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Shortest path embeddings of graphs on surfaces

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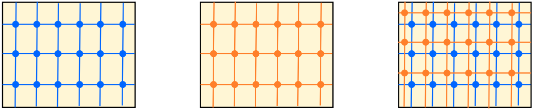
UMR 5216

Joint crossing numbers

- Let G_1 and G_2 be two graphs embeddable on a surface S of genus g.
- The *joint crossing number* $jcr(G_1, G_2)$ is the minimal number of crossings in two simultaneous and transverse embeddings of G_1 and G_2.



- Variant: *homeomorphic joint crossing number* when the combinatorial maps are fixed.

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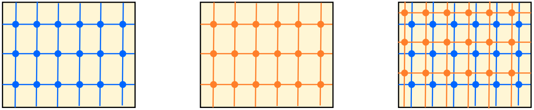

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Theorem (Negami)

There is a constant c *such that* $jcr(G_1, G_2) \leq cg|E(G_1)||E(G_2)|$.

Conjecture (Negami)

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Many different occurences

- [Negami '01]: Study of flips in triangulations.
- [Matoušek, Sedgwick, Tancer, Wagner '14]: 2 systems of curves $(\alpha_1, \ldots, \alpha_m)$, $(\beta_1, \ldots, \beta_n)$ on a surface with boundary. We seek a homeomorphism Φ such that the number of crossing between α_i and $\Phi(\beta_j)$ is as small as possible.
  $\rightarrow$ Complexity bound for embeddability of 2-complexes into $\mathbb{R}^3$.
- Similar question in [Geelen, Huynh, Richter '13] towards explicit bounds for graph minors.
- By duality, this amounts to drawing G_1 "on" G_2^* without using too many edges.
  $\rightarrow$ Useful for specific topological decompositions, e.g., canonical systems of loops [Lazarus, Pocchiola, Vegter, Verroust '01], pants decompositions [É. Colin de Verdière, Lazarus '07], octagonal decompositions [É. Colin de Verdière, Erickson '10].

What is known on Negami's conjecture

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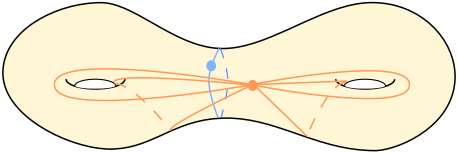

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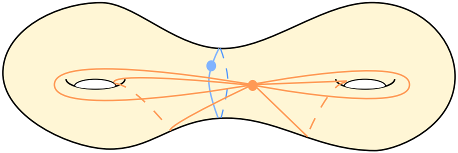

What about simple graphs?

Fàry's Theorem

Theorem (Fàry-Wagner)

Any simple planar graph can be embedded in the plane with straight lines.

- Straight lines cross at most once!

Plane	$\rightarrow$	Surface
Euclidean metric	$\rightarrow$	Some Riemannian metric?
Straight line	$\rightarrow$	Shortest path.

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Question

For each surface S, does there exist a metric such that any simple graph embeddable on S can be embedded with the edges drawn as shortest paths?

Main question

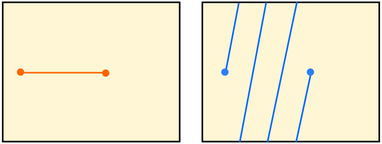
Question

For each surface S, does there exist a metric such that any simple graph embeddable on S can be embedded with the edges drawn as shortest paths?

- Shortest paths cross at most once!
  $\rightarrow$ If the answer is affirmative, Negami's conjecture is true.
- We call such an embedding a *shortest path embedding*, and such a metric a *universal shortest path metric*.
- We focus on constant-curvature metrics: spherical, flat or hyperbolic.

Related work I

We could ask for *geodesics* instead of shortest paths.



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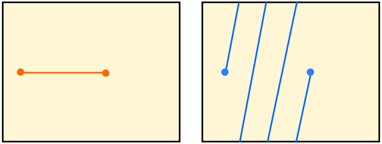
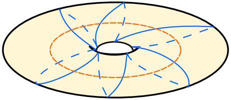


Theorem (Y. Colin de Verdière)

Let S be a surface endowed with a metric of nonpositive curvature. Then any simple graph G embeddable on S can be embedded such that the edges are drawn as geodesics.

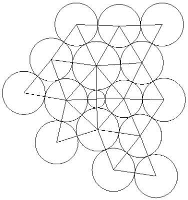
- The proof is à la Tutte: put springs on the edges and relax.
- But geodesics may cross an arbitrary number of times.

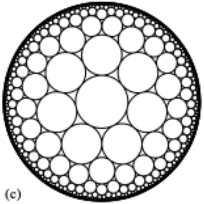


Related work II

Theorem (Circle packing theorem, Koebe-Andreev-Thurston, Mohar, Y. Colin de Verdière)

For any triangulated graph G on a surface S, there exists a conformal metric on S such that G can be represented as the contact graph of circles. This representation is unique up to Möbius transformation.

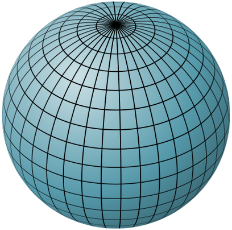


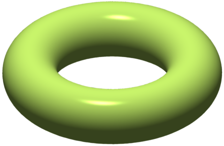


(c)

- In particular, the edges of G are shortest paths.
- But the metric depends on the graph!

The good





The sphere and the projective plane

Theorem

The usual round metrics for the sphere and the projective plane are universal shortest path metrics.

- For the sphere: There is only one conformal metric, so circle-pack everything.
- For the projective plane: Circle pack the spherical cover and quotient it (possible because of uniqueness).

The torus

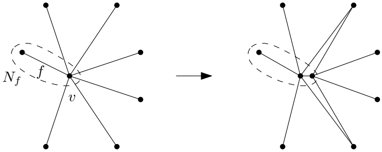
Theorem

The unit square flat metric on the torus is a universal shortest path metric.

- A triangulation of the torus is *reducible* if there is an edge e such that the contraction T/e is still a triangulation.

Lemma

If T/e admits a shortest path embedding, then so does T.



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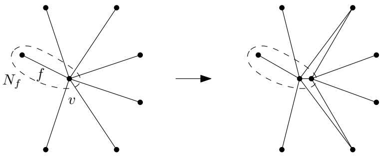
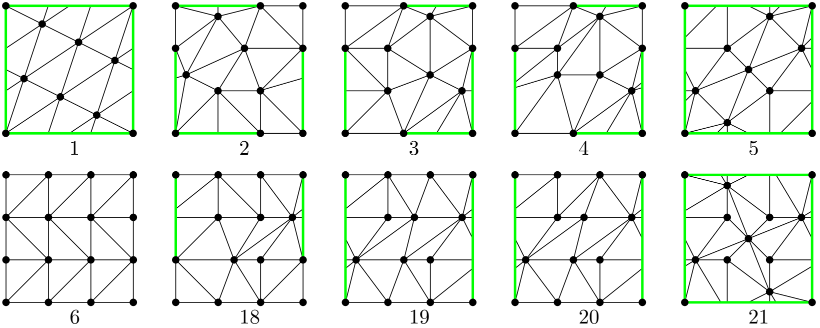


$\rightarrow$ It suffices to find shortest path embeddings for the 21 irreducible triangulations of the torus [Lawrencenko '87].

Irreducible triangulations of the torus

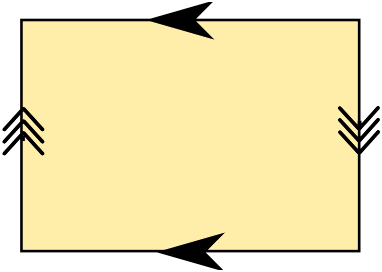


The bad

The Klein bottle

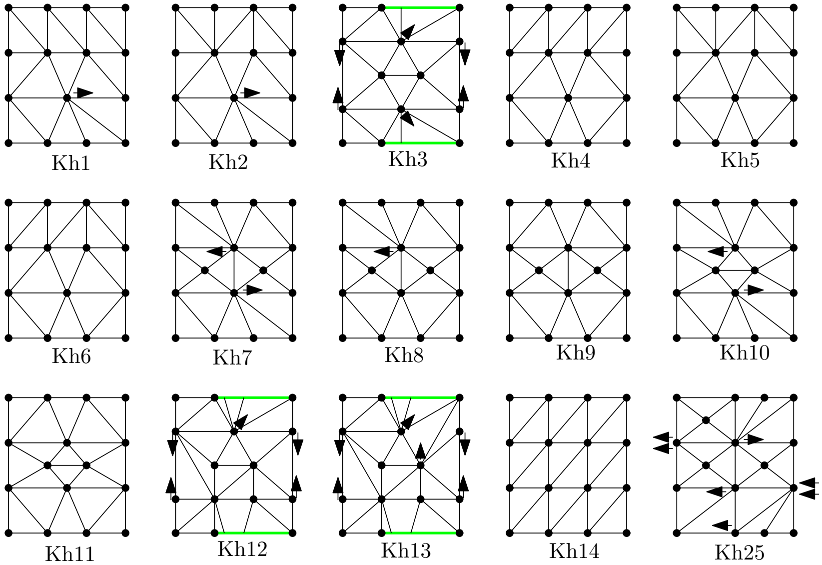
Theorem

The $1 \times \sqrt{4/3} + \varepsilon$ *flat metric with scheme* $aba^{-1}b$ *is a universal shortest path metric.*

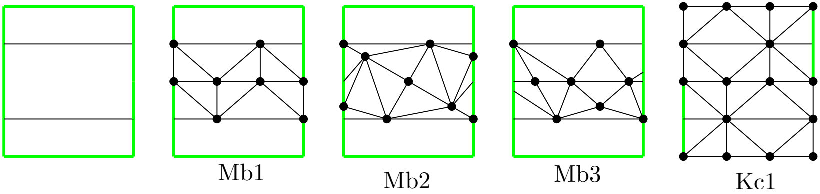


Let us check the 29 irreducible triangulations of the Klein bottle.

Some irreducible triangulations of the Klein bottle...



...and some more



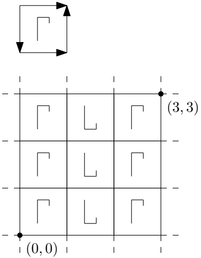
But...

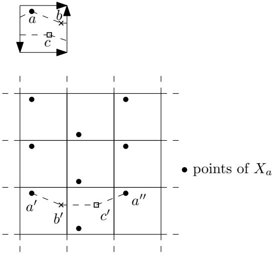
Theorem

The 1×1 flat metric with scheme $aba^{-1}b$ is not a universal shortest path metric.

<u>Proof</u>:

- Kc_1 has a unique embedding in the Klein bottle.
- This embedding contains a non-trivial separating cycle of length 3. Such a cycle needs to have "horizontal" length at least 2.





The ugly

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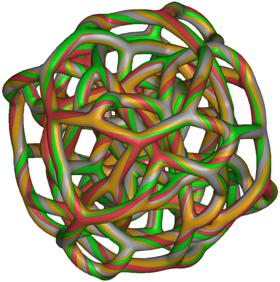

Positive genus

- Irreducible triangulations become non-tractable. (396784 for S_2)
- Relax the problem: look for embeddings with concatenations of k shortest paths $\rightarrow$ *k-universal shortest path metrics*.
- Hard to find any metric for which it does not work. Maybe they all do?

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Theorem

With probability tending to 1 as $g \rightarrow \infty$, a random hyperbolic metric is not a universal shortest path metric.

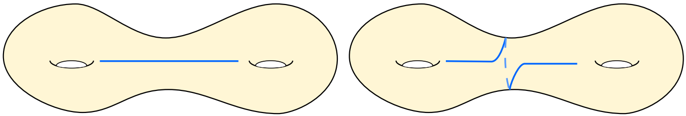
Actually, the result is stronger:

Theorem

For any $\varepsilon > 0$, with probability tending to 1 as $g \rightarrow \infty$, a random hyperbolic metric is not a $O(g^{1/3-\varepsilon})$-shortest path metric.

Random metric?

- The space of hyperbolic metrics up to isotopy on a surface of genus g is the *Teichmüller space* $\mathcal{T}_g$ of the surface.
- For our problem, two hyperbolic metrics related by an isometric homeomorphism are equivalent.

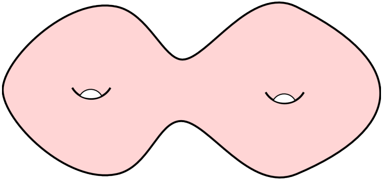


$\rightarrow$ We quotient by the action of the group of homeomorphisms (the *Mapping class group*).

- We obtain the *Moduli space* $\mathcal{M}_g$.
- This moduli space can be endowed with the *Weil-Petersson* metric, for which $\mathcal{M}_g$ has finite volume. $\rightarrow$ Probability space.

Pants decomposition

A *pants decomposition* is a family of disjoint closed curves on a surface cutting it into *pairs of pants*.



Properties of random metrics

Theorem (Mirzakhani)

With probability tending to 1*, the diameter of a surface with a random hyperbolic metric is* $O(\log g)$.

- Follows from precise estimates on the volume of moduli space.

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Theorem (Guth, Parlier and Young)

For any $\varepsilon > 0$*, a pants decomposition of a surface with a random hyperbolic metric has length* $\Omega(g^{7/6-\varepsilon})$ *with probability tending to* 1*.*

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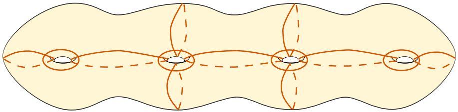
- Follows from Wolpert's formula, linking the Weill-Petersson form with pants decompositions.
- We build a graph G with $O(g)$ edges containing a pants decomposition (in all of its possible embeddings) and a $O(g^{1/6-\varepsilon})$ lower bound follows.
- We get to $O(g^{1/3-\varepsilon})$ with a bit more work.

A relaxed upper bound

Theorem

For every $g > 1$, there exists a $O(g)$-universal shortest path hyperbolic metric on the orientable surface of genus g.

- Starting tool: *octagonal decompositions*.

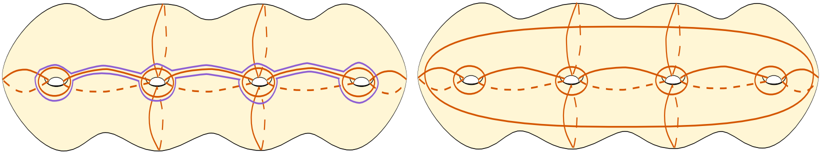


Theorem (É. Colin de Verdière, Erickson)

Let G be a graph embedded on S_g, there exists an octagonal decomposition Γ such that each edge of G crosses each curve of Γ a constant number of times.

From octagons to hexagons

- We upgrade these to *hexagonal decompositions*.

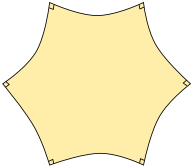


Corollary

Let G be a graph embedded on S_g, there exists a hexagonal decomposition Δ such that each edge of G crosses the curves of Δ at most $O(g)$ times.

The hyperbolic metric

- We endow each hexagon with the hyperbolic metric of equilateral right-angled hexagons.



- We reembed G separately in each hexagon with shortest paths. $\rightarrow$ We need a hyperbolic Tutte theorem with a boundary.

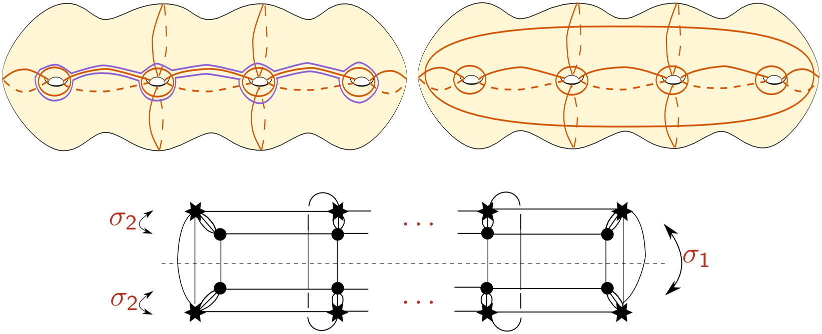
Theorem (Almost Y. Colin de Verdière)

Let G be a graph embedded as a triangulation in a hyperbolic hexagon H endowed with the metric m_H. If there are no dividing edges in G, then G can be embedded with geodesics, with the vertices on the boundary of H in the same positions as in the initial embedding.

The exchange argument

Lemma

Geodesics in the hexagons are shortest paths in the surface.



We mirror shortest paths until they stay in a single hexagon.

The unknown

- For positive genus, do there exist universal shortest path metrics?
  $\rightarrow$ Possible candidates: extremal metrics (Buser-Sarnak, Philips-Osgood-Sarnak).
- Can we prove lower bounds for explicit metrics?
- Can we find two simple graphs G_1, G_2 embeddable on S such that in any embedding, some edge of G_1 crosses some edge of G_2 twice?

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