Maximal Acyclic Subtournaments

Serge Gaspers (formerly LIRMM)

Matthias Mnich

(Eindhoven Univ of Technology, The Netherlands)
Tournaments & Acyclicity

- **Tournament**: orientation of a complete graph.

- A tournament is **acyclic / transitive** if it does not contain any directed cycles.

- **Maximal** acyclic subtournament: not properly contained in any other acyclic subtournament.
Problem and Results

• How many maximal acyclic subtournaments can a tournament with $n$ vertices have?

• **Upper Bound**
  Moon (1971): at most $1.717^n$.

• **Lower Bound**
  Moon (1971): at least $1.4570^n$.

Application:
Banks winner in Elections
Strong Tournaments & Acyclicity

Folklore: tournament acyclic $\iff$ no 3-cycles.

A tournament $T$ is strong if there is a directed path between any pair of vertices.

$$T = S_1 + S_2 + \ldots + S_r \rightarrow f(T) = \prod f(S_i)$$
Lower bound: $21^{n/7} \approx 1.5448^n$

Payley digraph of quadratic residues mod 7
Fano plane generator
Upper Bound I: Branching

Property: Every vertex beats \( \leq n-2 \) other vertices.

Example Case: \( v \) beats \( n-2 \) other vertices.

\[
\begin{align*}
&b \notin W: \text{source } v, \quad f(n-2) \text{ many} \\
&b \in W, v \in W: \text{source } (b,v), \quad f(n-4) \text{ many} \\
&b \in W, v \notin W: \text{ source } c_1 \text{ or } c_2, \quad 2f(n-3) \text{ many}
\end{align*}
\]

\[
f(T) \leq f(n-2) + 2f(n-3) + f(n-4) \leq f(n) \text{ for } f(n) = 1.6181^n
\]
Upper Bound II: Scores

- The **score** of a vertex $v$ is its outdegree.
- The **score sequence** of a tournament is the non-decreasing sequence of out-degrees: $s = s(T) = (s_1,\ldots,s_n)$ with $s_1 \leq \ldots \leq s_n$
- **Landau** (1953, “On dominance and the structure of animal societies”):  
  
  \[
  \sum_{v=1}^{k} s_v \geq \binom{k}{2} + 1 \quad \text{for all } k = 1, \ldots, n-1, \text{ and}
  \]
  
  \[
  \sum_{v=1}^{n} s_v = \binom{n}{2}
  \]
Upper Bound II: Special Sequences

\[ S_n = \{ s = (s_1, ..., s_n) : 3 \leq s_1 \leq ... \leq s_n \leq n-4 \}. \]

\[ G : S_n \rightarrow \mathbb{IR}_+, \quad G(s) = \sum \beta^{s_i} \]

\[ \sigma = (3,3,3,3,3,5,7,7,7,7,7) \quad \text{if } n = 11 \]

\[ \sigma = (3,3,3,3,3,3,8,8,8,8,8,8) \quad \text{if } n = 12 \]

\[ \sigma = (3,3,3,3,3,3,6,9,9,9,9,9,9) \quad \text{if } n = 13 \]

\[ \sigma = (3,3,3,3,3,3,4,7,8,\ldots, n-9,n-8,n-5,n-4,n-4,n-4,n-4,n-4,n-4,n-4) \quad \text{if } n \geq 14. \]

**Lemma:** \( G(s) \leq G(\sigma) \) for all \( s \in S_n \).
Upper Bound II: Convexity

**Lemma:** $G(s) \leq G(\sigma)$ for all $s \in S_n$.

Technical proof by strict convexity of $G$.

$$\sigma = (3,3,3,3,3,3,6,9,9,9,9,9,9) \quad \text{if } n = 13$$

**Corollary:**

$$f(T) \leq G(s) \leq G(\sigma) = 6\beta^3 + \beta^6 + 6\beta^9 \leq \beta^n$$

for $\beta \geq 1.6259$
Lemma: $G(s) \leq G(\sigma)$ for all $s \in S_n$.

$\sigma = (3,3,3,3,3,3,4,7,8,...,n-9,n-8,n-5,n-4,n-4,n-4,n-4,n-4,n-4,n-4,n-4)\$

Prove claims based on $G$ being strictly convex:
Let $s \in S_n$ be a maximizer of $G$. Then...

Claim 1: score $c$ appears multiple times $\rightarrow c \in \{3,n-4\}.$
Claim 2: scores $c \in \{3,n-4\}$ appear 2-6 times each.
Claim 3: scores $c \in \{3,n-4\}$ appear exactly 6 times.
Claim 4: $s = \sigma.$
Summary

\( f(n) \): maximum number of maximal acyclic subtournaments of a tournament with \( n \) vertices

- Moon (1971): \( 1.4570^n \leq f(n) \leq 1.717^n \)
- Gaspers & M. (2009): \( 1.5448^n \leq f(n) \leq 1.667^n \)
- Exact bounds for small tournaments.
Open Problems (and Conjectures)

• We can improve the bound slightly at the expense of a much longer proof. New technique for better upper bounds?
• Conjecture: $f(n) = 1.5448^n$.
• More general digraphs?
• Approach applicable to Tournament Dominating Sets?