Generalized power domination in regular graphs

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Electrical system management

Problem:
Monitor all vertices and edges of a network with PMU (Phase Measurement Units) using rules:

1. a PMU monitors its vertex and its incident edges
2. vertex incident to a monitored edge $\Rightarrow$ monitored (Ohm law)
3. edge joining 2 monitored vertices $\Rightarrow$ monitored (Ohm law)
4. degree $d$ monitored vertex incident to $d - 1$ monitored edges $\Rightarrow d^{th}$ edge monitored (Kirchhoff law).

Equivalent rules:
Monitor all vertices of the network ($\Rightarrow$ edges monitored from 3)

- **domination**: a PMU monitors the closed neighborhood of its vertex $(1 + 2)$
- **propagation**: degree $d$ monitored vertex with $d - 1$ monitored neighbours $\Rightarrow d^{th}$ neighbour monitored $((3 + 4) + 2)$. 
Example: $\gamma_P(P_4 \square P_5) \leq 2$
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Propagation 2
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Does $\gamma_P(G)$ decrease when you

- add edges?
- delete edges?
- delete vertices?
- add vertices?
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Introduction

Definition of power domination

Difficulties...

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$\Rightarrow$ No obvious heredity
Monitored vertices

Definition:

$G$ a graph, $S$ a subset of vertices
The set $\mathcal{P}^i(S)$ of vertices monitored by $S$ at step $i$
is defined by

- (domination)

$$\mathcal{P}^0(S) = N[S]$$

- (propagation)

$$\mathcal{P}^{i+1}(S) = \left\{ N[v] \mid v \in \mathcal{P}^i(S), \left| N[v] \setminus \mathcal{P}^i(S) \right| \leq 1 \right\}$$
Monitored vertices

Definition: [CDMR2012]

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$k = 2, \mathcal{P}^0(S)$
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$k = 2, P^2(S)$
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The diagram illustrates the concept for $k = 2$, $\mathcal{P}^{>3}(S)$. The nodes and edges represent the monitored vertices and their dependencies in the graph.
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Definition of power domination

Generalized power domination

Problem
Given a graph $G$, find its $k$-power domination number $\gamma_{P,k}(G)$

$$\gamma_{P,k}(G) = \text{smallest size of } S \text{ such that } P^\infty(S) = V(G).$$

- generalizes power domination ($\gamma_{P,1} = \gamma_P$)
- generalizes domination ($\gamma_{P,0} = \gamma$)
- helps to understand how power-domination is related to domination:
  - critical graphs: $(k + 1)$-crowns
  - general bounds
  - common linear algorithm on trees (and bounded treewidth)
  - other bounds for families of graphs...
Common general bound

For $G$ connected of order $n$

**Lemma**
If $\Delta(G) \leq k + 1$, $\gamma_{P,k}(G) = 1$

**Lemma**
Otherwise, there exist a minimum $k$-power dominating set containing only vertices of degree $\geq k + 2$

**Theorem**
If $G$ is of order $n \geq k + 2$, then $\gamma_{P,k}(G) \leq \frac{n}{k + 2}$
Relation between $\gamma_{P,k}$ for different $k$

**Question**

Clearly, $\gamma_{P,k}(G) \geq \gamma_{P,k+1}(G)$. Can we say more?
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Obs : No
For any sequence $(x_k)_k > 0$ finite and non-increasing, there exist $G$ such that $\gamma_{P,k}(G) = x_k$.

\[ K_7 \]

A graph for the sequence $(7, 5, 5, 3, 2)$
On regular graphs

Theorem [Zhao, Kang, Chang, 2006]
$G$ connected claw-free cubic $\Rightarrow \gamma_P(G) \leq \frac{n}{4}$.

Theorem [CDMR2012]
$G$ connected claw-free $(k + 2)$-regular
$\Rightarrow \gamma_{P,k}(G) \leq \frac{n}{k+3}$.

both with equality iff $G$ is isomorphic to the graph:
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both with equality iff $G$ is isomorphic to the graph:

Theorem [DHLMR2012+]
$G$ connected $(k + 2)$-regular, $G \neq K_{k+2,k+2}$, $\Rightarrow \gamma_{P,k}(G) \leq \frac{n}{k+3}$.
(A, B)-configurations

Let G be a connected (k + 2)-regular graph.

▶ For each vertex taken, find $k + 3$ new monitored vertices typically: its neighbours $\Rightarrow$ a 2-packing.
(A, B)-configurations

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- For each vertex taken, find \(k + 3\) new monitored vertices typically: its neighbours \(\Rightarrow\) a 2-packing.
- then look for obstructions... = (A, B)-configurations:
  - \(\exists\) a monitored vertex \(v(\in B)\) that has unmonitored neighbours \((\in A)\).
  - \(v\) does not propagate so at least \(k + 1\),
  - \(v\) is monitored so at least one monitored neighbour.

Definition: (A, B)-configurations

(P1). \(|A| \in \{k + 1, k + 2\}\).

(P2). \(B = N(A) \setminus A\).

(P3). \(d_A(v) = k + 1\) for each vertex \(v \in B\).

(P4). \(B\) is an independent set.
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- We can add more:

(P5). \(d_B(v) \geq 1\) for each vertex \(v \in A\).

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(P7). \(|B| \leq k + 2\).

- then we show they can’t intersect too much... exemple \(A \cap A' > 1\).

- Remains some family \(\mathcal{F}_k\)...
Final trick

- Remove from $G$ any edge not in a $C_3$ or a $C_4$.
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- Remove from $G$ any edge not in a $C_3$ or a $C_4$.
- Every $F_k$ in $G$ remain and is isolated: take a vertex in each
- take a vertex in every other $(A, B)$-configurations.
- Complete into a maximal packing of $G$.
- Propagate, then increase the set iterately: possible since no
  $(A, B)$-configurations left...
Summary

Recall that if $\Delta(G) \leq k + 1$, $\gamma_{P,k}(G) = 1$
We proved:

**Theorem [DHLMR2012+]**

$G$ connected $(k+2)$-regular, $G \neq K_{k+2,k+2}$, $\Rightarrow \gamma_{P,k}(G) \leq \frac{n}{k+3}$.

What next? Another bound? (I think not $\frac{n}{r+1}$)
Thanks for your attention.

DHLMR2012+ : Dorbec, Henning, Lowenstein, Montassier, Raspald, manuscript