Generalized power domination in regular graphs

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Graph protection Workshop, 2012 July 8th

Electrical system management

Problem :

Monitor all vertices and edges of a network with PMU (Phase Measurement Units) using rules :

1. a PMU monitors its vertex and its incident edges
2. vertex incident to a monitored edge $\Rightarrow$ monitored (Ohm law)
3. edge joining 2 monitored vertices $\Rightarrow$ monitored (Ohm law)
4. degree d monitored vertex incident to $d-1$ monitored edges $\Rightarrow$ d^{th} edge monitored (Kirchhoff law).

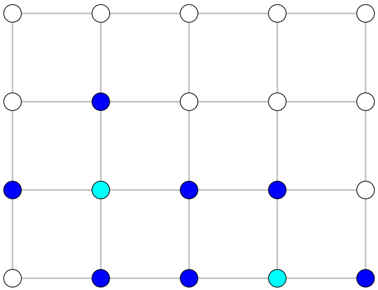
Equivalent rules :

Monitor all vertices of the network ($\Rightarrow$ edges monitored from 3)

domination a PMU monitors the closed neighborhood of its vertex $(1+2)$

propagation degree d monitored vertex with $d-1$ monitored neighbours $\Rightarrow$ d^{th} neighbour monitored $((3+4)+2)$.

Example : $\gamma_P(P_4 \,\square\, P_5) \leq 2$



Domination

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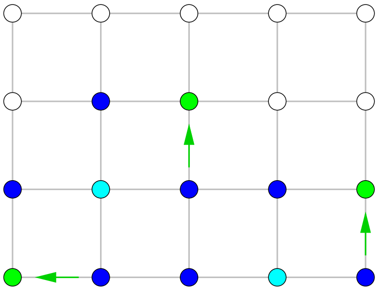

Propagation 1

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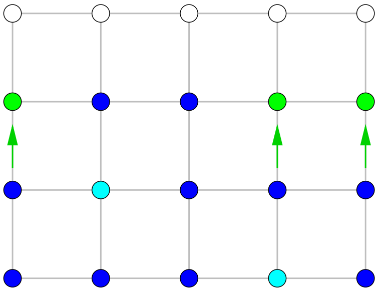
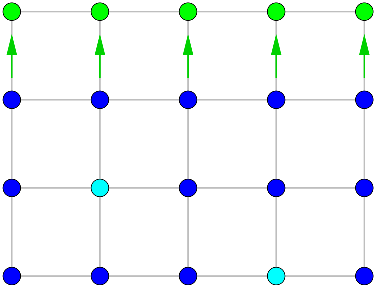


Propagation 2

Example : $\gamma_{\mathrm{P}}(P_4 \,\Box\, P_5) \leq 2$



Propagation 3

Difficulties...

Does $\gamma_{\mathrm{P}}(G)$ decrease when you

- add edges ?
- delete edges ?
- delete vertices ?
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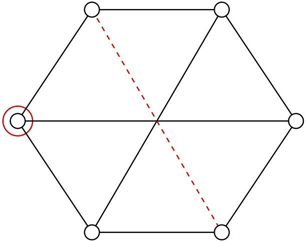

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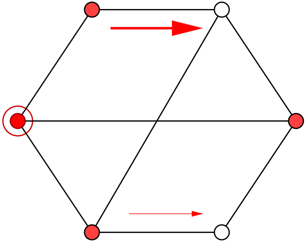

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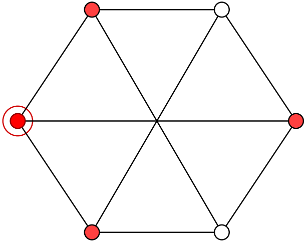

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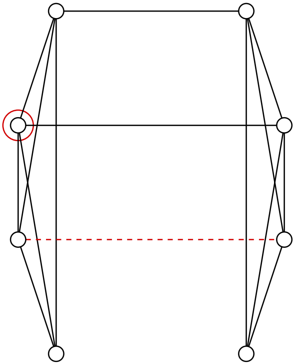

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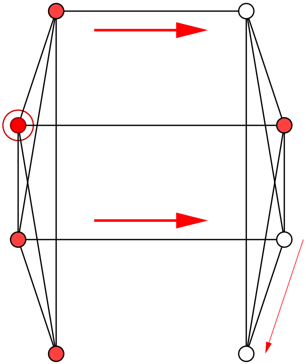

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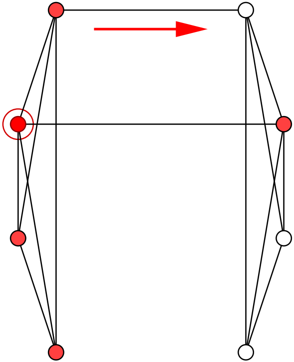

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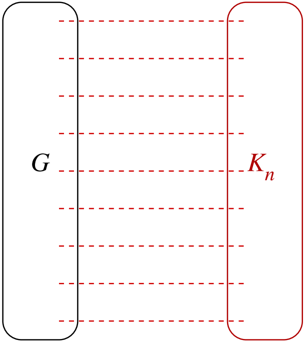

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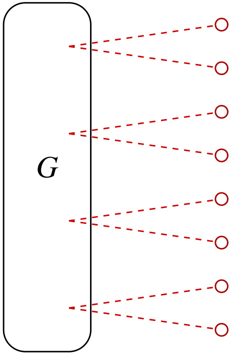

$\Rightarrow$ No obvious heredity

Monitored vertices

Definition :

G a graph, S a subset of vertices
The set $\mathcal{P}^i(S)$ of vertices monitored by S at step i
is defined by

- (domination)

$$\mathcal{P}^0(S) = N[S]$$

- (propagation)

$$\mathcal{P}^{i+1}(S) = \left\{ N[v] \;\middle|\; \begin{array}{l} v \in \mathcal{P}^i(S), \\ \left|N[v] \setminus \mathcal{P}^i(S)\right| \leq 1 \end{array} \right\}$$

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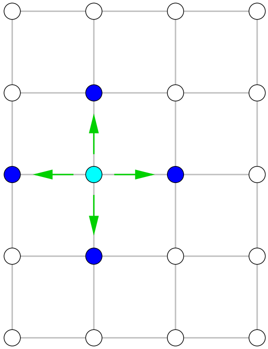

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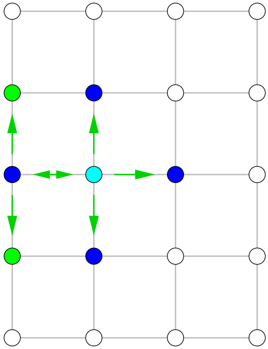

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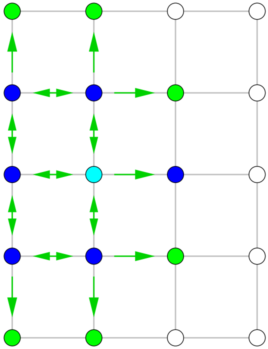

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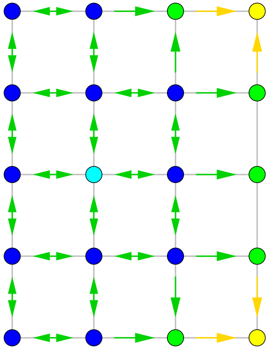


Generalized power domination

Problem

Given a graph G, find its k-power domination number $\gamma_{\mathrm{P,k}}(G)$
$= $ smallest size of S such that $\mathcal{P}^{\infty}(S) = V(G)$.

- generalizes power domination ($\gamma_{\mathrm{P},1} = \gamma_{\mathrm{P}}$)
- generalizes domination ($\gamma_{\mathrm{P},0} = \gamma$)
- helps to understand how power-domination is related to domination :
  - critical graphs : $(k+1)$-crowns
  - general bounds
  - common linear algorithm on trees (and bounded treewidth)
  - other bounds for families of graphs...

Common general bound

For G connected of order n

Lemma

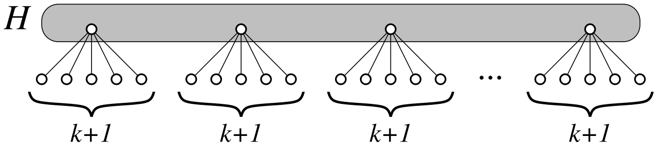
If $\Delta(G) \leq k+1$, $\gamma_{\mathrm{P,k}}(G) = 1$

Lemma

Otherwise, there exist a minimum k-power dominating set containing only vertices of degree $\geq k+2$

Theorem

If G is of order $n \geq k+2$, then $\gamma_{\mathrm{P,k}}(G) \leq \dfrac{n}{k+2}$



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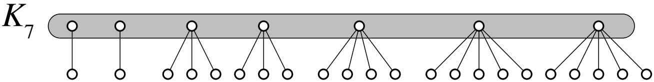
Obs : No

For any sequence $(x_k)_k > 0$ finite and non-increasing, there exist G such that $\gamma_{\mathrm{P,k}}(G) = x_k$.



A graph for the sequence $(7, 5, 5, 3, 2)$

On regular graphs

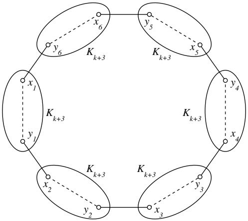
Theorem [Zhao,Kang,Chang,2006]

G connected claw-free cubic $\Rightarrow \gamma_{\mathrm{P}}(G) \leq \frac{n}{4}$.

Theorem [CDMR2012]

G connected claw-free $(k+2)$-regular
$\Rightarrow \gamma_{\mathrm{P,k}}(G) \leq \frac{n}{k+3}$.

both with equality iff G is isomorphic to the graph :



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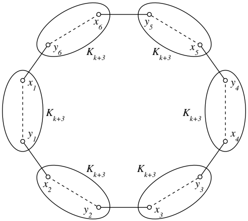

Theorem [DHLMR2012+]

G connected $(k+2)$-regular, $G \neq K_{k+2,k+2}$, $\Rightarrow \gamma_{\mathrm{P,k}}(G) \leq \frac{n}{k+3}$.

(A, B)-configurations

Let G be a connected $(k+2)$-regular graph.

- For each vertex taken, find $k+3$ new monitored vertices

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  - $\exists$ a monitored vertex $v (\in B)$ that has unmonitored neighbours $(\in A)$.
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Definition : (A, B)-configurations

(P1). $|A| \in \{k+1, k+2\}$.
(P2). $B = N(A) \setminus A$.
(P3). $d_A(v) = k+1$ for each vertex $v \in B$.
(P4). B is an independent set.

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(P5). $d_B(v) \geq 1$ for each vertex $v \in A$.
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- then we show they can't intersect too much... exemple $A \cap A' > 1$.
- Remains some family $\mathcal{F}_k$...

Final trick

- Remove from G any edge not in a C_3 or a C_4.
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- Remove from G any edge not in a C_3 or a C_4.
- every $\mathcal{F}_k$ in G remain and is isolated : take a vertex in each
- take a vertex in every other (A, B)-configurations.
- complete into a maximal packing of G.
- propagate, then increase the set iterately : possible since no (A, B)-configurations left...

Summary

Recall that if $\Delta(G) \leq k+1$, $\gamma_{\mathrm{P,k}}(G) = 1$
We proved :

Theorem [DHLMR2012+]

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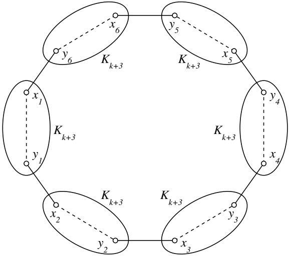

What next ? Another bound ? (I think not $\frac{n}{r+1}$)

Thanks for your attention.

CDMR2012 : Chang, Dorbec, Montassier, Raspaud, Discrete Appl. Math.
DHLMR2012+ : Dorbec, Henning, Lowenstein, Montassier, Raspaud, manuscript