max}}$ for all points x.

- The WP hypothesis is often violated, e.g. by homogeneous surfaces, plus λ may vary with x!

The original Retinex model of Land (1964)

- Edwin Land, the inventor of the polaroid mechanism, localized the WP hypothesis and searched for the patch with highest intensity on paths, giving birth to the (in)famous **Retinex model**.

Figure: Edwin Herbert Land.

Gray world hypothesis - Buchsbaum (1980)

- Gray World (GW) hypothesis: the average reflectance in a visual scene is gray (1/2 in normalized scale $0-1$);
- Using this hypothesis:

$$\bar{I} = \frac{1}{N}\sum_x I(x) = \frac{1}{N}\sum_x \lambda\rho(x) = \lambda\frac{1}{N}\sum_x \rho(x) = \lambda\cdot\frac{1}{2};$$

- Hence, $\lambda = 2\bar{I}$ and $\rho(x) = \frac{I(x)}{2\bar{I}}$ for each point of the scene;
- Again, the GW hypothesis is often violated, e.g. by homogeneous surfaces, plus λ may vary with x!

The ACE algorithm

- Developed at the university of Milan in 2003;
- It implements a local version of the GW hypothesis;
- In some cases the results of ACE and Retinex are comparable, in some others one prevails over the other with respect to ability to remove color cast, detail enhancement, etc.
- Their direct equations are very difficult to compare.

Retinex Vs. ACE

- Retinex equations:

$$\boxed{\tilde{I}^{\text{Retinex}}(x) = \frac{1}{N}\sum_{k=1}^{N}\prod_{t_k=1}^{n_k-1}\delta_k(R_{t_k})} \qquad \forall x \in \Omega,$$

where $\delta_k : \mathbb{R}^+ \to \mathbb{R}^+$, $k = 1, \ldots, N$, are defined in this way: $\delta_k(R_0) \equiv 1$ and, for $t_k = 1, \ldots, n_k - 1$,

$$\delta_k(R_{t_k}) = \begin{cases} R_{t_k} & \text{if } 0 < R_{t_k} \le 1 - \varepsilon \\ 1 & \text{if } 1 - \varepsilon < R_{t_k} < 1 + \varepsilon \\ R_{t_k} & \text{if } 1 + \varepsilon \le R_{t_k} \le \frac{1+\varepsilon}{\prod_{m_k=0}^{t_k-1}\delta_k(R_{m_k})} \\ \frac{1}{\prod_{m_k=0}^{t_k-1}\delta_k(R_{m_k})} & \text{if } R_{t_k} > \frac{1+\varepsilon}{\prod_{m_k=0}^{t_k-1}\delta_k(R_{m_k})}. \end{cases}$$

- ACE equations

$$\boxed{\tilde{I}^{\text{ACE}}(x) = \frac{1}{2} + \frac{R_c^{\alpha,w}(x)}{2M_c^{\alpha,w}}} \qquad \forall x \in \Omega,$$

with

$$R^{\alpha,w}(x) = \int_\Omega w(x,y)\,\mathrm{s}_\alpha(I\,(x) - I\,(y))\,dy,$$

where the weighting function $w : \Omega \times \Omega \to (0, +\infty)$ is a normalized, i.e. $\int_\Omega w(x,y)\,dy = 1$, decreasing function of $\|x - y\|$, and $\mathrm{s}_\alpha : [-1,1] \to [-1,1]$ is the function

$$\mathrm{s}_\alpha(t) = \begin{cases} -1 & \text{if } -1 \le t \le -\frac{1}{\alpha} \\ \alpha t & \text{if } -\frac{1}{\alpha} < t < \frac{1}{\alpha} \\ +1 & \text{if } \frac{1}{\alpha} \le t \le 1 \end{cases}$$

$\alpha > 1$, the slope of s_α, is a parameter of the model, and $M_c^{\alpha,w} = \max_{x\in\Omega}\{R_c^{\alpha,w}(x)\}$.

Summary of the most important phenomenological features of color vision (4)

- **Weber-Fechner's law of contrast perception**: a dark-adapted human observer is put in a dim room in front of a white screen on which a narrow beam of light is thrown in the center of the visual field;

- The luminous intensity I of the beam is increased very slowly and the observer is asked to tell whether he/she could perceive an intensity change called **JND** for *Just Noticeable Difference*, ΔL.

- Approximately it holds that: $\boxed{\frac{\Delta I}{I} = \text{const.}}$, which implies greater sensitivity to variations in dim light conditions.

Resume of the first part

- The HVS has important features:
  - Adaptation to the average luminance level
  - local *spatial contrast* enhancement
  - color constancy
  - Weber-Fechner's law for *intensity contrast* perception.

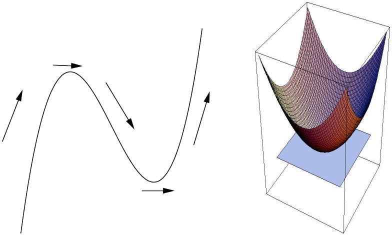
- Models that try to implement directly (some of) these properties are really *difficult to compare* and to *analyze* in terms of image features as contrast of tone dispersion;

- We will see in the third part that, embedding these models in a variational framework, we can better understand their property and compare them easily;

- But first...we must introduce the variational calculus...after a well deserved coffee break...

Part two: introduction to variational principles in image processing and computer vision

- Remember *Fermat's optimization principle of ordinary Calculus*: *Maxima and minima* of smooth functions $\vec{x} \mapsto f(\vec{x})$ can be located only at points where **gradient is 0**;



- The directional derivatives $D_v f(\vec{x})$ along every direction v in the extremals $\vec{x}$ must be 0.

The image function

- The **image function**:

$$\vec{I}: \begin{array}{ccc} \Omega & \longrightarrow & [0,1] \times [0,1] \times [0,1] \\ x & \mapsto & (I_R(x), I_G(x), I_B(x)) \end{array}$$

  where:
  - Ω: spatial support of the image;
  - $x = (x_1, x_2)$: pixel position in Ω;
  - $I_c(x)$: normalized intensity of the pixel in the Red, Green and Blue channel, respectively;
- For simplicity we will avoid the subscript c and write simply $I(x)$.

- Variational methods in imaging don't deal *directly* with image functions, but with **functionals** of image functions;

- A functional E maps a function to a real number;
- $\mathcal{F}$ space of image functions, then

$$\begin{array}{rccc} E: & \mathcal{F} & \longrightarrow & \mathbb{R} \\ & I & \mapsto & E(I) \end{array}$$

- Fermat's condition $f'(\vec{x}) = 0$ is substituted by $\delta E(I) = 0$, where δ is called **first variation** or *Gateaux derivative* (generalization of directional derivative).

Directional derivative Vs. First variation

- Directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$:

$$D_v f(x_0) = \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h} \equiv \frac{d}{dh}\Big|_{h=0} f(x + hv);$$

$$\boxed{D_v f(x_0) = 0, \ \forall v \in \mathbb{R}^n}, \quad \textbf{Stationarity}.$$

- First variation of a functional $E : \mathcal{F} \to \mathbb{R}$:

$$\delta E(I, J) = \lim_{h \to 0} \frac{E(I + hJ) - E(I)}{h} \equiv \frac{d}{dh}\Big|_{h=0} E(I + hJ).$$

$$\boxed{\delta E(I, J) = 0, \ \forall J \in \mathcal{F}}, \quad \textbf{‘Euler-Lagrange equations’}.$$

Min and Argmin of a functional

- Let $I^* \in \mathcal{F}$ such that:

$$E(I^*) = \min_{I \in \Omega} E(I)$$

- Then, we write:

$$I^* = \text{argmin } E(I)$$

- In a *convex* context the Euler-Lagrange equations determine the minima, in general they are just necessary conditions for minima.

- Resuming:

Image Function $I : \Omega \to [0,1]$	Energy Functional $E : \mathcal{F} \to \mathbb{R}$
Domain: $x \in \Omega$	Domain: $I \in \mathcal{F}$
Range: $I(x) \in [0,1]$	Range: $E(I) \in \mathbb{R}$
Stationarity: $I'(x) = 0$	Stationarity: $\delta E(I) = 0$

- **Riesz representation theorem**: under some suitable hypothesis every continuous linear functional can always be represented as a suitable **integral** (or finite **sum**).

Variational models in digital image processing

- A variational model in imaging consists in the **selection of a suitable functional** whose extremization induces some **desired effects** on an image
- Examples:
  - Mumford-Shah functional for image **segmentation**;
  - Total variation functional for **denoising**;
  - Quadratic functional for **regulator mechanisms**;
  - and so on. . .

Example from imaging: variational denoising

- J: image corrupted with additive noise: $J = I + n$;
- I: uncorrupted (unknown) image, n: noise image;
- Noise generates high gradients, so we want to minimize them in order to denoise the image;

- Quadratic denoising functional:

$$E(I) = \frac{1}{2}\int_{\Omega} \left(|\nabla J(x)|^2 + \lambda |I(x) - J(x)|^2\right)\, dx;$$

- Total variation:

$$E(I) = \int_{\Omega} \left(|\nabla J(x)| + \frac{\lambda}{2}|I(x) - J(x)|^2\right)\, dx.$$

Example from physics: motion of Newtonian particles in a conservative field

- Problem: trajectory $\vec{q}(t)$ of a particle, in a conservative field, in time $[t_0, t_1]$

- Lagrange functional:

$$L(\vec{q}(t)) = \int_{t_0}^{t_1} [T(\vec{q}(t)) - V(\vec{q}(t))]\, dt$$

  - $T(\vec{q}(t)) = \frac{1}{2} m \|\dot{\vec{q}}(t)\|^2$: particle's **kinetic energy**
  - $V(\vec{q}(t))$: particle's **potential energy**, it depends on particle's position and on the forces $\vec{F}$ acting on it

- Argmin of $L(\vec{q}(t))$: $\vec{q}(t)$ satisfying the differential equation

$$\vec{F}(t) = m\ddot{\vec{q}}(t), \; \forall t \in [t_0, t_1] \qquad \textbf{Newton's law!}$$

- **Interpretation**: the motion of a Newtonian particle minimizes the difference between its actual (kinetic) and potential energy between t_0 and t_1!

Classical histogram equalization

- We will show that a particularly important variational functional is that related to histogram equalization;
- For a 'continuous' digital image, if $\lambda \in [0, 1]$ is a generic intensity level, then the normalized **histogram** of I computed in λ is:

$$h(\lambda) = \frac{1}{|\Omega|} \mathrm{Area}\{x \in \Omega \mid I(x) = \lambda\} \qquad \lambda \in [0, 1],$$

- It's the **occurrence probability** of the level λ in the image, that is, how many times the level λ appears in the image.

Classical histogram equalization

Figure: Histograms of a low contrast (first column) high contrast (second column) image.

Classical histogram equalization

- The normalized **cumulative histogram** of I computed in λ, $H(\lambda)$, is:

$$H(\lambda) = \frac{1}{|\Omega|}\,\mathrm{Area}\{y \in \Omega \,|\, I(x) \leq \lambda\} \qquad \lambda \in [0,1],$$

the **probability to find a pixel with intensity** $\leq \lambda$;

- Of course, the relationship between h and H is:

$$\boxed{H(\lambda) = \int_0^\lambda h(t)\,dt}, \quad \boxed{H'(\lambda) = h(\lambda)},$$

i.e. H **is the integral function of** h in the interval $[0,1]$ and the first derivative of H in each level gives the histogram of that level.

Classical histogram equalization

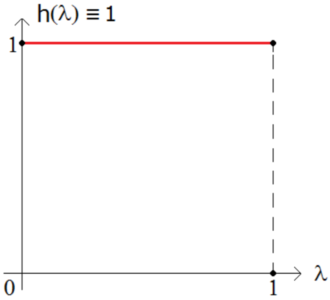
- An image is said to be **equalized** if it has the *same occurrence probability for all levels*, i.e. if $h(\lambda) \equiv k$, the value of k can be determined by integrating h:

$$1 \underset{\text{h is normalized!}}{=} \int_0^1 h(\lambda)\, d\lambda = \int_0^1 k\, ds = k \int_0^1 ds = k,$$

  which implies $k = 1$ (1 must not be interpreted as a probability of 100%, but $\frac{100\%}{|\Omega|}$ since the histogram is normalized).

- The equalization condition on the histogram $h(\lambda) \equiv 1$ can be traduced to the condition $H(\lambda) = \int_0^\lambda d\lambda = \lambda$ on the cumulative histogram.

Classical histogram equalization



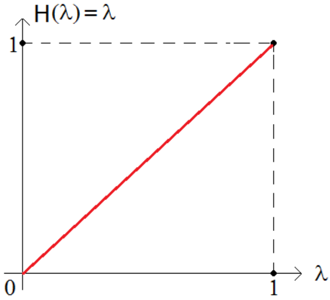


Figure: Histogram and cumulative histogram of an equalized image.

Classical histogram equalization

- Many times, an image does not have a balanced number of intensities over the range $[0,1]$: some values appear many times (where the histogram has a **peak**), others less frequently and some level may never appear;
- If an image has equalized histogram, then all the levels appear with the same frequency of occurrence;
- Such an image carries a larger amount of (quantitative, not necessarily qualitative) information.

Classical histogram equalization

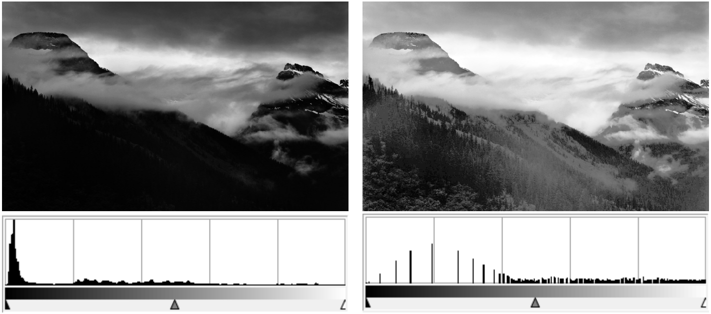


Figure: A famous picture of Ansel Adams before (left) and after (right) histogram equalization. Of course in the digital domain a perfect equalization is almost never impossible to achieve, so that approximations must be considered.

Classical histogram equalization

- Classical **histogram equalization** amounts to the transformation

$$\boxed{\lambda \underset{\varphi}{\longmapsto} \mu = H(\lambda) : \text{Histogram equalization}};$$

- Thus, classical histogram equalization is implemented by changing each level λ of the original image into the value μ of the normalized cumulative histogram H in λ;
- Histogram equalization can produce nice results, but it can also destroy images. Typically in low-key images, for which the cumulative histogram of dark levels is already close to 1, so that contrast of bright levels can even be decreased by histogram equalization!

Classical histogram equalization

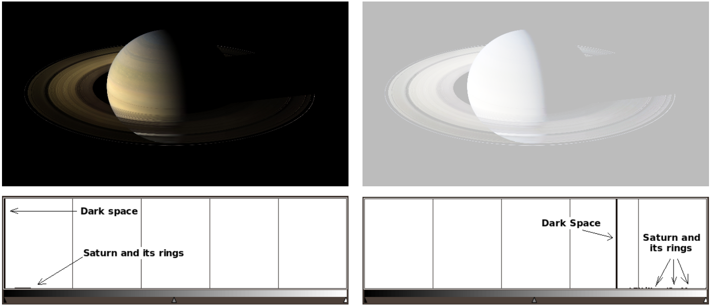


Figure: Effect of histogram equalization (right) on a low key image (left). Notice that the histogram of the 'equalized' image on the right starts exactly at the level defined by the normalized cumulated histogram in the level 0 of the original image.

Classical histogram equalization

- What about histogram equalization in color images?
- In general, equalization in the three independent chromatic channels can be dangerous, since unnatural color can be generated by the unrelated stretching of the three histograms;
- The equalization of only the luminance channel in general avoids this problem but it can have a minor impact on the image.

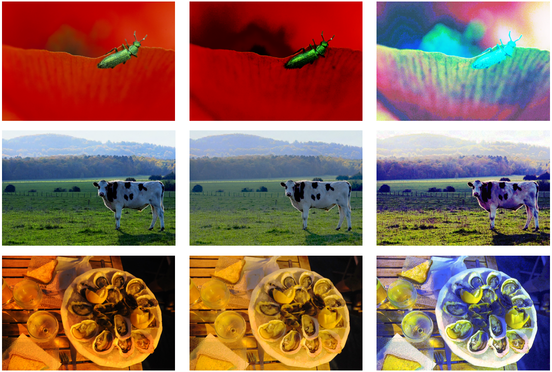


Figure: *Left*: original color image. *Center*: histogram equalization of the luminance. *Right*: histogram equalization of the three independent chromatic channels.

From classical to variational histogram equalization

- We have seen that a ‘nice’ histogram equalization of color images is *not a trivial task* to perform;
- We will see that the HVS automatically performs an equalization of light information, so we could take advantages of the HVS properties to implement a more sound histogram equalization.
- To understand that, we must embed histogram equalization in a variational framework and better understand its action on image features.

Variational histogram equalization

- Result of V. **Caselles** and G. **Sapiro** obtained in the paper '*Histogram modification via differential equations*', J. Diff. Eq., vol. 135, no. 2, pp. 238266, 1997.

- They proved that the a stationary image for this functional

$$E_{\text{hist eq}}(I) \equiv 2\int_{\Omega}\left(I(x)-\frac{1}{2}\right)^2 dx - \frac{1}{|\Omega|}\iint_{\Omega^2}|I(x)-I(y)|\, dxdy.$$

is an image I^* with equalized histogram: $H(I^*(x)) = I^*(x)$ $\forall x \in \Omega$.

- Plus, a *gradient descent algorithm* with $I_0 =$ initial image, converges to a unique (equalized) image, argmin of $E_{\text{hist eq}}(I)$.

Meaning of $E_{\text{hist eq}}(I)$

$$E_{\text{hist eq}}(I) \equiv 2\int_{\Omega}\left(I(x)-\frac{1}{2}\right)^2 dx \;-\; \frac{1}{|\Omega|}\iint_{\Omega^2}|I(x)-I(y)|\,dxdy;$$

- **Two opposing terms**:
  - The **quadratic dispersion** (L^2 distance) around the middle gray: $D(I) \equiv 2\int_{\Omega}\left(I(x)-\frac{1}{2}\right)^2 dx$
  - A term that gives a measure of **global Michelson contrast**: $C(I) \equiv -\frac{1}{|\Omega|}\iint_{\Omega^2}|I(x)-I(y)|\,dxdy$
- Min $E(I)$: **optimal balance between global contrast maximization and minimization of quadratic dispersion to middle gray**, a highly non-trivial result!

Interpretation of variational histogram equalization in terms of perceptual features

- The Caselles-Sapiro's result is **profound** and it can be related to the perceptual features previously discussed;
- In fact, also in human vision we have a balance between contrast enhancement and control of dispersion around an average luminance value;
- However there are some *fundamental differences*: human contrast enhancement is *local* and *color constancy* and *Weber-Fechner's law* must be taken into account.

Break

Coffee break?

Part three: variational PICE algorithms and further research

Variational models of perceptually-inspired contrast enhancement of color images

- Basic idea: define a suitable ‘energy’ functional E of **digital image functions** such that its minimum is reached in correspondence of an **optimal image**:

The closest image to what we would perceive if we were embedded in the scene where the picture was taken and adapted to the illumination conditions.

Axioms for a perceptually inspired energy functional

Axioms 1: General structure of a perceptually inspired energy. The general structure of a perceptually inspired color correction energy functional is

$$E_w(I) = D(I) + C_w(I),$$

where

$$D(I) = \int_\Omega d(I(x))\, dx,$$

being $d : \mathbb{R} \to \mathbb{R}$ a differentiable function, is called *dispersion term* and

$$C_w(I) = \iint_{\Omega^2} w(x,y)\, c(I(x), I(y))\, dxdy,$$

being $w : \mathbb{R}^2 \to \mathbb{R}^+$ an induction weight function and $c : \mathbb{R}^2 \to \mathbb{R}$ a differentiable function, is called *contrast term*.

Axioms for a perceptually inspired energy functional

- The minimization of D must provide a **control of the dispersion around** μ, the average image intensity, (visual adaptation) **and around the original intensity values** (imperfection of human color constancy);

- The minimization of C_w must provide a local contrast enhancement, locality being induced by the weight function w.

Selection of the contrast term

- Recall that $I(x) = \lambda \cdot \rho(x)$, λ: illuminant, $\rho(x)$: reflectance
- c homogeneous of degree $n \in \mathbb{N}$ if:

$$c(\lambda\rho(x), \lambda\rho(y)) = \lambda^n \, c(\rho(x), \rho(y)) \qquad \forall \lambda \in \mathbb{R}^+$$

- If $n = 0$, i.e. c is **homogeneous function of degree 0**, then c intrinsically disregards the illuminant λ
- λ, in general, is not globally homogeneous, but locally yes...

Selection of the contrast term

- Spatial contrast between two pixels doesn't depend on their order, the function c must be **symmetric** in the exchange of its arguments;
- Finally, to avoid the inversion of contrast, c must be **monotonically increasing**;
- *Axiom 2. The contrast function c is a* **monotonically increasing, homogeneous function of degree 0, symmetric** *in the exchange of its arguments.*

The second axioms in the Weber-Fechner domain

- A natural function that accomplishes axioms 2 is any monotone function of

$$\frac{\min(I(x), I(y))}{\max(I(x), I(y))}.$$

- Remarkably, if Weber-Fechner's law is taken into account, the basic contrast variable is univocally determined by this function!

  Axiom 2'. c is a monotone function of the basic contrast variable $\frac{\min(I(x),I(y))}{\max(I(x),I(y))}$, $x, y \in \Omega$.

Modification of Caselles-Sapiro's functional to take into account perceptual features

- So, in R.Palma, E.Provenzi, M. Bertalmío and V. Caselles: '*A perceptually inspired variational framework for color enhancement*', IEEE PAMI, 31 (3), 458-474, March 2009, we considered this class of contrast functionals

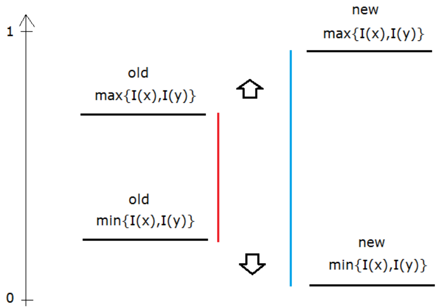
$$C_w^{\varphi}(I) = \iint_{\Omega^2} w(x,y)\,\varphi\left(\frac{\min\{I(x),I(y)\}}{\max\{I(x),I(y)\}}\right)\, dxdy$$

- w is a spatially decreasing symmetric kernel, i.e. a Gaussian;
- φ, a functional parameter of the model, it can be any smooth increasing function.

The contrast term of a perceptual functional

$$C_w^{\varphi}(I) = \iint_{\Omega^2} w(x,y)\,\varphi\left(\frac{\min\{I(x),I(y)\}}{\max\{I(x),I(y)\}}\right)\,dxdy$$

- Minimizing $c^{\varphi}(I(x),I(y)) \equiv \frac{\min\{I(x),I(y)\}}{\max\{I(x),I(y)\}}$ means decreasing the minimum and increasing the maximum, i.e. intensifying contrast;



$$C_w^{\mathrm{id}}(I) := \frac{1}{4} \iint_{\Omega^2} w(x,y)\, \frac{\min(I(x), I(y))}{\max(I(x), I(y))}\, dxdy,$$

$$\begin{aligned}\delta C_w^{\mathrm{id}}(I) = &-\frac{1}{2} \int_\Omega w(x,y) \frac{I(y)}{I(x)^2}\, \mathrm{sign}^+(I(x) - I(y))\, dy \\ &+ \frac{1}{2} \int_\Omega w(x,y) \frac{1}{I(y)}\, \mathrm{sign}^+(I(y) - I(x))\, dy;\end{aligned}$$

$$\mathrm{sign}^+(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

$\varphi \equiv \log$

$$C_w^{\log}(I) := \frac{1}{4} \iint_{\Omega^2} w(x,y) \log\left(\frac{\min(I(x), I(y))}{\max(I(x), I(y))}\right) dxdy,$$

$$\delta C_w^{\log}(I) = -\frac{1}{2} \int_{\Omega} w(x,y) \frac{1}{I(x)} \operatorname{sign}(I(x) - I(y))\, dy;$$

Variational models of perceptually-inspired contrast enhancement of color images

- $\delta C_w^{\varphi}(I)$ has **dimension -1** in terms of I, so we must search for a dispersion term whose variation has a coherent dimension;
- Good candidate: **entropy**, i.e

$$D^{\mathcal{E}}_{\alpha,\beta}(I) \equiv \alpha \int_{\Omega} \left[\mu \log \frac{\mu}{I(x)} - (\mu - I(x)) \right] dx + \beta \int_{\Omega} \left[I_0(x) \log \frac{I_0(x)}{I(x)} - (I_0(x) - I(x)) \right] dx;$$

- Statistical interpretation of entropy: **measure of disorder**. So minimizing the entropy around I_0 and the average intensity μ means constraining the intensity values of the image around them.

Variational models of perceptually-inspired contrast enhancement of color images

- By minimizing the energy $E^{\varphi}_{w,\alpha,\beta}(I) = C^{\varphi}_{w}(I) + D^{\mathcal{E}}_{\alpha,\beta}(I)$ we **balance two opposite effects**, as in our visual system:
  - Reduction of the dynamic range towards the average luminance (performed mainly by the eyes), and attachment to the original image (imperfection of human color constancy);
  - Local contrast enhancement (performed by neuron activity inhibition/excitation);
- A gradient descent scheme to find the minimum of $E^{\varphi}_{w,\alpha,\beta}(I)$ provides a **computational algorithm** that corrects color in digital images in a perceptually-sound way.

A discrete gradient descent scheme

- A convenient method is given by the *discrete gradient descent* with respect to $\log I$. The continuous equation is

$$\partial_t \log I = -\delta E^{\varphi}_{w,\alpha,\beta}(I)$$

- t is an evolution parameter, $\log I$ simply changes the speed of the gradient descent and consists in using the relative entropy as a metric, instead of the usual quadratic distance. Useful because of the logarithmic derivative

$$\partial_t \log I = \frac{1}{I}\partial_t I,$$

  which allows restoring the important property of homogeneity of degree 0 simply multiplying by I both sides of the Euler-Lagrange equations.

- Computational cost $\mathcal{O}(N^2)$, reduced to $\mathcal{O}(N \log N)$ with a suitable approximation.

Minimum of $E^{\log}_{w,\alpha,\beta}(I)$

- The computational equation for $\varphi = \log$:

$$I^{k+1}(x) = \frac{I^k(x) + \Delta t \left(\frac{\alpha}{2} + \beta I_0(x) + \frac{1}{2} R^{\log}_{I^k}(x) \right)}{1 + \Delta t(\alpha + \beta)},$$

  where

$$R^{\log}_{I^k}(x) := \int_\Omega w(x,y)\,\mathrm{sign}(I^k(x) - I^k(y))dy.$$

- If we set $\alpha = 1$, $\beta = 0$ and we smooth the signum to a sigmoid **we obtain the equations of ACE**! M. Bertalmío, V. Caselles, E. Provenzi, A. Rizzi '*Perceptual Color Correction Through Variational Techniques*', IEEE TIP, 2007.

Minimum of $E^{\mathrm{id}}_{w,\alpha,\beta}(I)$

- The computational equation for $\varphi = $id:

$$I^{k+1}(x) = \frac{I^k(x) + \Delta t\left(\frac{\alpha}{2} + \beta I_0(x) + \frac{1}{2} R^{\mathrm{id}}_{I^k}(x)\right)}{1 + \Delta t(\alpha + \beta)},$$

$$\begin{aligned} R^{\mathrm{id}}_{I^k}(x) := & \int_\Omega w(x,y) \frac{I^k(y)}{I^k(x)} \,\mathrm{sign}^+(I^k(x) - I^k(y))\, dy \\ & - \int_\Omega w(x,y) \frac{I^k(x)}{I^k(y)} \,\mathrm{sign}^+(I^k(y) - I^k(x))\, dy, \end{aligned}$$

- In the paper: M. Bertalmío, V. Caselles, E. Provenzi '*Issues About Retinex Theory and Contrast Enhancement*', IJCV, 2009, it has been proven that this corresponds to a **symmetric continuous version of Retinex**.

Results

Original image of a tapestry in the Amboise castle with color cast

Filtered image with the perceptually-inspired variational method

Results

Image after histogram equalization

Results

Original image of Lena

Results

Image of Lena filtered with our algorithm

Results

Image after histogram equalization

Histogram comparison

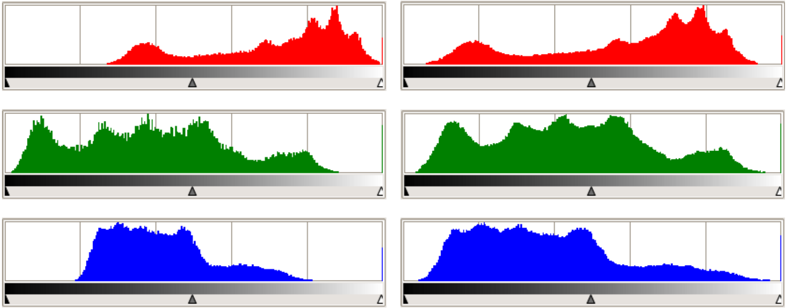
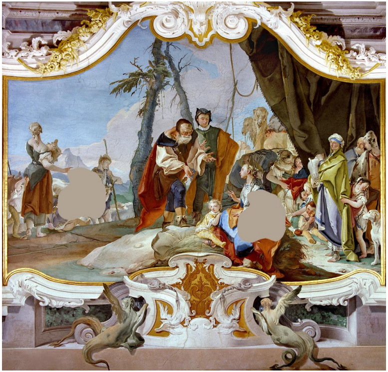


Figure: *Left* and *Right*: histograms of the RGB channels of the *original* and *filtered* Lena image, respectively (far from being equalized)

Applications to art: filling lacunae

- We can decrease the perception of lacunae in frescos by *minimizing* the perceptual contrast (with Luca Grementieri)



A beautiful fresco of **Tiepolo** with two artificial lacunae, painted with different shades of gray.

Conclusions

1. The powerful '**view from above**' provided by variational principles allowed building **a general framework for perceptual models**, where they **can be compared** and their **action on important image features** as local contrast and dispersion can be clearly understood;

2. Within the variational framework, we can **embed already existing models** and generate new ones, simply by changing the analytical shape of the functional parameter φ;

3. Thanks to the fundamental result of Caselles-Sapiro, we can also **compare perceptual models with histogram equalization**, understand and overcome its limitations;

Thanks

Thanks a lot for your attention!