Independence Concepts in Imprecise Probability Exercises

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Exercise

Consider a variable X with 3 possible values x_1, x_2 and x_3. Suppose the following assessments are given:

$$p(x_1) \leq p(x_2) \leq p(x_3)\,;$$

$$p(x_i) \geq 1/20 \quad \text{for } i \in \{1, 2, 3\};$$

$$p(x_3|x_2 \cup x_3) \leq 3/4.$$

Show the credal set determined by these assessments in baricentric coordinates.

Exercise

- A closed convex credal set is completely characterized by the associated lower expectation.
- But given a lower expectation, many credal sets generate it.
- Usually only the maximal closed convex set is chosen.
- **Exercise:** Given the assessments in the previous exercise, find two credal sets that yield the same lower expectation.

Exercise

Credal set $\{P_1, P_2\}$:

$$P_1(s_1) = 1/8, \quad P_1(s_2) = 3/4, \quad P_1(s_3) = 1/8,$$

$$P_2(s_1) = 3/4, \quad P_2(s_2) = 1/8, \quad P_2(s_3) = 1/8,$$

Acts $\{a_1, a_2, a_3\}$:

	s_1	s_2	s_3
a_1	3	3	4
a_2	2.5	3.5	5
a_3	1	5	$4.$

Which one to select?
And if we take convex hull of credal set?

Exercise

- Urn with $m > 0$ balls, numbered from 1 to m
- r balls are red and $m - r$ balls are black.
- n samples with replacement.
- ω is a numbered sequence produced this way.
- m^n possible numbered sequences.
- Assume uniformity: $P(\omega) \geq (1 - \epsilon)m^{-n}$.
- What is the lower probability that k balls are red?

Exercise

What is the largest credal set that satisfies exchangeability of two binary variables?

Exercise

- Suppose we have 4 binary variables that are exchangeable.
- What are the conditions on the probabilities $P(X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4)$?

Exercise

- Suppose we have 4 binary variables that are exchangeable.
- Suppose $P(0000) = 1/10$ and $P(1111) = 1/2$.
- Draw the credal set.

Exercise

Draw the credal set $K(X, Y)$ given the structural assessments:

- X and Y are exchangeable.
- X and Y are the first two variables in a sequence of three exchangeable variables.
- X and Y are the first two variables in a sequence of five exchangeable variables.
- X and Y are the first two variables in a sequence of infinitely many exchangeable variables.

Exercise

Prove decomposition, weak union and contraction for stochastic independence.

Exercise

- Consider a finite possibility space.
- Suppose $K(Y)$ is a singleton.
- Suppose $P(X)$, $K(X|Y \in B)$ are “almost” vacuous in that $P(X \in A|\cdot) > 0$ is the only constraint.
- Show that Y is epistemically irrelevant to X, but X is not epistemically irrelevant to Y.
- This is an extreme case of *dilation*!
- Construct an example that is not so extreme but that stills fails symmetry.

Exercise

Prove:

- Kuznetsov independence implies epistemic independence.
- Epistemic independence does not imply Kuznetsov independence.

Exercise

Consider

- Two binary variables X and Y.
- $P(X = 0) \in [2/5, 1/2]$ and $P(Y = 0) \in [2/5, 1/2]$.
- Epistemic independence of X and Y: $K(X, Y)$ is convex hull of

$$[1/4, 1/4, 1/4, 1/4], [4/25, 6/25, 6/25, 9/25],$$

$$[1/5, 1/5, 3/10, 3/10], [1/5, 3/10, 1/5, 3/10],$$

$$[2/9, 2/9, 2/9, 1/3], [2/11, 3/11, 3/11, 3/11],$$

Write down the linear constraints that must be satisfied by $K(X, Y)$.

Exercise

Due to de Campos and Moral (1995).

- X and Y are binary.
- $K(X,Y)$ is the convex hull of two distributions P_1 and P_2 such that $P_1(X=0,Y=0) = P_2(X=1,Y=1) = 1$.

Show:

- X and Y are strongly independent.
- Neither Y is type-5 irrelevant to X, nor X is type-5 irrelevant to Y.

Exercise

Show that strict and strong independence satisfy all graphoid properties.

Exercise

Show:

- Epistemic independence satisfies decomposition and weak union in finite spaces.
- Epistemic irrelevance satisfies: if Y is epistemically irrelevant to X and W is epistemically irrelevant to X given Y then (W, Y) are epistemically irrelevant to X.
- Kuznetsov independence satisfies decomposition.