Fixed parameter tractability and kernels for feedback set problems on generalization of tournaments

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Joint work with

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Problem (FVS (FAS))

Given a digraph D, find a minimum $F \subset V(D)$ ($F \subset A(D)$) s.t. $D - F$ is acyclic?

It is known that Both problems are NP-complete (even if restricted to tournaments).

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Problem (Parametrized feedback set problems)

Given D and $k \in \mathbb{N}$, is there $F \subset V(D)$ ($F \subset A(D)$) s.t. $|F| \leq k$ and $D - F$ is acyclic?

Theorem (Chen,Liu,Lu,O'Sullivan,Razgon)

Parametrized FVS (and thus FAS) is FPT.

Problem

Given D and $k, g \in \mathbb{N}$, is there $F \subset V(D)$ s.t. $|F| \leq k$ and $g(D - F) > g$?

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A sunflower with h petals is a collection of sets $S_1, ..., S_h$ s.t. $\forall i \neq j \; S_i \cap S_j = Y$.

Lemma (Erdös-Rado)

Among $d!k^d$ sets of size $\leq d$ there is a sunflower with $k + 1$ petals.

Theorem

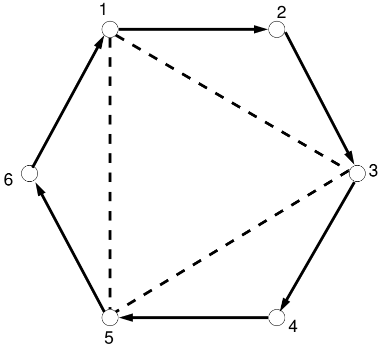
The previous problem has an $O(g \cdot g! \cdot k^g)$ kernel.

Proof sketch.

- Make a list $\mathcal{F}$ of all the cycles of length $\leq g$
- For every sunflower with $> k+1$ petals in $\mathcal{F}$. Delete one of the petals. (If the $Y = \emptyset$ answer NO)
- If no sunflower is found there are $O((k+1)^g \cdot g!)$ sets. Output the digraph induced by all the edges of the sets (size $O(k^g \cdot g! \cdot g)$).

□

- D acyclic iff $g(D) > 2\alpha(D) + 1$.



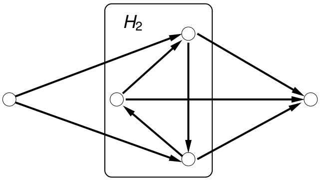
Corollary

k-FVS has an $O(k^{2\alpha+1})$ kernel for digraph with independence number α, (in particular $O(k^3)$ for tournaments)

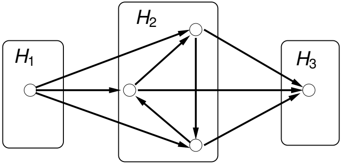
H an induced subdigraph of *D* is a module if

$$\forall a, b \in V(H),\ v \in V(D \setminus H) \quad \mu(va) = \mu(vb),\ \mu(av) = \mu(bv).$$

(If *D* is simple, we simply say that every vertex of *H* must have the same in and out neighborhood)



D is decomposable if $\exists$ partition of V into modules $H_1, ... H_s$, $s \geq 2$. We write $D = S[H_1, ..., H_s]$, where S is the adjacency (or quotient) digraph of $H_1, ... H_s$.



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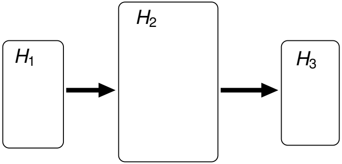
$S =$ 

Decomposable digraphs

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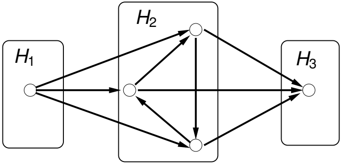
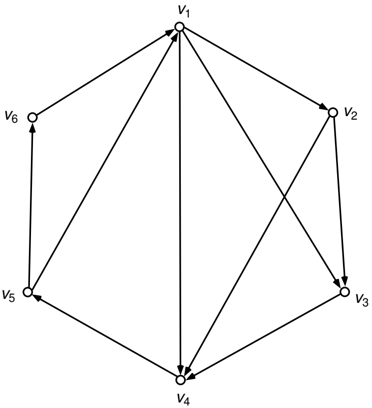


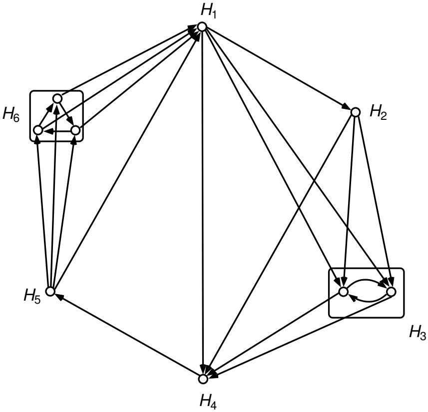
Φ class of digraphs. D is totally Φ-decomposable if either $D \in \Phi$ or $D = S[H_1, ..., H_s]$, with $S \in \Phi$ and H_i totally Φ-decomposable, $i = 1, ..., s$.
The digraph in the figure is totally Φ-decomposable with $\Phi = P_3 \cup C_3 \cup P_1$

- D is round if we can label its vertices $v_1, \ldots, v_n$ so that $\forall\ i$, $N^+(v_i) = \{v_{i+1}, \ldots, v_{i+d^+(i)}\}$ and $N^-(v_i) = \{v_{i-d^-(i)}, \ldots, v_{i-1}\}$.



Round digraphs

- D is round if we can label its vertices $v_1, ..., v_n$ so that $\forall\ i$, $N^+(v_i) = \{v_{i+1}, ..., v_{i+d^+(i)}\}$ and $N^-(v_i) = \{v_{i-d^-(i)}, ..., v_{i-1}\}$.
- D is round decomposable if $D = R[H_1, ..., H_r]$, where R is a round digraph and $H_1, ..., H_r$ are semicomplete digraphs.



FVS on round (decomposable)

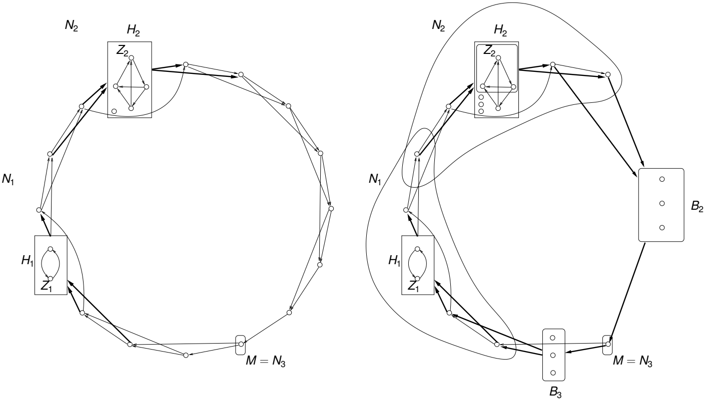
Observation

FVS is poly on round digraphs (One among $(N^+(v), N^-(v))_{v \in V}$ *must be killed, and it is enough)*

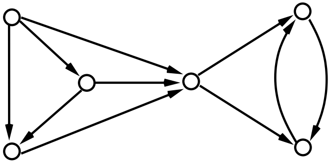
Theorem

k-FVS has an $O(k^3)$ kernel on round decomposable digraphs.

1. Decompose $D = R[H_1, ..., H_r]$.
2. Find non-trivial modules $K_1, , , , K_h$ and kernelize each of them (keep the size $> k$ if it was before)
3. Find a min FVS M for Q.
4. Keep M the K_i's kernels and the $2k$ modules around them (k left and k right).
5. Contract the gaps into I_{k+1}.



Locally semicomplete digraphs (LSD): $\forall x \in V, N^+(x), N^-(x)$ are semicomplete :



Theorem (Guo)

A connected LSD is either

- *round decomposable, or*
- *Every cycle induces a cycle on ≤ 4 vertices.*

Theorem

k-FVS has an $O(k^4)$ kernel on LSD.

FVS on totally Φ-decomposable

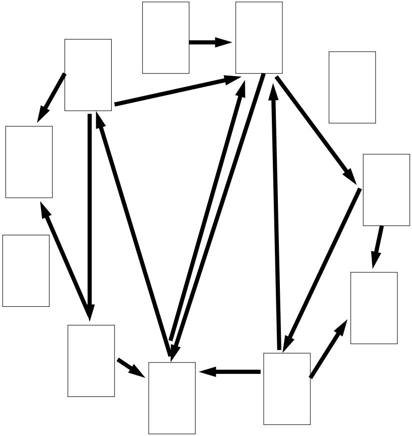
We say that a kernel is virgin if it contains all minimal solutions
Let Φ be s.t.

- $\exists$ poly algorithm to find total Φ-decomposition
- k-FVS has an $O(f(k))$ virgin kernel on Φ.

Theorem

k-FVS has a $O(k \cdot f(k))$ kernel on totally Φ-decomposable digraphs.

1. Decompose $D = Q[M_1, ..., M_q]$
2.
3.
4.



1. Decompose $D = Q[M_1, ..., M_q]$
2. Find recursively virgin kernels $K_1, ..., K_h$ for the cyclic modules.
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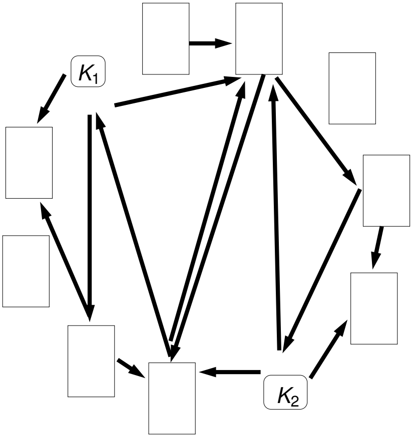

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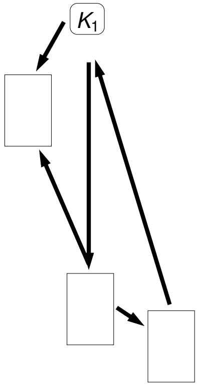
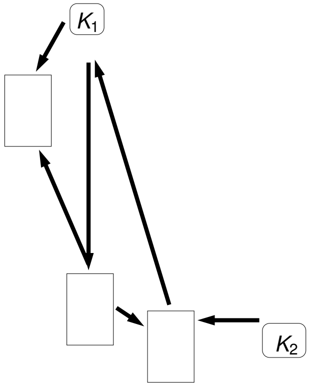


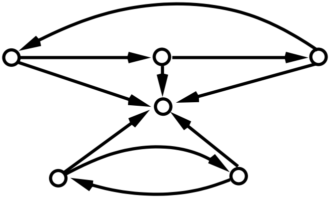
1. Decompose $D = Q[M_1, ..., M_q]$
2. Find recursively virgin kernels $K_1, ..., K_h$ for the cyclic modules.
3. Find a virgin kernel K for Q.
4. Output $(K \cup \bigcup_{K_i \not\trianglelefteq K} K_i, k)$.



Justification

- If YES, then $O(k)$ kernels recursively constructed.
- KERNEL $\subset$ ORIGINAL DIGRAPH
- VIRGINITY $\Rightarrow$ (YES KERNEL $\leftrightarrow$ YES ORIGINAL)

Quasi-transitive digraphs: $xy, yz \in A$ implies that $zx \in A$ or $xz \in A$:



Theorem (Bang-Jensen and Huang)

D be quasi-transitive, then either

- $D = T[H_1, \ldots, H_t]$, *T acyclic and* $H_1, \ldots, H_t$ *(strong) quasi-transitive, or*
- $D = S[Q_1, Q_2, \ldots, Q_s]$, *S semicomplete and* $Q_1, \ldots, Q_s$ *(non-strong) quasi-transitive.*

Observation

Quasi-transitive are totally Φ_1-decomposable, where

$$\Phi_1 = \{ \textit{Semicomplete} \cup \textit{Acyclic} \}$$

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FVS has an $O(k^4)$ *kernel on quasi-transitive digraphs.*

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Theorem

FVS has an $O(k^4)$ kernel on quasi-transitive digraphs.

We hit also other classes:

1. Directed cographs
2. Extended semicomplete digraphs

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A $2^{o(k)}$ algorithm for k-FAS is unlikely to exist

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Theorem (Bessy, Fomin, Gaspers, Paul, Perez, Saurabh, Thomassé)

There is an $O(k)$ kernel for k-FAS on tournaments (semicomplete digraphs).

Theorem (Alon, Lokshtanov, Saurabh)

There is a $2^{o(k)}$ algorithm for k-FAS on tournaments (semicomplete digraphs). (Best complexity $n^{O(1)}2^{O(\sqrt{k})}$ by Feige).

FAS on totally Φ-decomposable

A kernel (x', k') of (x, k) is tight if $k' = k$ and

$$\forall h \leq k,\ (x', h) \text{ is a YES} \Leftrightarrow (x, h) \text{ is a YES}$$

Let Φ s.t.

- $\exists$ poly algorithm for total Φ-decomposition
- k-FAS has an $O(f(k))$ tight kernel on Φ.

Theorem

k-FAS has an $O(k \cdot f(k))$ kernel on totally Φ-decomposable digraphs

FAS on totally Φ-decomposable

- Totally decompose D: Get $D_1, ..., D_p \in \Phi$
- Find non-acyclic digraphs in the decomposition $D_{i_1}, ..., D_{i_c}$
- Output $(D_{i_1} \cup .... \cup D_{i_c}, k)$.

Key lemma

Given $D = Q[M_1, ..., M_q]$, there is a min fas $F = F_1 \cup ... \cup F_q \cup F^*$ s.t. F_i is a min fas of M_i and F^* is a min fas of Q^D.

FAS on totally Φ-decomposable

$$\Phi_2 = \{ \text{ Semicomplete } \cup \text{ Acyclic } \cup \text{ Round } \}$$

Corollary

There is an $O(k^2)$ kernel for k-FAS on totally Φ_2-decomposable digraphs

In particular for quasi-transitive or extended semicomplete or directed cographs or round decomposable.
In fact there is an $O(k)$ kernel for totally Φ_2-decomposable.

Theorem

There is an $O(n^3 \cdot 2^{O(\sqrt{k}\log k)})$ algorithm for k-FAS has on lsd.

An lsd is either

- Round decomposable= round + semicomplete, or
- Has vertex set partitionable into two tournaments

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First case: Round part is poly semicomplete part reduces to second case

Fix a random partition $V_1, ..., V_l$ of V. $l = O(\sqrt{k})$.

Theorem (Alon, Lokshtanov, Saurabh)

$P($ "arcs of a fas of size $\leq k$ belong to different V_i's "$) \geq (2e)^{-\sqrt{k/8}}$

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$P($ "arcs of a fas of size $\leq k$ belong to different V_i's "$) \geq (2e)^{-\sqrt{k/8}}$

- Objective: Find a partition and a fas of size $\leq k$ with arcs belonging to different V_i's.
- Expected number of iterations is $O(2^{\sqrt{k}})$.
- Derandomize (use $\tilde{O}(2^{\sqrt{k}})$ iterations)

$p = (a_1, ..., a_l)$, $0 \leq a_i \leq |V_i|$
Define $FAS(p)$ = min fas of $D\langle p \rangle$.

$$FAS(p) = \min_{i \in [l]} (FAS(p - e_i) + d^+_{D\langle p \rangle}(v_{i,a_i}))$$

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Do dynamic programming over a restricted table: size $O(n^2 \cdot 2^{O(\sqrt{k} \log k)})$.

Theorem

There is an $O(n^3 \cdot 2^{O(\sqrt{k} \log k)})$ algorithm for k-FAS on digraphs such that $V(D) = V_1 \cup V_2$, V_1, V_2 semicomplete.

Conjecture

There is a poly kernel for k-FAS on lsd

Problem

Is there a poly kernel for k-FAS on digraph with bounded independence number?

Open problems

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Problem

Is there a poly kernel for k-FAS (and thus k-FVS) on general digraphs?