Algorithms for Imprecise Probability
Part I

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Overview

- Part I: algorithms without independence (this talk).
- Part II: algorithms with independence (next talk, by Cassio).

Overview (some more)

- Part I: algorithms without independence (this talk).
  1. The basic linear fractional program.
  2. Dealing with probabilities that may be zero.
  3. Special important cases: neighborhoods, capacities, and the like.
  4. Decision making.
- Part II: algorithms with independence (next talk, by Cassio).

Easy warm-up

- Possibility space Ω with states ω; events are subsets of Ω.
- Random variables and indicator functions.
  - Bounded function $X : \Omega \to \Re$.
  - Special type: indicator function of event A:
    - Denoted by A as well.
    - $A(\omega) = 1$ if $\omega \in A$; 0 otherwise.

Axioms for expectations

EU1 If $\alpha \leq X \leq \beta$, then $\alpha \leq E[X] \leq \beta$.

EU2 $E[X + Y] = E[X] + E[Y]$.

Some consequences:

1. $X \geq Y \Rightarrow E[X] \geq E[Y]$.
2. $E[\alpha X] = \alpha X$.

Probabilities

- The *probability* $P(A)$ is $E[A]$.

- Properties of a probability measure:

  PU1 $P(A) \geq 0$.

  PU2 $P(\Omega) = 1$.

  PU3 If $A \cap B = \emptyset$, $P(A \cup B) = P(A) + P(B)$.

Conditional expectations/probabilities

- Conditional expectation of X given B,

$$E[X|B] = \frac{E[BX]}{P(B)} \quad \text{if } P(B) > 0.$$

- Bayes rule: If $P(B) > 0$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Algorithms: Boole (1854)

- Propositional formula ϕ:
  1. propositions
  2. operators ($\neg$, $\wedge$, $\vee$, $\rightarrow$).
- Take Ω as the set of 2^n truth assignments for n propositions.
- Interpret $P(\phi) \geq \alpha$ as

$$\sum_{\omega \models \phi} P(\omega) \geq \alpha.$$

Probabilistic satisfiability

- Given m assessments, is there a probability measure over Ω?
  - Each assessments is a linear constraint.
  - Must satisfy $P(\omega) \geq 0$ and $\sum_{\omega \in \Omega} P(\omega) = 1$.
- This is a *linear program*!
  - Derived first by Hailperin (1965).
- Somewhat surprisingly, NP-complete problem.
  - The same as usual satisfiability (!?!).
- Note: solution is at extreme points.

Exercise

Build linear program:

- $P(A) \geq \alpha$.
- $B \rightarrow C$.
- $P(B) = \beta$.

Can you give bounds for $P(A \wedge B \wedge C)$?

Solution

- $P(A) \geq \alpha$, $B \to C$, $P(B) = \beta$.

Define:

ω_i	A	B	C
1	0	0	0
2	0	0	1
3	0	1	0
4	0	1	1
5	1	0	0
6	1	0	1
7	1	1	0
8	1	1	1

Then ω_3 and ω_7 are impossible; and

$$p_5 + p_6 + p_8 = \alpha, \quad p_4 + p_8 = \beta, \quad p_i \geq 0, \quad \sum_i p_i = 1.$$

de Finetti's fundamental theorem

- Given m assessments over events H_i, is there a probability measure over them?
- And how about the allowed assessments over another event H_0?
- Theorem: $P(H_0)$ belongs to an interval with constraints given by other assessments
  - (and the usual $P(\omega) \geq 0$ and $\sum_{\omega \in \Omega} P(\omega) = 1$).
- This is a linear program.
  - Well, this is the same linear program as before (Gilio (1980)).

Exercise

Coletti and Scozzafava (1999).

- Take H_1, H_2, H_3.
- Assume $H_3 \subset H_1^c \cap H_2$.
- Assessments $P(H_1) = 1/2$, $P(H_2) = 1/5$, $P(H_3) = 1/8$.

Build linear program.

Exercise

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- Take H_1, H_2, H_3.
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Build linear program.

- $x_1 = P(A_1)$; $A_1 = H_1 \cap H_2 \cap H_3^c$.
- $x_2 = P(A_2)$; $A_2 = H_1 \cap H_2^c \cap H_3^c$.
- $x_3 = P(A_3)$; $A_3 = H_1^c \cap H_2 \cap H_3^c$.
- $x_4 = P(A_4)$; $A_4 = H_1^c \cap H_2 \cap H_3$.
- $x_5 = P(A_5)$; $A_5 = H_1^c \cap H_2^c \cap H_3^c$.

Exercise

Coletti and Scozzafava (1999).

- Take H_1, H_2, H_3.
- Assume $H_3 \subset H_1^c \cap H_2$.
- Assessments $P(H_1) = 1/2$, $P(H_2) = 1/5$, $P(H_3) = 1/8$.

Build linear program.

$$
\begin{array}{ccccccccccc}
x_1 & + & x_2 & & & & & & & = & 1/2 \\
x_1 & & & + & x_3 & + & x_4 & & & = & 1/5 \\
& & & & & & x_4 & & & = & 1/8 \\
x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 1 \\
x_1 & \geq 0, & x_2 & \geq 0, & x_3 & \geq 0, & x_4 & \geq 0, & x_5 & \geq 0. &
\end{array}
$$

Conditional probabilities

- Assessment $P(A|B) \geq \alpha$.
- Transform to (Hailperin (1965) and many others later):

$$P(A \wedge B) \geq \alpha P(B) \, .$$

- Or use the language of events.
- Still a linear program!

Exercise

Coletti and Scozzafava (1999).

- Take H_1, H_2, H_3.
- Assume $H_3 \subset H_1^c \cap H_2$.
- Assessments $P(H_1) = 1/2$, $P(H_2) = 1/5$, $P(H_3) = 1/8$.
- Also, $P(H_2 | H_1 \cup H_2) \geq 1/2$.

Build linear program.

Column generation

- Probabilistic satisfiability is

$$\min 0p$$

$$\text{subject to} \quad Ap \geq \alpha, p \geq 1.$$

- General problem minimizes cp.
- The difficulty is that p has 2^n elements (for a problem with n propositions).
- The usual technique is *column generation*.
  - That is, generate only those columns of A that are necessary
    - (at any given time, simplex only needs m columns where m is number of lines of A).

The mechanics of column generation

- Use the revised simplex algorithm.
  - That is, keep only a basis ($m \times m$).
  - Must decide whether to bring a column into the basis.
- Then choose the column using a nonlinear subproblem:
  - Solve $\min_j c_B A_B^{-1} A_j$.
  - Note that A_j contains a set of logical formulas.
  - This is a MAXSAT problem.
  - Replace:

$$X \wedge Y \doteq XY, \quad X \vee Y \doteq X + Y - XY, \quad \neg X \doteq 1 - X.$$

  - It can be reduced to *linear (integer) programming*!

Integer programming

- *Very* useful fact:
  - Consider product $a \times b$, where
    - $a \in [0, 1]$.
    - b is either 0 or 1.
  - Create a new variable c, replace $a \times b$ by c and add

$$0 \leq c \leq b;$$

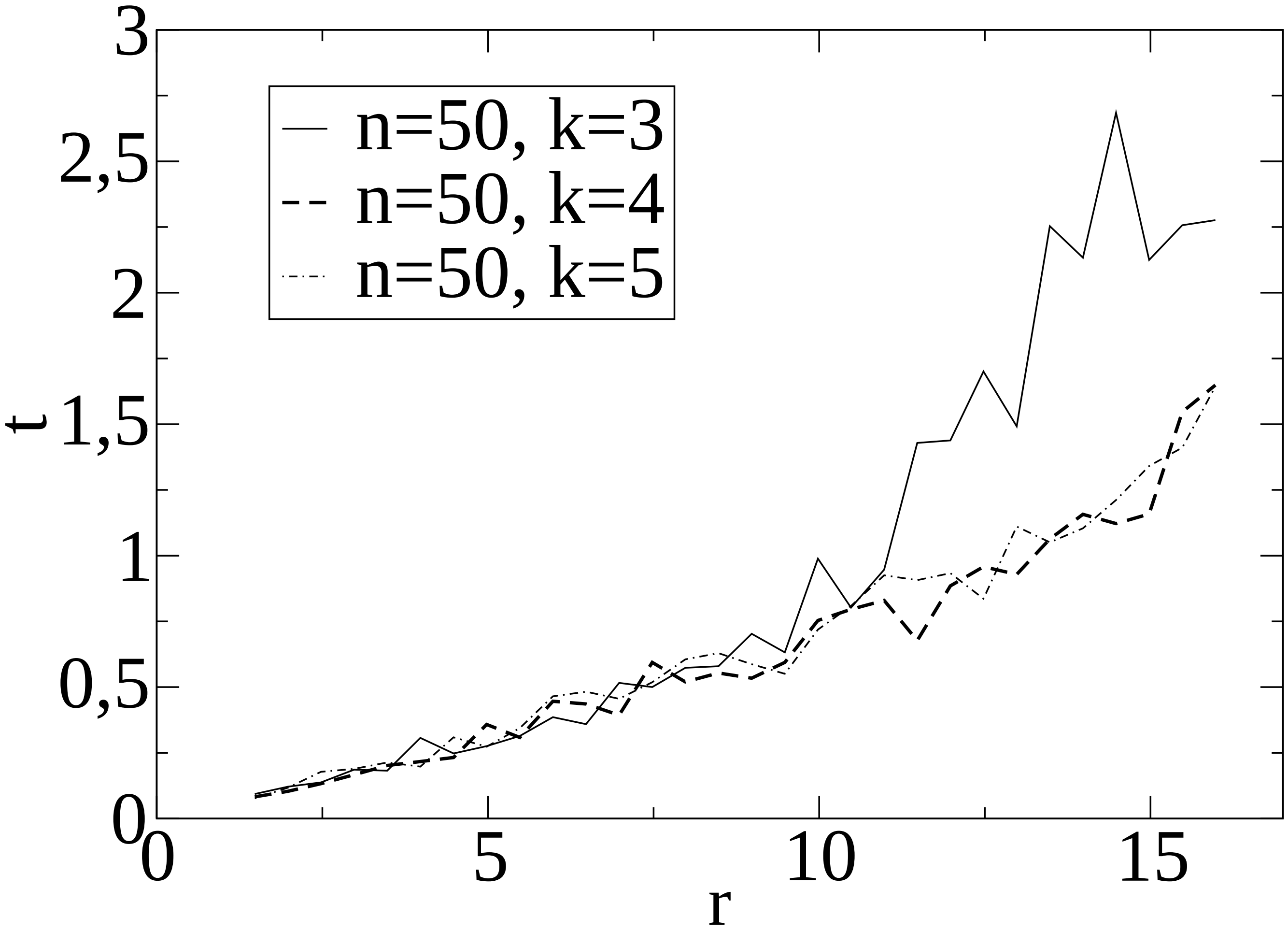
$$a - 1 + b \leq c \leq a.$$

- Now solve by linear (integer) programming!

PSAT with column generation

- Best results in the literature: hundreds of propositions, hundreds of assessments (Perron et al 2004), using lots of special tricks.
- There are also a few special cases that are “easy” and several variants, etc.
  - For instance, when formulas can be put in a “tree” structure (Andersen & Pretolani 1999).
  - Also if formulas can be organized in junction trees (van der Gaag 1991).
- (Also, approximation methods based on local search for large problems, but really no guarantees yet...)

Phase transitions?



Aside: PPL system

- Interface in Python, connects to CPLEX or free linear programming tools (at http://www.pmr.poli.usp.br/ltd/Software/PPL/index.html).

```
>>> s1 = 'a <=> (b?c)'
>>> s1
'a <=> (b?c)'
>>> s2 = PPL.toCNF(s1)
>>> s2
'((?b j a) & (?c j a) & (b j c j ?a))'
>>> PPL.p(s1, 0.5)
>>> s3 = 'd j (e & f) j g'
>>> PPL.p(s3, 0.3, 0.8)
>>> PPL.checkCoherence()
Coherent!
```

- Another package by Dickey (see SIPTA Newsletter).

Computing conditional probabilities

- Now suppose we wish $\underline{P}(A|B) = \min P(A|B)$.
- This is not a linear program (it is a linear fractional program).
- However, it can be solved through linear programming:
  - Charnes-Cooper transformation (similar solutions by White, Snow).
  - Dinkelbach-Jagannathan algorithm (similar solutions by Walley, Lavine).

Charnes-Cooper transformation

- Wish to solve:

$$\min_p \frac{\sum_i f_i \alpha_i p_i}{\sum_i \alpha_i p_i} \quad \text{s.t. } Ap \geq 0, \sum_i p_i = 1, p_i \geq 0.$$

  where $\sum_i \alpha_i p_i > 0$.

- Change variables to

$$q_i = \frac{p_i}{\sum_i \alpha_i p_i}.$$

- Now:

$$\min_q \sum_i f_i \alpha_i q_i \quad \text{s.t. } Aq \geq 0, \sum_i \alpha_i q_i = 1, q_i \geq 0.$$

Exercise

- Take H_1, H_2, H_3.
- Assume $H_3 \subset H_1^c \cap H_2$.
- Assessments $P(H_1) = 1/2$, $P(H_2) = 1/5$, $P(H_3) = 1/8$.

Build linear program to compute $\underline{P}(H_1|H_1 \cup H_2)$, applying the Charnes-Cooper transformation.

Solution

- Take H_1, H_2, H_3, assume $H_3 \subset H_1^c \cap H_2$.
- Assessments $P(H_1) = 1/2$, $P(H_2) = 1/5$, $P(H_3) = 1/8$.

Build linear program to compute $\underline{P}(H_1 | H_1 \cup H_2)$.

First,

$$\min(x_1 + x_2)/(x_1 + x_2 + x_3 + x_4) \quad \text{s.t.}$$

$$x_1 + x_2 = 1/2; \quad x_1 + x_3 + x_4 = 1/5; \quad x_4 = 1/8; \quad x_i \geq 0; \quad \sum_i x_i = 1.$$

Then

$$\min(x_1 + x_2)/(x_1 + x_2 + x_3 + x_4) \quad \text{s.t.}$$

$$x_1/2 + x_2/2 - x_3/2 - x_4/2 - x_5/2 = 0; \quad 4x_1/5 - x_2/5 + 4x_3/5 + 4x_4/5 - x_5/5 = 0;$$

$$-x_1/8 - x_2/8 - x_3/8 + 7x_4/8 - x_5/8 = 0; \quad x_i \geq 0; \quad \sum_i x_i = 1.$$

Solution

- Take H_1, H_2, H_3, assume $H_3 \subset H_1^c \cap H_2$.
- Assessments $P(H_1) = 1/2$, $P(H_2) = 1/5$, $P(H_3) = 1/8$.

Build linear program to compute $\underline{P}(H_1|H_1 \cup H_2)$.

First,

$$\min(x_1 + x_2)/(x_1 + x_2 + x_3 + x_4) \quad \text{s.t.}$$

$$x_1 + x_2 = 1/2; \quad x_1 + x_3 + x_4 = 1/5; \quad x_4 = 1/8; \quad x_i \geq 0; \quad \sum_i x_i = 1.$$

Then

$$\min(y_1 + y_2) \quad \text{s.t.}$$

$$y_1/2 + y_2/2 - y_3/2 - y_4/2 - y_5/2 = 0; \quad 4y_1/5 - y_2/5 + 4y_3/5 + 4y_4/5 - y_5/5 = 0;$$

$$-y_1/8 - y_2/8 - y_3/8 + 7y_4/8 - y_5/8 = 0; \quad y_i \geq 0; \quad \sum_{i=1}^{4} y_i = 1.$$

Larger example (based on Jaeger 1994)

- Take:
  AntarticBird $\rightarrow$Bird,
  FlyingBird $\rightarrow$Bird,
  Penguim $\rightarrow$Bird,
  FlyingBird $\rightarrow$Flies,
  Penguim $\rightarrow \neg$Flies,
  P(FlyingBird|Bird) $= 0.95$,
  P(AntarticBird|Bird) $= 0.01$,
  P(Bird) ≥ 0.2,
  P(FlyingBird $\vee$ Penguim|AntarticBird) ≥ 0.2,
  P(Flies|Bird) ≥ 0.8.
- Then
  P(FlyingBird|Bird $\wedge \neg$AntarticBird) $\in [0.949, 0.960]$,
  P(Penguim|$\neg$AntarticBird) $\in [0.000, 0.050]$.

Dinkelbach-Jagannathan for probability

- Note:
$$\lambda = \min \frac{P(A \cap B)}{P(B)},$$
iff
$$\min \left(P(A \cap B) - \lambda P(B)\right) = 0,$$
assuming $P(B) > 0$.
- The left side is strictly decreasing function of λ.
- So, we can bracket λ.

Dinkelbach-Jagannathan for expectation

- Also,

$$\lambda = \min \frac{E[f(X)B]}{P(B)},$$

iff

$$\min\left(E[f(X)B] - \lambda P(B)\right) = 0$$

or, rather,

$$\min E[(f(X) - \lambda)B] = 0;$$

that is,

$$\underline{E}[(f(X) - \lambda)B] = 0.$$

- This is Walley's Generalized Bayes Rule (GBR).
  - Walley proposed iteration:
    $\mu_{i+1} = \mu_i + 2\underline{E}[(f(X) - \mu_i)B] / (\overline{P}(B) + \underline{P}(B))$.

Lavine's algorithm

- In 1991, Lavine published a paper on robust statistics with the same algorithm, apparently unaware of the literature.
- Lavine's algorithm became quite popular.
- Until Lavine's algorithm, calculation of posterior lower expectations in robust statistics usually relied on very special arguments.
  - Often, minimax theory.

Now, imprecise likelihoods

- Suppose we have $K(X)$ ("prior") and $K(Y|X=x)$ for each x ("likelihood").
- Suppose $K(Y|X=x)$ is *separately specified* (important condition!).
- If $\underline{P}(Y=y)>0$, $\underline{E}[f(X)|Y=y]$ is the unique solution of the equation

$$\underline{E}[(f(X)-\lambda)p_\lambda(y|X)]=0,$$

where

$$p_\lambda(y|X)=\begin{cases}\underline{E}[y|x] & \text{if} \quad f(x)\geq\lambda\\ \overline{E}[y|x] & \text{if} \quad f(x)<\lambda\end{cases}$$

Dealing with imprecise likelihoods

$$\underline{E}[f(X)|Y=y] = \min_{p',p''} \left[\frac{\sum_i \left(f_i L_y(x_i) p'_i + f_i U_y(x_i) p''_i\right)}{\sum_j \left(L_y(x_j) p'_j + U_y(x_j) p''_j\right)} \right],$$

subject to:

$$A(p' + p'') \leq 0,$$

$$\sum_i (p'_i + p''_i) = 1, \qquad p'_i \geq 0, p''_i \geq 0.$$

Example (based on White 1986)

- Variable with 4 values $\{\theta_1, \theta_2, \theta_3, \theta_4\}$,

$$2.5p(\theta_1) \geq p(\theta_4) \geq 2p(\theta_1)\,,$$

$$10p(\theta_3) \geq p(\theta_2) \geq 9p(\theta_3)\,, \quad p(\theta_2) = 5p(\theta_4)\,.$$

- Also, bounds on likelihood:

$$\begin{aligned} L(x|\theta_1) &= 0.9, & L(x|\theta_2) &= 0.1125, \\ L(x|\theta_3) &= 0.05625, & L(x|\theta_4) &= 0.1125, \\ U(x|\theta_1) &= 0.95, & U(x|\theta_2) &= 0.1357, \\ U(x|\theta_3) &= 0.1357, & U(x|\theta_4) &= 0.1357. \end{aligned}$$

Example: solution

$\underline{P}(\theta_1|x) = \min_{p',p''}(0.9p_1' + 0.95p_1'')$,
$p' \geq 0$, $p'' \geq 0$,

$$\begin{bmatrix} -\frac{5}{2} & 0 & 0 & 1 \\ 2 & 0 & 0 & -1 \\ 0 & -1 & 0 & 5 \\ 0 & 1 & 0 & -5 \\ 0 & -1 & 9 & 0 \\ 0 & 1 & -10 & 0 \end{bmatrix} [p' + p''] \leq 0,$$

$F_1\alpha' + F_2\alpha'' = 1$, where
$F_1 = [0.9, 0.1125, 0.0562, 0.1125], F_2 = [0.95, 0.1357, 0.1357, 0.1357]$.
By linear programming: $\underline{P}(\theta_1|x) = 0.2881$.

Independence relations

1. We may easily face some “inferential vacuity”: A and B have no logical relation, $P(A) = 1/2$, $P(B) = 1/2$; then $P(A \wedge B) \in [0, 1/2]$.
2. Introduce independence to reduce inferential vacuity...
   - A and B independent, $P(A) = 1/2$, $P(B) = 1/2$; then $P(A \wedge B) = 1/4$.
3. Independence leads to
   - *nonlinear* constraints.
   - open problems concerning complexity.
4. Idea: organize independence relations using graphs.
   - This will take us to credal networks and the like; this is for **other** talks.

Credal sets

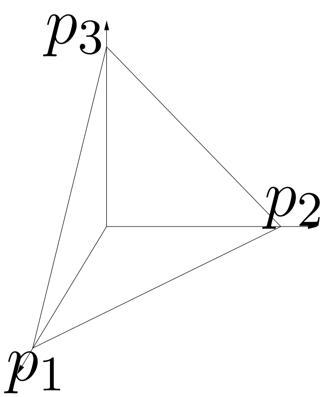
- So far, Boolean and categorical variables, with linear programming.
- Some general terminology and understanding helps.
- A credal set is a set of probability measures (distributions).
- A credal set is usually defined by a set of *assessments*.

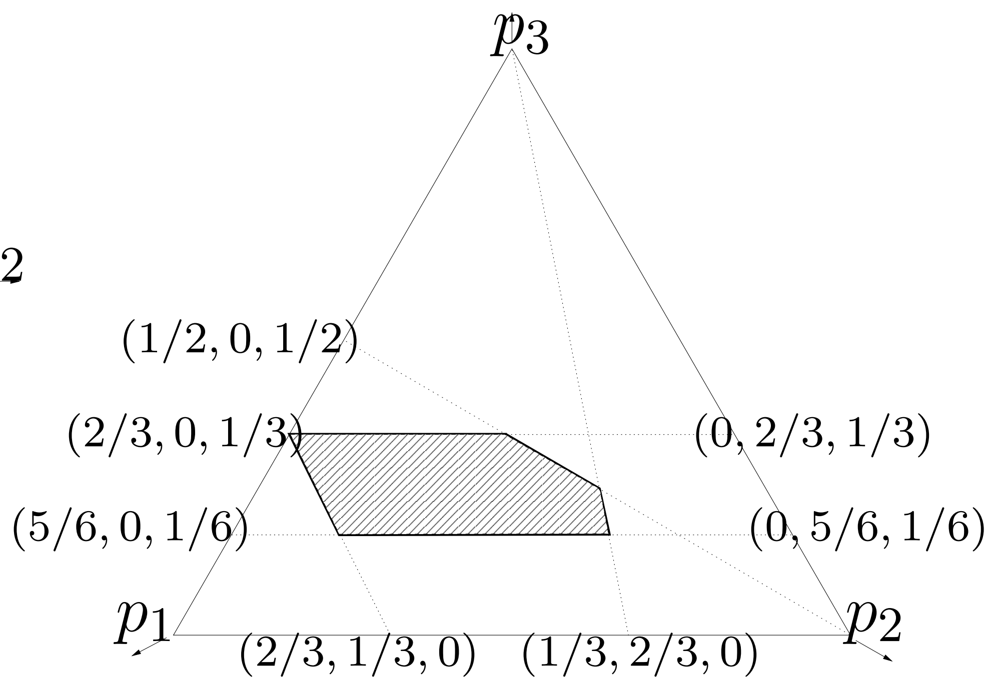
Example:

1. $\Omega = \{\omega_1, \omega_2, \omega_3\}$.
2. $P(\omega_i) = p_i$.
3. $p_1 > p_3$, $2p_1 \geq p_2$, $p_1 \leq 2/3$ and $p_3 \in [1/6, 1/3]$.
4. Take points $P = (p_1, p_2, p_3)$.

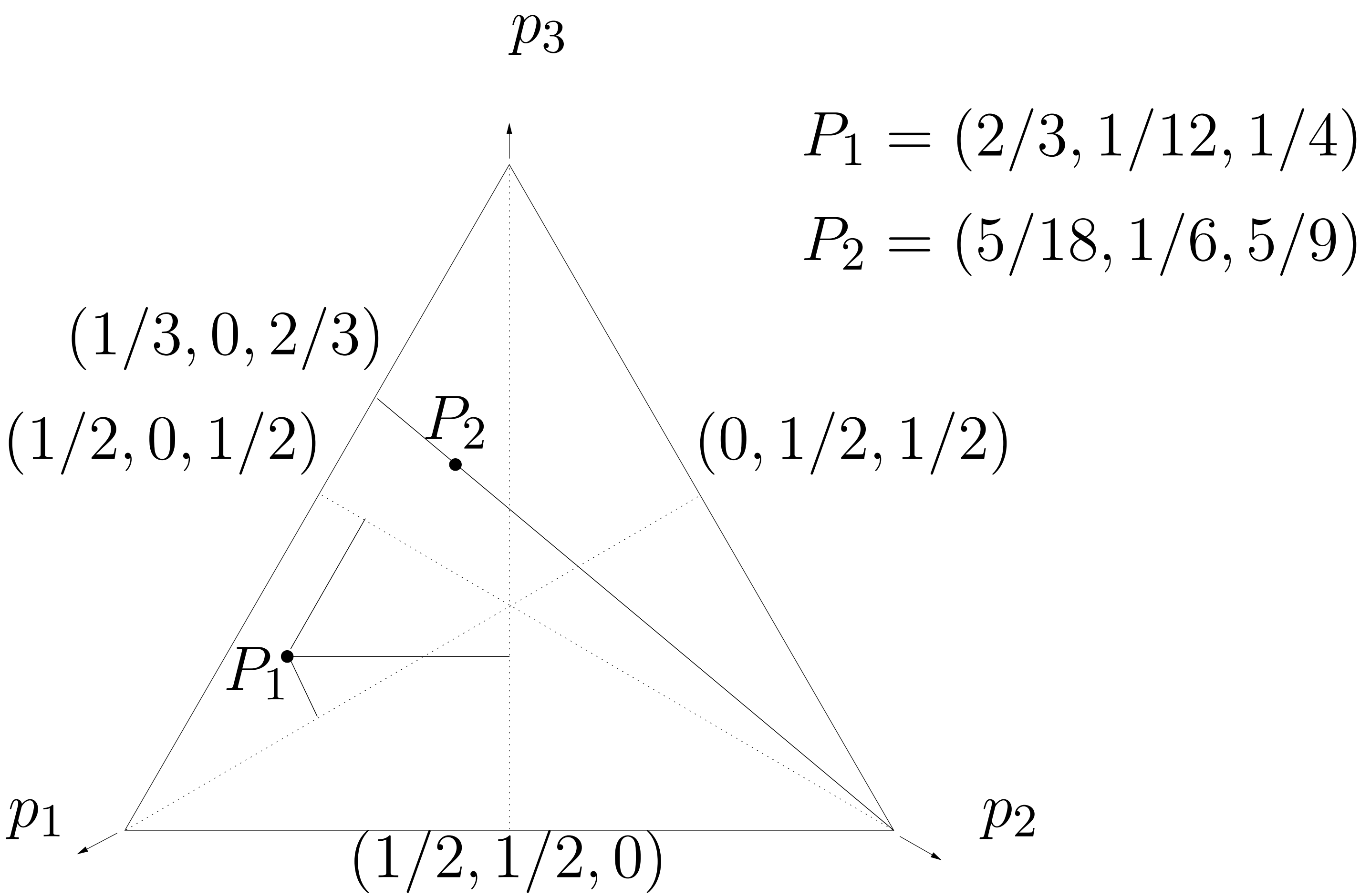
Some geometry

1. $\Omega = \{\omega_1, \omega_2, \omega_3\}$.
2. $P(\omega_i) = p_i$.
3. $p_1 > p_3$, $2p_1 \geq p_2$, $p_1 \leq 2/3$ and $p_3 \in [1/6, 1/3]$.
4. Take points $P = (p_1, p_2, p_3)$.





Baricentric coordinates



The coordinates of a distribution are read on the lines bissecting the angles of the triangle.

Exercise

Consider a variable X with 3 possible values x_1, x_2 and x_3. Suppose the following assessments are given:

$$p(x_1) \leq p(x_2) \leq p(x_3)\,;$$

$$p(x_i) \geq 1/20 \quad \text{for } i \in \{1,2,3\};$$

$$p(x_3|x_2 \cup x_3) \leq 3/4.$$

Show the credal set determined by these assessments in baricentric coordinates.

The basics of credal sets

- Credal set with distributions for X is denoted $K(X)$.
- Given credal set $K(X)$:
  - $\underline{E}[X] = \inf_{P \in K(X)} E_P[X]$.
  - $\overline{E}[X] = \sup_{P \in K(X)} E_P[X]$.
- For closed convex credal sets, lower and upper expectations are attained at vertices.
- A closed convex credal set is completely characterized by the associated lower expectation.
  - That is, there is only one lower expectation for a given closed convex credal set.
- The set of conditional distributions from a convex credal set is convex.

Exercise

Suppose the following judgements are stated:

	ω_1	ω_2	ω_3	
X_1	-1	0	1	Desirable
X_2	0	2	-1	Desirable

Here "desirable" means $E[X] \geq 0$.
Draw the credal set defined by such assessments.
What can be said about the desirability of

	ω_1	ω_2	ω_3
X_3	1	-1	-1
X_4	-2	4	1

Back to algorithms

- There are details on conditional probabilities that must be analyzed.
- Before, a little more on probabilistic logic: moving to first order.

First-order probabilistic logic

- Now we have constants, relations, functions, quantifiers:
  man(Socrates) $\vee$ mortal(Socrates)
  $\forall x$: man(x) $\rightarrow$ mortal(x).
- There are few general techniques here: too many variations.
- Nilsson (1986) advocated: $P(\phi) \geq \alpha$ where ϕ is sentence.
  - This can be solved by linear programming... but there are *decidability* questions.
  - Recent study by Jaumard et al (2007) for decidable fragments.

Example (Jaumard et al 2007)

Assessments:

- $P(\forall x : \exists y : t(x,y) \wedge s(y)) = 0.9$.
- $P(\exists x : \neg r(x)) = 0.6$.
- $P(\exists y : \neg s(y)) = 0.6$.
- $P(\forall x : \forall y : \neg t(x,y) \wedge r(x) \wedge s(y)) = 0.7$.

Compute $P(\exists x : \exists y : \neg t(x,y))$.

- Only 12 possible worlds (elements in the Lindenbaum algebra).
- Possible to apply linear program; extension to column generation method is open problem.

Other proposals

- A different proposal is to impose probabilities over the domain:
  - "Probability that a randomly selected bird flies is no smaller than 0.9."
- There has been great interest in this kind of probabilistic logic for
  - probabilistic logic programming;
  - the *semantic web*;
  - probabilistic databases.
- Most algorithms are for languages that can be translated to Bayesian networks.
- Few general algorithms (good starting point is the work of Thomas Lukasiewicz).

Zero probabilities

- This is one of the most embarassing challenges in the world of credal sets.
- In the standard theory of probabilities, it is easy to ignore null events (events with probability zero).
  - Such events “will never happen”.

Zero probabilities

- This is one of the most embarassing challenges in the world of credal sets.
- In the standard theory of probabilities, it is easy to ignore null events (events with probability zero).
  - Such events “will never happen”.
- But now there may be events with zero lower probability and nonzero upper probability.
  - For instance, if $P(B) \leq \alpha$, then $P(B)$ may be zero.
- So, we may observe A and we need to say something about $P(A|B)$.
- This issue has drawn steady interest in the community, but it is not easy to understand.

Zeroes in linear fractional programs

- Note: the linear fractional programs we discussed before required $P(B) > 0$.
  - If $\overline{P}(B) = 0$, then programs become unfeasible.
- They compute:

$$\min E_P[f(X)|B]$$

where P belongs to

$$\{P : P(B) > 0\}.$$

Full conditional measures

- The most elegant solution is to consider *full probability measures*.
- A full probability measure is a function $P(\cdot|\cdot)$ on $\mathcal{E} \times \mathcal{E} \backslash \emptyset$ where $\mathcal{E}$ is an algebra of events, such that
  - $P(A|C) = 1$;
  - $P(A|C) \geq 0$ for all A;
  - $P(A \cup B|C) = P(A|C) + P(B|C)$ when $A \cap B = \emptyset$;
  - $P(A \cap B|C) = P(A|B \cap C)\, P(B|C)$ when $B \cap C \neq \emptyset$.
- Full probability measures allow $P(A|C)$ to be defined even if $P(C) = 0$!

The Krauss-Dubins representation

- We can partition Ω into events $L_0, \ldots, L_K$, $K \leq N$,
- such that the full conditional measure is represented as a sequence of strictly positive probability measures $P_0, \ldots, P_K$, where the support of P_i is restricted to L_i.
- $P(A|B) = P(A|B \cap L_B)$, where L_B is the "layer" where B has nonzero probability.
- This representation has been advocated by Coletti & Scozzafava.

Example (note: $P(A) = 0$, but $P(B|A) = \beta$):

	A	A^c
B	$\lfloor \beta \rfloor_1$	α
B^c	$\lfloor 1 - \beta \rfloor_1$	$1 - \alpha$

Exercise

Consider assessments:

- $P(A) \geq 1/2$.
- $P(A^c \cap B^c) = 1/2$.
- $P(C|A^c \cap B) = 1/3$.

What is the Krauss-Dubins representation?
What is $P(C|B)$?
What is $P(C^c|A^c \cap B)$?

Coletti-Scozzafava's method

- Run the usual linear program with assessments $P(A_i|B_i) \geq \alpha_i$.
- If all B_i have $P(B_i) > 0$ for all feasible solutions, stop (solution has been found).
- Otherwise:
  - Collect those B_i with $P(B_i) = 0$ for all feasible solutions.
  - Then build another linear program *only* with those assessments with these B_i.
  - Repeat until there are no more assessments (inference is vacuous).

Improving the algorithm

- Coletti-Scozzafava's method has been optimized and expanded by Vantaggi, Capotorti and others.
  - Idea is to quickly detect/exploit zero probabilities.
  - Check coherence (CkC) package: http://www.dipmat.unipg.it/˜upkd/paid/software.html
  - Vantaggi has dealt with independence as well.

- Overall, many tests to make, to detect whether events may are null.

Other approaches

- Sequence of $2m$ direct linear programs in the worst case (Walley, Pelessoni, Vicig (1999, 2004)).
  - But still, necessary to run additional linear programs to check whether to proceed.
  - Possible to divide number of linear programs by 2, by examining slack variables (Cozman 2002).
- All of this is to check “coherence” in a strong sense.
  - There are weaker concepts of “coherence”.

Changing gears: Classes of credal sets

- General assessments are flexible (too flexible?) but are hard to handle for general spaces.
- Possible strategy is to focus on a few canonical ways to define credal sets.
- There are many!
  - Neighborhoods.
  - Capacities.
  - Boxes.
- A great deal of this work is found in the literature on robust statistics.
  - Usually, some mix of linear fractional programming (Dinkelbach-Jagannathan algorithm), minimax theory, and creativity with particular problems.

The classic ϵ-contaminated

- Credal set based on P_0 and $\epsilon \in (0, 1)$:

$$\{(1-\epsilon)P_0 + \epsilon Q : \text{ any } Q\}.$$

- Old model, originally from robust frequentist statistics (Tukey, then Huber).

Exercise

- If $K(X)$ is an ϵ-contaminated class, what are

$$\underline{E}[f(X)]\,, \quad \overline{E}[f(X)]?$$

  - If P_0 is always nonzero, what is

$$\underline{P}(A|B)\,, \quad \overline{P}(A|B)?$$

- If one gives a measure L such that

$$L(\Omega) < 1,$$

  is this an ϵ-contaminated class?
  If so, what are P_0 and ϵ?

Solution

- If $K(X)$ is an ϵ-contaminated class,

$$\underline{E}[f(X)] = (1-\epsilon)E_0[f(X)], \quad \overline{E}[f(X)] = (1-\epsilon)E_0[f(X)]+\epsilon.$$

- If one gives a measure L such that

$$L(\Omega) < 1,$$

this an ϵ-contaminated class

$$\{(1-\epsilon)(L/L(\Omega)) + \epsilon Q\},$$

where $\epsilon = 1 - L(\Omega)$.

Other neighborhoods

- Total variation class:

$$\{P : |P(A) - R(A)| \leq \epsilon\}.$$

(Exercise: Find lower/upper probabilities for event A.)

- Neighborhoods for several metrics; with several contaminations (given moments, given quantiles, given modes); from conjugate families (well-known example is Imprecise Dirichlet Model).
- Bose (1994): several contaminations at once,

$$\{(1 - \epsilon)P + \epsilon_1 q_1 + \cdots + \epsilon_n q_n : q_i \in K_i\}.$$

Density bounded classes

- Given two measures L and U such that

$$L \leq U, \quad L(\Omega) \leq 1, \quad U(\Omega) \geq 1,$$

take the set

$$\{P : L \leq P \leq U\}.$$

- Lower/upper probabilities are easy to compute. For instance,

$$\underline{P}(A) = \max\left(L(A), 1 - U(A^c)\right).$$

- *Constant* bounded class if $kL = P_0 = U/k$ for some P_0, $k > 1$.

Density ratio classes

- Given two measures L and U such that $L(A) \leq U(A)$ for every event A,

$$\{P = \mu/\mu(\Omega) : L \leq \mu \leq U\}.$$

  - That is, you "draw" μ between L and U, then normalize it.
- Equivalent definition: set of distributions such that for every A and B,

$$\frac{L(A)}{U(B)} \leq \frac{P(A)}{P(B)} \leq \frac{U(A)}{L(B)}.$$

Facts about density ratio classes

- Posterior probability:

$$\underline{P}(A|B) = \frac{L(A \cap B)}{L(A \cap B) + U(A^c \cap B)},$$

$$\overline{P}(A|B) = \frac{U(A \cap B)}{U(A \cap B) + L(A^c \cap B)}.$$

- Posterior from single likelihood: just multiply L and U by likelihood.
- There are bracketing algorithms for computing lower/upper expectations.

Constant density ratio class

- Set of distributions P such that

$$\frac{P(A)}{P(B)} \leq \alpha \frac{P_0(A)}{P_0(B)},$$

  for distribution P_0 and $\alpha > 1$.
- Class is preserved by conditioning/marginalization!

One great (obscure) idea

- Wasserman and Kadane (1982) observed that for some classes (total variation, constant bounded, constant ratio),
  - it is possible to sample from the “center” P_0 of the neighborhood, and compute lower expectations.

- One of the few cases where a sampling algorithm has been applied to credal sets.

- It would be nice to see other sampling methods, but hard to imagine how to do it.

And 2-monotone capacities

- If a credal set satisfies

$$\underline{P}(A \cup B) \geq \underline{P}(A) + \underline{P}(B) - \underline{P}(A \cap B)\,,$$

  it is *2-monotone*.

- Examples: ϵ-contaminated, total variation, density bounded.
- Define $\overline{F}_X(x) = \overline{P}(X \leq x)$; then

$$\underline{E}[X] = \int_{-\infty}^{\infty} x \; d\overline{F}_X(x).$$

- Also,

$$\underline{P}(A|B) = \frac{\underline{P}(A \cap B)}{\underline{P}(A \cap B) + \overline{P}(A^c \cap B)}.$$

And belief functions

- A capacity that is infinitely monotone; that is, for any n,

$$\underline{P}(\cup_{i=1}^{n} A_i) \geq \sum_{J \subset 1,\ldots,n} (-1)^{|J|+1} \underline{P}(\cap_{i \in J} A_i) \,.$$

- These are basic entities in Dempster-Shafer theory.
- They can always be expressed as a probability mass assignment and a multi-valued mapping.
- Useful:

$$\underline{E}[X] = \sum_{A} m(A) \inf_{\omega \in A} X(\omega).$$

Probability boxes (p-boxes)

- Take two nondecreasing functions $\underline{F}$ and $\overline{F}$ such that

$$\underline{F} \leq \overline{F}.$$

- The set of distributions such that

$$\{P : \underline{F} \leq F \leq \overline{F}\}.$$

  is a p-box.
- There has been work on risk assessment and reliability analysis with p-boxes: often discretization of continuous possibility spaces and then linear programming.

Changing gears: Decision making

- Set of acts $\mathcal{A}$, need to choose one.
  - There are several criteria!
- Γ-*minimax*:

$$\arg\max_{X \in \mathcal{A}} \underline{E}[X] \,.$$

- *Maximality*: maximal elements of the partial order $\succ$. That is, X is *maximal* if

  there is no $Y \in \mathcal{A}$ such that $E_P[Y - X] > 0$ for all $P \in K$.

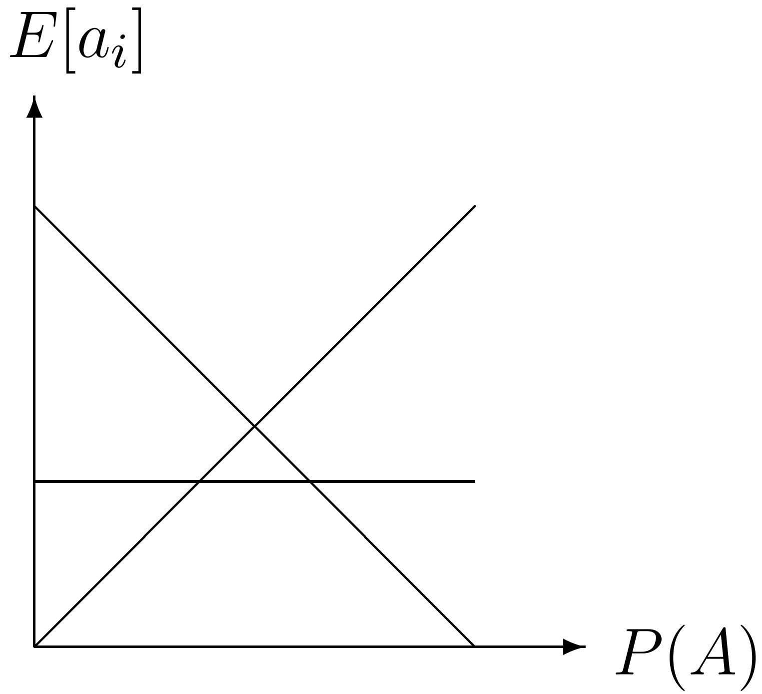
- *E-admissibility*: maximality for at least a distribution. That is, X is *E-admissible* if

  there is $P \in K$ such that $E_P[X - Y] \geq 0$ for all $Y \in \mathcal{A}$.

- Maximax, interval dominance, etc.

Comparing criteria

Three acts: $a_1 = 0.4$; $a_2 = 0/1$ if A/A^c; $a_3 = 1/0$ if A/A^c.



$P(A) \in [0.3, 0.7]$.
Γ-minimax: a_1; Maximal: all of them; E-admissible: $\{a_2, a_3\}$.

Exercise

Credal set $\{P_1, P_2\}$:

$$P_1(s_1) = 1/8, \quad P_1(s_2) = 3/4, \quad P_1(s_3) = 1/8,$$

$$P_2(s_1) = 3/4, \quad P_2(s_2) = 1/8, \quad P_2(s_3) = 1/8,$$

Acts $\{a_1, a_2, a_3\}$:

	s_1	s_2	s_3
a_1	3	3	4
a_2	2.5	3.5	5
a_3	1	5	4.

Which one to select?

Solution

$P_1(s_1) = 1/8,\; P_1(s_2) = 3/4, ; P_1(s_3) = 1/8, \quad P_2(s_1) = 3/4,\; P_2(s_2) = 1/8,\; P_2(s_3) = 1/8.$

Acts $\{a_1, a_2, a_3\}$:

	s_1	s_2	s_3
a_1	3	3	4
a_2	2.5	3.5	5
a_3	1	5	4.

Then:

$$
\begin{aligned}
E_1[a_1] &= 3/8 + 18/8 + 4/8 = 25/8; \\
E_1[a_2] &= 2.5/8 + 21/8 + 5/8 = 28.5/8; \\
E_1[a_3] &= 1/8 + 15/8 + 4/8 = 35/8. \\
E_2[a_1] &= 18/8 + 3/8 + 4/8 = 25/8; \\
E_2[a_2] &= 15/8 + 3.5/8 + 5/8 = 23.5/8 \\
E_2[a_3] &= 2/8 + 5/8 + 4/8 = 11/8.
\end{aligned}
$$

A quick discussion

- Limited to finite set of acts.

- Consider Γ-minimax:
  - Compute $\underline{E}[a_i]$ for each act.
  - Select act with highest $\underline{E}[a_i]$.
- (Considerable minimax theory in Berger's book (1985).)

Maximality

- Find Γ-minimax solution a_0.
- For each other act $a_i \neq a_0$, verify whether

$$E_P[a_0 - a_i] \geq 0;$$

  for all P; if so, discard a_i.
- That is, verify whether

$$\underline{E}[a_0 - a_i] \geq 0.$$

E-admissibility

- For each act a_i:
  - Collect all constraints that must be satisfied by P.
  - Add constraints

    $$E_P[a_i - a_j] \geq 0$$

    for every $a_j \neq a_i$.
  - If all these constraints can be satisfied for some P, then a_i is E-admissible.

- This scheme can be extended to problems with mixed acts (Utkin and Augustin 2005).

Exercise

Credal set $\{P_1, P_2\}$:

$$P_1(s_1) = 1/8, \quad P_1(s_2) = 3/4, \quad P_1(s_3) = 1/8,$$

$$P_2(s_1) = 3/4, \quad P_2(s_2) = 1/8, \quad P_2(s_3) = 1/8,$$

Acts $\{a_1, a_2, a_3\}$:

	s_1	s_2	s_3
a_1	3	3	4
a_2	2.5	3.5	5
a_3	1	5	4.

Which one to select?
And if we take convex hull of credal set?

Solution

$$P_1(s_1) = 1/8,\ P_1(s_2) = 3/4, ; P_1(s_3) = 1/8, \quad P_2(s_1) = 3/4,\ P_2(s_2) = 1/8,\ P_2(s_3) = 1/8.$$

Acts $\{a_1, a_2, a_3\}$:

	s_1	s_2	s_3
a_1	3	3	4
a_2	2.5	3.5	5
a_3	1	5	4.

- Consider $P = \alpha P_1 + (1-\alpha)P_2$.
- Then:

$$E_P[a_2 - a_1] = 10\alpha - 3 \geq 0; \quad \alpha \geq 3/10.$$

- And:

$$E_P[a_2 - a_3] = -30\alpha + 17 \geq 0; \quad \alpha \leq 17/30.$$

Conclusion

- Goal of this talk: overview of some central ideas without independence.
- Basic tool is linear programming (column generation, etc).
  - Full conditional measures require special tools.
- There are many special kinds of credal sets with associated algorithms: neighborhoods, capacities, etc.
- Decision making (severa criteria) requires such calculations.