Edge coloring: not so simple on multigraphs?

Marthe Bonamy

May 10, 2012
Edge coloring

\[ \chi : \text{Minimum number of colors to ensure that } a \neq b. \]

\[ \chi' : \text{Minimum number of colors to ensure that } a \neq b. \]

\[ \Delta : \text{Maximum degree of the graph.} \]

\[ \Delta \leq \chi' \leq 2\Delta - 1. \]
$\chi$: Minimum number of colors to ensure that

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Simple graphs

**Theorem (Vizing '64)**

For any simple graph $G$, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

**Theorem (König '16, Sanders Zhao '01)**

For any simple graph $G$, if $G$ is bipartite, or $G$ is planar with $\Delta(G) \geq 7$, then $\chi'(G) = \Delta(G)$.

**Theorem (Erdős Wilson '77)**

Almost every simple graph $G$ verifies $\chi'(G) = \Delta(G)$.

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## Simple graphs

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It is NP-complete to compute $\chi'$ on simple graphs.
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### Theorem (Holyer '81)

It is **NP-complete** to compute $\chi'$ on simple graphs.

### Theorem (Misra Gries '92 (Inspired from the proof of Vizing’s theorem))

For any simple graph $G = (V, E)$, a $(\Delta + 1)$-edge-coloring can be found in $O(|V| \times |E|)$. 
Both theorems are optimal!

$\Delta = 2p.$  
$\chi' = 3p.$  
$\mu = p.$  

$\mu$: Maximum number of edges sharing the same endpoints.
**Multigraphs**

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**Theorem (Shannon ‘49)**

For any multigraph \( G \),
\[ \chi'(G) \leq \frac{3\Delta(G)}{2}. \]

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For any multigraph \( G \),
\[ \chi'(G) \leq \Delta(G) + \mu(G). \]
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Linear relaxation of edge coloring

$M(G)$: set of all matchings.

$w : M(G) \rightarrow \{0; 1\}$.

$\forall e \in E, \sum_{M \mid e \in M} w(M) = 1.$
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\[ w' \text{ optimal solution } \iff \]
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\( w' \) optimal solution \( \iff \)
\[ \sum_M w'(M) = \chi'_f(G). \]

\( \chi'_f(G) \): Minimum number of colors to ensure that
\[ a_1 a_2 \ldots a_p b_1 b_2 \ldots b_p \Rightarrow \forall i, j, a_i \neq b_j. \]

Property \( \chi'_f(G) = \inf_p \chi'_p(G) \).
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Edge colorings and matchings

Edge coloring $\approx$ Decomposition into matchings.
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Maximum size of a matching: $\left\lfloor \frac{|V|}{2} \right\rfloor$. 
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$$\chi'(G) \geq \max \frac{|E(H)|}{\left\lfloor \frac{|V(H)|}{2} \right\rfloor} = m(G).$$

$m(G)$: density of $G$.

Theorem (Edmonds '65)

For any multigraph $G$, $\chi'(G) = \max (\Delta(G), m(G))$.

Conjecture (Goldberg '73)

For any multigraph $G$, $\chi'(G) \leq \max (\Delta(G) + 1, \lceil m(G) \rceil)$.

Which would be optimal.

And is computable in polynomial time.

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Goldberg’s conjecture

Theorem (Yu ’08)

For any multigraph $G$, $\chi'(G) \leq \max(\Delta(G) + \sqrt{\frac{\Delta(G)}{2}}, w(G))$.

Theorem (Kurt ’09)

For any multigraph $G$, $\chi'(G) \leq \max(\Delta(G) + \frac{\Delta(G) + 22}{24}, w(G))$. 
Seymour’s conjecture

Conjecture (Seymour)

For any \( k \)-regular planar multigraph \( G \) s.t. every odd subset \( H \) has \( d(H) \geq k \) verifies \( \chi'(G) = k \).

- \( k = 3 \iff 4\text{CT. (Tait)} \)
- \( k = 4, 5 \) (Guenin)
- \( k = 6 \) (Dvorak, Kawarabayashi, Kral)
- \( k = 7 \) (Edwards, Kawarabayashi)
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Sketch of the proof.
Conclusion

Conjecture (Jensen Toft ’95)

For any simple graph $G$ on an even number of vertices,

\[ \chi'(G) = \Delta(G) \text{ or } \chi'(\overline{G}) = \Delta(\overline{G}). \]
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For any simple graph $G$ on an even number of vertices, 

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Thanks for your attention.

Any questions?