Independence Concepts in Imprecise Probability
Exercises

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Exercise

Consider a variable $X$ with 3 possible values $x_1$, $x_2$ and $x_3$. Suppose the following assessments are given:

\[ p(x_1) \leq p(x_2) \leq p(x_3); \]

\[ p(x_i) \geq 1/20 \quad \text{for } i \in \{1, 2, 3\}; \]

\[ p(x_3|x_2 \cup x_3) \leq 3/4. \]

Show the credal set determined by these assessments in baricentric coordinates.
Exercise

A closed convex credal set is completely characterized by the associated lower expectation.

But given a lower expectation, many credal sets generate it.

Usually only the maximal closed convex set is chosen.

Exercise: Given the assessments in the previous exercise, find two credal sets that yield the same lower expectation.
Exercise

Credal set \( \{P_1, P_2\} \):

\[
P_1(s_1) = 1/8, \quad P_1(s_2) = 3/4, \quad P_1(s_3) = 1/8, \\
P_2(s_1) = 3/4, \quad P_2(s_2) = 1/8, \quad P_2(s_3) = 1/8,
\]

Acts \( \{a_1, a_2, a_3\} \):

<table>
<thead>
<tr>
<th></th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>(s_3)</th>
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<tbody>
<tr>
<td>(a_1)</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>(a_2)</td>
<td>2.5</td>
<td>3.5</td>
<td>5</td>
</tr>
<tr>
<td>(a_3)</td>
<td>1</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

Which one to select?
And if we take convex hull of credal set?
Exercise

- Urn with $m > 0$ balls, numbered from 1 to $m$
- $r$ balls are red and $m - r$ balls are black.
- $n$ samples with replacement.
- $\omega$ is a numbered sequence produced this way.
- $m^n$ possible numbered sequences.
- Assume uniformity: $P(\omega) \geq (1 - \epsilon)m^{-n}$.
- What is the lower probability that $k$ balls are red?
Exercise

What is the largest credal set that satisfies exchangeability of two binary variables?
Exercise

- Suppose we have 4 binary variables that are exchangeable.

- What are the conditions on the probabilities $P(X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4)$?
Exercise

Suppose we have 4 binary variables that are exchangeable.

Suppose $P(0000) = 1/10$ and $P(1111) = 1/2$.

Draw the credal set.
Exercise

Draw the credal set $K(X,Y)$ given the structural assessments:

- $X$ and $Y$ are exchangeable.
- $X$ and $Y$ are the first two variables in a sequence of three exchangeable variables.
- $X$ and $Y$ are the first two variables in a sequence of five exchangeable variables.
- $X$ and $Y$ are the first two variables in a sequence of infinitely many exchangeable variables.
Exercise

Prove decomposition, weak union and contraction for stochastic independence.
Exercise

Consider a finite possibility space.

Suppose $K(Y)$ is a singleton.

Suppose $P(X)$, $K(X|Y \in B)$ are “almost” vacuous in that $P(X \in A|\cdot) > 0$ is the only constraint.

Show that $Y$ is epistemically irrelevant to $X$, but $X$ is not epistemically irrelevant to $Y$.

This is an extreme case of *dilation*!

Construct an example that is not so extreme but that still fails symmetry.
Exercise

Prove:

- Kuznetsov independence implies epistemic independence.
- Epistemic independence does not imply Kuznetsov independence.
Consider

- Two binary variables $X$ and $Y$.
- $P(X = 0) \in [2/5, 1/2]$ and $P(Y = 0) \in [2/5, 1/2]$.
- Epistemic independence of $X$ and $Y$: $K(X, Y)$ is convex hull of

  \[
  [1/4, 1/4, 1/4, 1/4], [4/25, 6/25, 6/25, 9/25],
  [1/5, 1/5, 3/10, 3/10], [1/5, 3/10, 1/5, 3/10],
  [2/9, 2/9, 2/9, 1/3], [2/11, 3/11, 3/11, 3/11],
  \]

Write down the linear constraints that must be satisfied by $K(X, Y)$. 

Due to de Campos and Moral (1995).

- $X$ and $Y$ are binary.
- $K(X, Y)$ is the convex hull of two distributions $P_1$ and $P_2$ such that $P_1(X = 0, Y = 0) = P_2(X = 1, Y = 1) = 1$.

Show:

- $X$ and $Y$ are strongly independent.
- Neither $Y$ is type-5 irrelevant to $X$, nor $X$ is type-5 irrelevant to $Y$. 
Exercise

Show that strict and strong independence satisfy all graphoid properties.
Exercise

Show:

- Epistemic independence satisfies decomposition and weak union in finite spaces.
- Epistemic irrelevance satisfies: if $Y$ is epistemically irrelevant to $X$ and $W$ is epistemically irrelevant to $X$ given $Y$ then $(W, Y)$ are epistemically irrelevant to $X$.
- Kuznetsov independence satisfies decomposition.