Fixed parameter tractability and kernels for feedback set problems on generalization of tournaments

Alessandro Maddaloni
University of Southern Denmark

LIRMM, February 7

Joint work with
Jørgen Bang-Jensen and Saket Saurabh
Problem (FVS (FAS))

Given a digraph $D$, find a minimum $F \subset V(D)$ ($F \subset A(D)$) s.t. $D - F$ is acyclic?

It is known that Both problems are NP-complete (even if restricted to tournaments).
Problem (FVS (FAS))

*Given a digraph $D$, find a minimum $F \subset V(D)$ ($F \subset A(D)$) s.t. $D - F$ is acyclic?*

It is known that **Both problems are NP-complete (even if restricted to tournaments).**

Problem (Parametrized feedback set problems)

*Given $D$ and $k \in \mathbb{N}$, is there $F \subset V(D)$ ($F \subset A(D)$) s.t. $|F| \leq k$ and $D - F$ is acyclic?*

Theorem (Chen, Liu, Lu, O’Sullivan, Razgon)

*Parametrized FVS (and thus FAS) is FPT.*
Problem

Given $D$ and $k$, $g \in \mathbb{N}$, is there $F \subset V(D)$ s.t. $|F| \leq k$ and $g(D - F) > g$?
Problem

Given $D$ and $k, g \in \mathbb{N}$, is there $F \subset V(D)$ s.t. $|F| \leq k$ and $g(D - F) > g$?

A **sunflower** with $h$ petals is a collection of sets $S_1, ..., S_h$ s.t.
\[ \forall i \neq j \ S_i \cap S_j = Y. \]

**Lemma (Erdös-Rado)**

Among $d!k^d$ sets of size $\leq d$ there is a sunflower with $k + 1$ petals.
Theorem

The previous problem has an $O(g \cdot g! \cdot k^g)$ kernel.

Proof sketch.

- Make a list $\mathcal{F}$ of all the cycles of length $\leq g$
- For every sunflower with $> k + 1$ petals in $\mathcal{F}$. Delete one of the petals. (If the $Y = \emptyset$ answer NO)
- If no sunflower is found there are $O((k + 1)^g \cdot g!)$ sets. Output the digraph induced by all the edges of the sets (size $O(k^g \cdot g! \cdot g)$).
D acyclic iff \( g(D) > 2\alpha(D) + 1 \).

**Corollary**

\( k \)-FVS has an \( O(k^{2\alpha+1}) \) kernel for digraph with independence number \( \alpha \), (in particular \( O(k^3) \) for tournaments)
$H$ an induced subdigraph of $D$ is a module if

$$\forall a, b \in V(H), \; v \in V(D \setminus H) \quad \mu(va) = \mu(vb), \; \mu(av) = \mu(bv).$$

(If $D$ is simple, we simply say that every vertex of $H$ must have the same in and out neighborhood)
$D$ is decomposable if $\exists$ partition of $V$ into modules $H_1, \ldots, H_s$, $s \geq 2$. We write $D = S[H_1, \ldots, H_s]$, where $S$ is the adjacency (or quotient) digraph of $H_1, \ldots, H_s$. 

The digraph in the figure is totally $\Phi$-decomposable with $\Phi = P_3 \cup C_3 \cup P_1$. 
**Decomposable digraphs**

*Definition:* A digraph $D$ is **decomposable** if there exists a partition of $V$ into modules $H_1, \ldots, H_s$, $s \geq 2$. We write $D = S[H_1, \ldots, H_s]$, where $S$ is the adjacency (or quotient) digraph of $H_1, \ldots, H_s$.

The digraph in the figure is totally $\Phi$-decomposable with $\Phi = P_3 \cup C_3 \cup P_1$. 

\[ S = \begin{array}{c}
H_1 \\
\downarrow \\
H_2 \\
\downarrow \\
H_3 
\end{array} \]
A digraph $D$ is **decomposable** if there exists a partition of $V$ into modules $H_1, \ldots, H_s$, with $s \geq 2$. We write $D = S[H_1, \ldots, H_s]$, where $S$ is the adjacency (or quotient) digraph of $H_1, \ldots, H_s$.

![Graph Diagram](image)

For a **totally decomposable** class of digraphs $\Phi$, $D$ is **totally $\Phi$-decomposable** if either $D \in \Phi$ or $D = S[H_1, \ldots, H_s]$, with $S \in \Phi$ and $H_i$ totally $\Phi$-decomposable, $i = 1, \ldots, s$.

The digraph in the figure is totally $\Phi$-decomposable with $\Phi = P_3 \cup C_3 \cup P_1$. 
A digraph $D$ is round if we can label its vertices $v_1, ..., v_n$ so that $\forall \ i$, 
\[ N^+(v_i) = \{ v_{i+1}, ..., v_{i+d^+(i)} \} \] and 
\[ N^-(v_i) = \{ v_{i-d^-(i)}, ..., v_{i-1} \}. \]
Round digraphs

- $D$ is round if we can label its vertices $v_1, ..., v_n$ so that $\forall i$, $N^+(v_i) = \{v_{i+1}, ..., v_{i+d^+(i)}\}$ and $N^-(v_i) = \{v_{i-d^-(i)}, ..., v_{i-1}\}$.

- $D$ is round decomposable if $D = R[H_1, ..., H_r]$, where $R$ is a round digraph and $H_1, ..., H_r$ are semicomplete digraphs.
**Observation**

*FVS is poly on round digraphs (One among \((N^+(v), N^-(v))_{v \in V}\) must be killed, and it is enough)*

**Theorem**

*k-FVS has an \(O(k^3)\) kernel on round decomposable digraphs.*
1. Decompose $D = R[H_1, \ldots, H_r]$. 
2. Find non-trivial modules $K_1, \ldots, K_h$ and kernelize each of them (keep the size $> k$ if it was before). 
3. Find a min FVS $M$ for $Q$. 
4. Keep $M$ the $K_i$'s kernels and the $2k$ modules around them ($k$ left and $k$ right). 
5. Contract the gaps into $I_{k+1}$. 

\[ Z_2 = N_2 \] 
\[ Z_1 = N_1 \] 
\[ H_1 = H_2 \] 
\[ B_2 = B_3 \] 
\[ M = N_3 \]
Locally semicomplete digraphs (LSD): \( \forall x \in V, N^+(x), N^-(x) \) are semicomplete:

**Theorem (Guo)**

A connected LSD is either

- round decomposable, or
- Every cycle induces a cycle on \( \leq 4 \) vertices.

**Theorem**

\( k\)-FVS has an \( O(k^4) \) kernel on LSD.
We say that a kernel is **virgin** if it contains all minimal solutions.

Let $\Phi$ be s.t.

- $\exists$ poly algorithm to find total $\Phi$-decomposition
- $k$-FVS has an $O(f(k))$ virgin kernel on $\Phi$.

**Theorem**

$k$-FVS has a $O(k \cdot f(k))$ kernel on totally $\Phi$-decomposable digraphs.
1. Decompose $D = Q[M_1, ..., M_q]$
1  Decompose $D = \mathbb{Q}[M_1, \ldots, M_q]$
2  Find recursively virgin kernels $K_1, \ldots, K_h$ for the cyclic modules.
1. Decompose $D = Q[M_1, ..., M_q]$
2. Find recursively virgin kernels $K_1, ..., K_h$ for the cyclic modules.
3. Find a virgin kernel $K$ for $Q$. 

![Diagram with arrows and rectangles]
1. Decompose $D = Q[M_1, \ldots, M_q]$
2. Find recursively virgin kernels $K_1, \ldots, K_h$ for the cyclic modules.
3. Find a virgin kernel $K$ for $Q$.
4. Output $(K \cup \bigcup_{K_i \neq K} K_i, k)$. 
If YES, then $O(k)$ kernels recursively constructed.

- KERNEL ⊂ ORIGINAL DIGRAPH
- VIRGINITY $\Rightarrow$ (YES KERNEL $\leftrightarrow$ YES ORIGINAL)
Quasi-transitive digraphs: \( xy, yz \in A \) implies that \( zx \in A \) or \( xz \in A \):

\[
\begin{align*}
\text{Theorem (Bang-Jensen and Huang)} \\
D \text{ be quasi-transitive, then either} \\
\bullet \ D = T[H_1, \ldots, H_t], \ T \text{ acyclic and } H_1, \ldots, H_t \text{ (strong)} \text{ quasi-transitive, or} \\
\bullet \ D = S[Q_1, Q_2, \ldots, Q_s], \ S \text{ semicomplete and } Q_1, \ldots, Q_s \text{ (non-strong) quasi-transitive.}
\end{align*}
\]
### Observation

*Quasi-transitive are totally $\Phi_1$-decomposable, where*

\[ \Phi_1 = \{ \text{Semicomplete} \cup \text{Acyclic} \} \]
Observation

Quasi-transitive are totally $\Phi_1$-decomposable, where

$$\Phi_1 = \{ \text{Semicomplete} \cup \text{Acyclic} \}$$

Observation

There is a virgin $O(k^3)$ kernel for FVS on $\Phi_1$

Theorem

FVS has an $O(k^4)$ kernel on quasi-transitive digraphs.
Observation

Quasi-transitive are totally $\Phi_1$-decomposable, where

$$\Phi_1 = \{ \text{Semicomplete} \cup \text{Acyclic} \}$$

Observation

There is a virgin $O(k^3)$ kernel for FVS on $\Phi_1$

Theorem

FVS has an $O(k^4)$ kernel on quasi-transitive digraphs.

We hit also other classes:

1. Directed cographs
2. Extended semicomplete digraphs
Observation (Speckenmeyer)

A $2^{o(k)}$ algorithm for k-FAS is unlikely to exist
Observation (Speckenmeyer)

A $2^{o(k)}$ algorithm for $k$-FAS is unlikely to exist

Theorem (Bessy, Fomin, Gaspers, Paul, Perez, Saurabh, Thomassé)

There is an $O(k)$ kernel for $k$-FAS on tournaments (semicomplete digraphs).

Theorem (Alon, Lokshtanov, Saurabh)

There is a $2^{o(k)}$ algorithm for $k$-FAS on tournaments (semicomplete digraphs). (Best complexity $n^{O(1)}2^{O(\sqrt{k})}$ by Feige).
A kernel \((x', k')\) of \((x, k)\) is **tight** if \(k' = k\) and

\[
\forall h \leq k, \ (x', h) \text{ is a YES } \iff (x, h) \text{ is a YES}
\]

Let \(\Phi\) s.t.
- \(\exists\) poly algorithm for total \(\Phi\)-decomposition
- \(k\)-FAS has an \(O(f(k))\) tight kernel on \(\Phi\).

**Theorem**

\(k\)-FAS has an \(O(k \cdot f(k))\) kernel on totally \(\Phi\)-decomposable digraphs
Totally decompose $D$: Get $D_1, ..., D_p \in \Phi$
Find non-acyclic digraphs in the decomposition $D_{i_1}, ..., D_{i_c}$
Output $(D_{i_1} \cup .... \cup D_{i_c}, k)$.

**Key lemma**

Given $D = Q[M_1, ..., M_q]$, there is a min fas $F = F_1 \cup ... \cup F_q \cup F^*$
s.t. $F_i$ is a min fas of $M_i$ and $F^*$ is a min fas of $Q^D$. 
\( \Phi_2 = \{ \text{Semicomplete} \cup \text{Acyclic} \cup \text{Round} \} \)

Corollary

There is an \( O(k^2) \) kernel for \( k \)-FAS on totally \( \Phi_2 \)-decomposable digraphs

In particular for quasi-transitive or extended semicompelte or directed cographs or round decomposable. In fact there is an \( O(k) \) kernel for totally \( \Phi_2 \)-decomposable.
Theorem

There is an $O(n^3 \cdot 2^{O(\sqrt{k} \log k)})$ algorithm for k-FAS has on lsd.

An lsd is either

- Round decomposable = round + semicomplete, or
- Has vertex set partitionable into two tournaments.
Theorem

There is an $O(n^3 \cdot 2^{O(\sqrt{k \log k})})$ algorithm for $k$-FAS has on lsd.

An lsd is either
- Round decomposable = round + semicomplete, or
- Has vertex set partitionable into two tournaments

First case: Round part is poly semicomplete part reduces to second case
Fix a random partition $V_1, \ldots, V_l$ of $V$. $l = O(\sqrt{k})$.

**Theorem (Alon, Lokshtanov, Saurabh)**

$P(\text{"arcs of a fas of size } \leq k \text{ belong to different } V_i\text{'s }) \geq (2e)^{-\sqrt{k/8}}$
Fix a random partition $V_1, \ldots, V_l$ of $V$. $l = O(\sqrt{k})$.

**Theorem (Alon, Lokshtanov, Saurabh)**

$$P\left( \text{"arcs of a fas of size } \leq k \text{ belong to different } V_i \text{'s "} \right) \geq (2e)^{-\sqrt{k/8}}$$

- **Objective**: Find a partition and a fas of size $\leq k$ with arcs belonging to different $V_i$’s.
- Expected number of iterations is $O(2^{\sqrt{k}})$.
- Derandomize (use $\tilde{O}(2^{\sqrt{k}})$ iterations)
\( p = (a_1, \ldots, a_l), \ 0 \leq a_i \leq |V_i| \)
Define \( FAS(p) = \min \) fas of \( D\langle p \rangle \).

\[
FAS(p) = \min_{i \in [l]} (FAS(p - e_i) + d^+_D(p)(v_i, a_i))
\]
\( p = (a_1, ..., a_l), \ 0 \leq a_i \leq |V_i| \)
Define \( FAS(p) = \min \text{ fas of } D\langle p \rangle \).

\[
FAS(p) = \min_{i \in [l]} (FAS(p - e_i) + d^+_D(v_i, a_i))
\]

Do dynamic programming over a restricted table: size \( O(n^2 \cdot 2^{O(\sqrt{k} \log k)}) \).

**Theorem**

There is an \( O(n^3 \cdot 2^{O(\sqrt{k} \log k)}) \) algorithm for \( k \)-FAS on digraphs such that \( V(D) = V_1 \cup V_2, \ V_1, V_2 \) semicomplete.
Conjecture

There is a poly kernel for $k$-FAS on lsd

Problem

Is there a poly kernel for $k$-FAS on digraph with bounded independence number?
| **Conjecture** | There is a poly kernel for k-FAS on lsd |
| **Problem** | Is there a poly kernel for k-FAS on digraph with bounded independence number? |
| **Problem** | Is there a poly kernel for k-FAS (and thus k-FVS) on general digraphs? |