Induced immersions

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Some graph operations

- Vertex deletion;
- Edge deletion;
- Edge contraction;
- Vertex dissolution;
- Lift.

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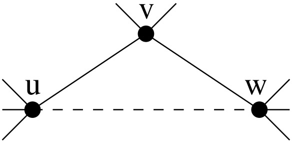
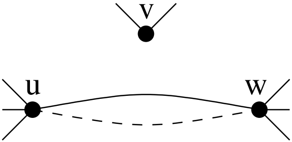
- Edge deletion;
- Edge contraction;
- Vertex dissolution;
- **Lift**.

Lift



Lift



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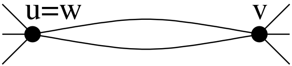

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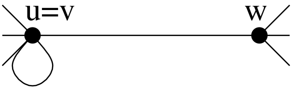

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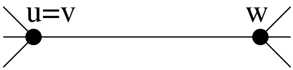

Containment relations

Containment Relation	VD	ED	EC	VDi	L
Minor	yes	yes	yes	yes	no
Induced minor	yes	no	yes	yes	no
Topological minor	yes	yes	no	yes	no
Induced topological minor	yes	no	no	yes	no
Immersion	yes	yes	no	yes	yes
Induced immersion	yes	no	no	yes	yes

Table: Some containment relations.

Some terminology

Definition

A containment relation $R(G, H)$ is:

- FPT if there exists an algorithm that decides R in time $f(|H|) \cdot poly(|G|)$, for some function f and every pair of graphs G and H;
- XP if there exists an algorithm that decides R in time $|G|^{f(|H|)}$, for some function f and every pair of graphs G and H;
- NP-complete for fixed H (aka paraNP-complete) if there exists a fixed graph H for with deciding $R(G, H)$ is NP-complete;

Complexity of containment relations

Containment Relation	VD	ED	EC	VDi	L	Complexity
Minor	yes	yes	yes	yes	no	FPT
Induced minor	yes	no	yes	yes	no	NP-C (fixed H)
Topological minor	yes	yes	no	yes	no	FPT
Induced topological minor	yes	no	no	yes	no	NP-C (fixed H)
Immersion	yes	yes	no	yes	yes	FPT
Induced immersion	yes	no	no	yes	yes	?

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Immersion	yes	yes	no	yes	yes	FPT
Induced immersion	**yes**	**no**	**no**	**yes**	**yes**	**XP**

Table: Some containment relations.

Alternative definition for immersion

Observation

Let G and H be two multigraphs. G contains H as an immersion if and only if there exists a set of vertices S of G, a bijection ϕ from $V(H)$ to $V(G)$, and a map α from $E(H)$ to paths in G such that:

- *for every edge e of H with endpoint u, v, the path $\alpha(e)$ has endpoints $\alpha(u), \alpha(v)$;*
- *for every $e \neq f \in E(H), \alpha(e)$ and $\alpha(f)$ are edge-disjoint.*

Finding edge-disjoint paths

Definition (EDGE-DISJOINT PATHS problem)

Let G be a graph and $(s_1, t_1), \ldots, (s_k, t_k)$ be k pairs of vertices of G (the *terminals*). The k EDGE-DISJOINT PATHS problem asks if there exists paths $P_1, \ldots, P_k$ in G such that for every i, P_i has endpoints s_i and t_i and $P_1, \ldots, P_k$ are mutually edge-disjoint.

Theorem (Kawarabayashi, Kobayashi, Reed, JCTB, 2011)

The k EDGE-DISJOINT PATHS problem can be solved in $O(n^2)$ time for every fixed k.

Some definitions about (induced) immersions

Definition (H-model)

Let G and H be two multigraphs such that G contains H as an induced immersion. Given a sequence of lifts $\mathcal{L}$, a set of vertices $S \subseteq V(G)$ and a bijection ϕ from $V(H)$ to S, we say that $(S, \mathcal{L}, \phi)$ is an H-*model* of G if ϕ is an isomorphism from H to $G'[S]$, where $G' = G \vee \mathcal{L}$.

XP algorithm for IMMERSION

- Try all choices for S;
- Try all choices for ϕ;
- Find missing edges using DISJOINT PATHS;
- Delete bad edges.

A first idea to solve INDUCED IMMERSION

- Try all choices for S;
- Try all choices for ϕ;
- Find missing edges using DISJOINT PATHS;
- **Delete bad edges**.

A first idea to solve Induced Immersion

- Try all choices for S;
- Try all choices for ϕ;
- Find missing edges using Disjoint Paths;
- **Delete bad edges**.

Question

Would we be happy if there were no bad edges?

The reduced case

Lemma

If $G[S]$ with isomorphim ϕ to H does not have any bad edges, then there is an H-model $(S, \mathcal{L}, \phi)$ of G iff we can find edge-disjoint paths for the missing edges.

How can we get rid of the bad edges?

Lemma

Let G be a graph and let H be a multigraph such that G contains H as an induced immersion, and let $(S, \mathcal{L}, \phi)$ be an H-model of G. Then there exists a sequence $\mathcal{L}^*$ of lifts that satisfies the following four properties:

(i) *$\mathcal{L}^*$ is short (at most $|E(G[S])|$ lifts);*

(ii) *for every $\{uv, vw\} \in \mathcal{L}^*$, uv or vw is bad;*

(iii) *there are no bad edges in $G \vee \mathcal{L}^*$;*

(iv) *$G \vee \mathcal{L}^*$ contains H as an induced immersion.*

How do we find $\mathcal{L}^*$?

Lemma

Let G and H be two multigraphs such that G contains H as an induced immersion, and let $(S, \mathcal{L}, \phi)$ be an H-model of G. For any edge uv of $G[S]$ that appears in $\mathcal{L}$, there exists an H-model $(S, \mathcal{L}', \phi)$ of G such that uv appears in the first lift in $\mathcal{L}'$.

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Let G and H be two multigraphs such that G contains H as an induced immersion, and let $(S, \mathcal{L}, \phi)$ be an H-model of G. For any edge uv of $G[S]$ that appears in $\mathcal{L}$, there exists an H-model $(S, \mathcal{L}', \phi)$ of G such that uv appears in the first lift in $\mathcal{L}'$.

In other words: we can start by "killing all the bad edges"!

Outline of the algorithm

- Try all choices for S ($|V(G)|^{|V(H)|}$ choices);
- Try all choices for ϕ ($|V(H)|!$ choices);
- Try all sequences $\mathcal{L}^*$ that kill all the bad edges $((|V(H)|^2 \cdot |V(G)|)^{|E(G[S])|}$ choices);
- Find missing edges using Disjoint Paths (can be checked in $f(|E(H)|) \cdot |V(G)|^2$ time);

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Theorem

The H-INDUCED IMMERSION problem can be solved in time $O(h(|V(H)| + |E(H)|) \cdot |V(G)|^{|V(H)|^2+2}$, for some function h.

Finally...

Theorem

The H-INDUCED IMMERSION problem can be solved polynomial time for every fixed multigraph H.

New question

Question

What is the structure of graphs that do not contain some "small" graph H as an induced immersion?

Example of similar results for other containment relations

Theorem (Robertson, Seymour, GM V)

Let $\mathcal{C}$ be a set of graphs. There exists a constant t such that every graph in $\mathcal{C}$ has treewidth at most t if and only if there exists a planar graph H such that no graph in $\mathcal{C}$ contains H as a minor.

Corollary

Let $\mathcal{C}$ be a set of graphs. There exists a constant t such that every graph in $\mathcal{C}$ has treewidth at most t if and only if there exists a subcubic planar graph H such that no graph in $\mathcal{C}$ contains H as an immersion/topological minor.

Our result for induced immersion

Theorem

Let H be a multigraph with maximum degree at most 2. Every multigraph with "big" treewidth contains H as an induced immersion.

Tools for the proof: (1) The wall

Theorem (Robertson, Seymour, Thomas, Quicly Excluding a Planar Graph)

For $r \geq 1$, every graph with "big" treewidth contains W_r as an immersion.

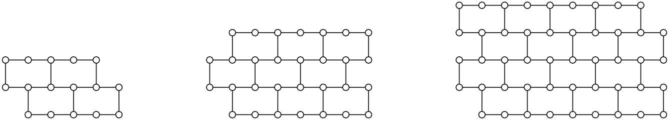

Figure: Elementary walls of height 2, 3 and 4.

Tools for the proof: (2) Ramsey's theorem

Theorem (Ramsey's theorem)

A graph that has neither a clique nor an independent set of size more than k must have at most 2^{2k-3} vertices.

Tools for the proof: (3) Some additional lemmata

Lemma

Let G and H be two multigraphs. If G contains a large clique as a subgraph, then G contains H as an induced immersion.

Lemma

Let G be a multigraph, and let H be a multigraph of maximum degree at most 2. If G contains an elementary wall W as a subgraph such that $G[V(W)]$ contains a large independent set, then G contains H as an induced immersion.

Sketch of the proof

- If G has big enough treewidth, then it contains a large elementary wall as an immersion;
- Let W be the graph obtained by applying only the lifts and vertex deletions: W contains a large elementary wall as a spanning subgraph;
- If W contains a big clique as a subgraph, then G contains H as an induced immersion. Otherwise, W must contain a big independent set, by Ramsey's theorem;
- Hence W contains H as an induced immersion.

Further possible results...

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There are 3 typical problems when studying a containment relation:

- Complexity of deciding the relation;
- Structure of graphs excluding a fixed pattern;
- Well-quasi-orders.

Comparison between immersion and induced immersion

	Immersion	Induced immersion
Complexity	FPT	XP
Structure	H is planar subcubic $\Updownarrow$ bounded treewidth	H has maximum degree 2 $\Downarrow$ bounded treewidth
Well-quasi-order	general graphs are WQO	?

Table: Immersion vs. induced immersion.

Thank you very much

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