Contact System of Segments for Planar Graphs.

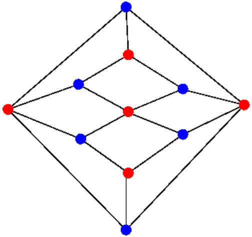
Daniel Gonçalves

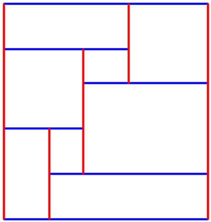
18 Janvier 2016

- Bipartite Planar Graphs
  - 2 Slopes & 2-Orientations
  - Prescribing Shapes
  - (New) Proof
- 3-Colorable Planar Graph
- Future Work

Theorem (Hartman et al. '91, and de Fraysseix et al. '94)

Every bipartite planar graph has a contact system by horizontal and vertical segments.





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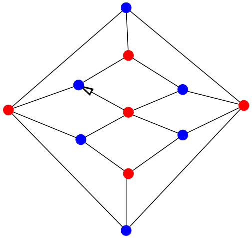

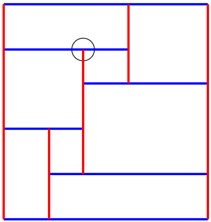

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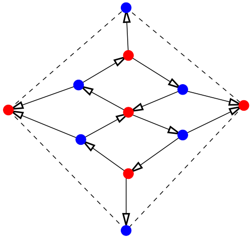

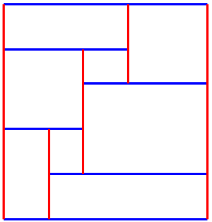

Folklore ?

These representations are in bijection with 2-orientations
(i.e. orientation of inner edges s.t. $\forall$ inner v, $d^+(v) = 2$).

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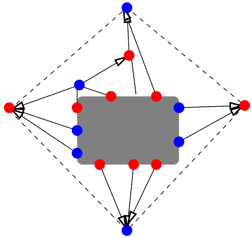

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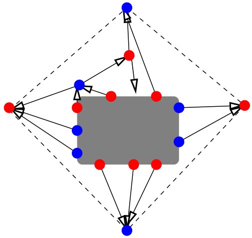

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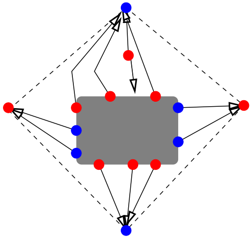

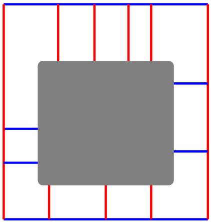

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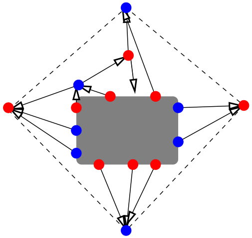

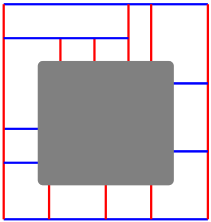

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Further Properties

Theorem (Felsner & Knauer)

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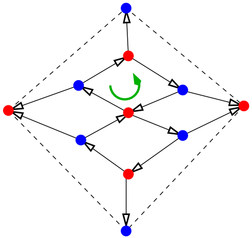

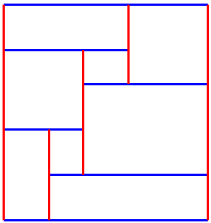

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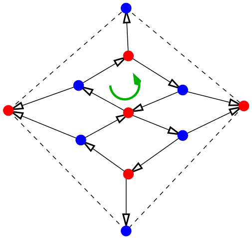

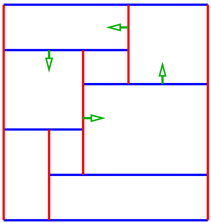

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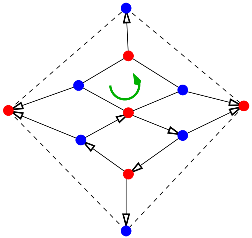

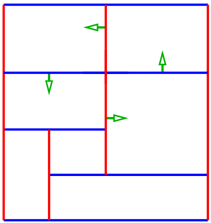

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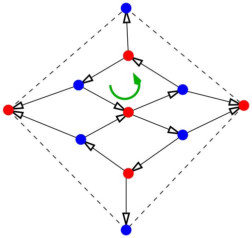

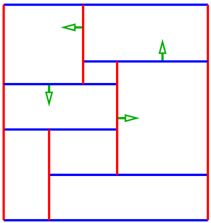

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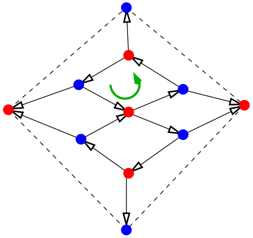

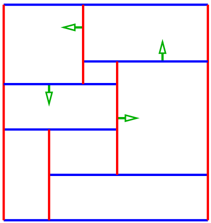

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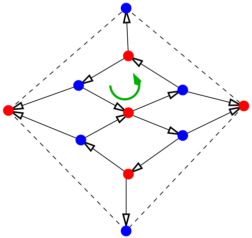

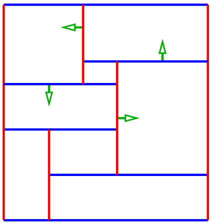

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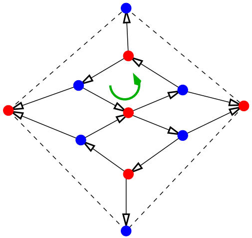

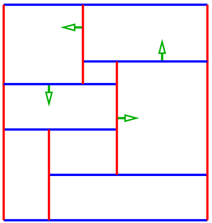

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- Induction on nb of inner faces

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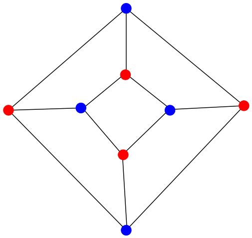
Square faces (by Felsner)

- Sometimes faces & segments are degenerated!



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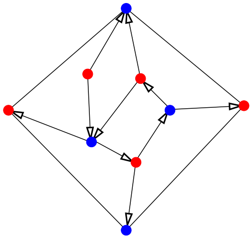

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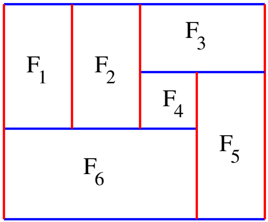

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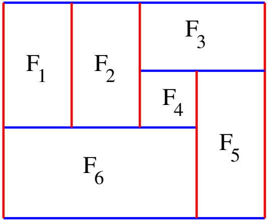

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\begin{array}{ccccccl}
F_1 & & & & & +F_6 & = 2 \\
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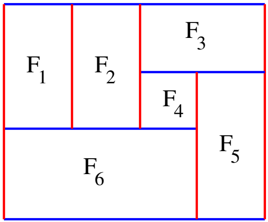

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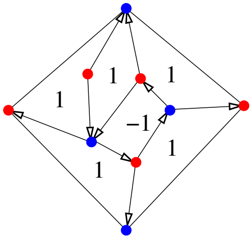

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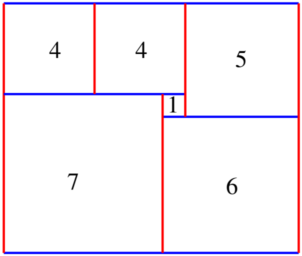
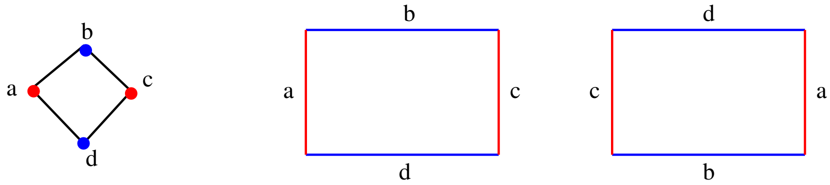


Using signed sizes

- A face *abcd* has two types of realizations.



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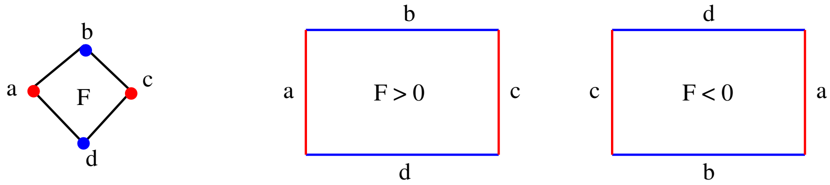

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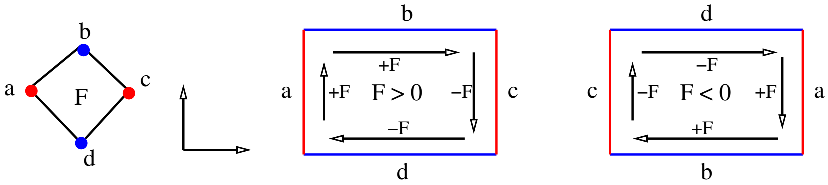

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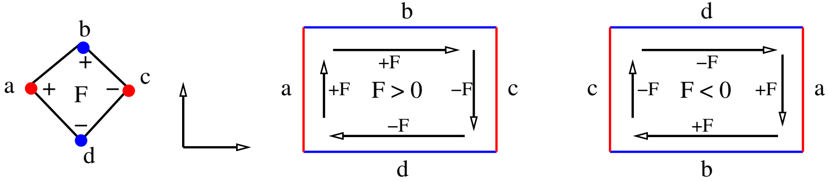

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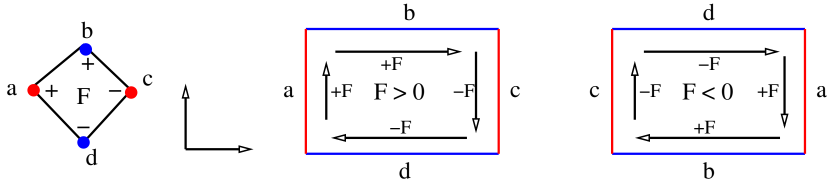

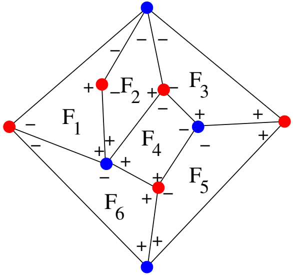

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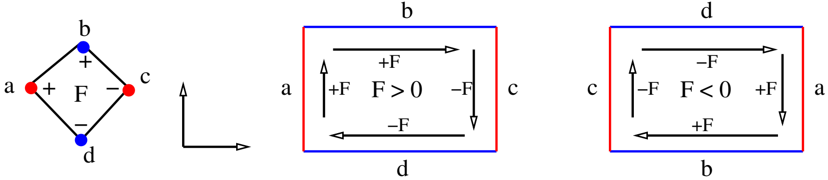

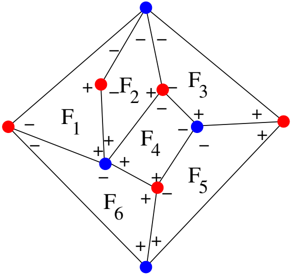

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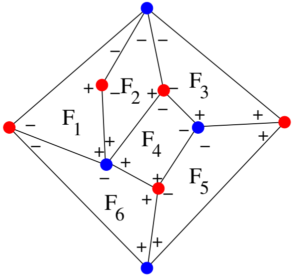

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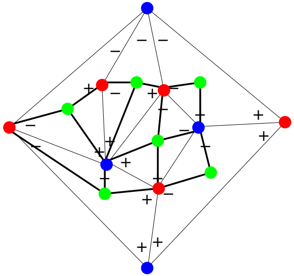

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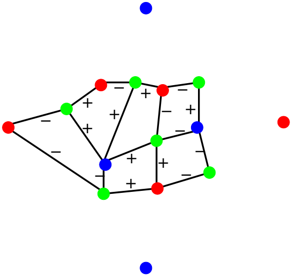

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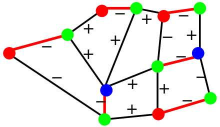

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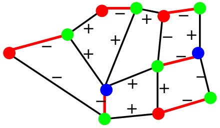

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- The set of perf. matchings in $Bip(M)$ is connected by C_4-flips.

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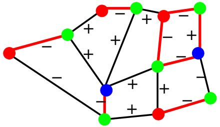

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- $\exists\sigma$ s.t. $0 \neq \prod_i M_{i,\sigma(i)}$ (i.e. $\exists$ matching in $Bip(M)$).
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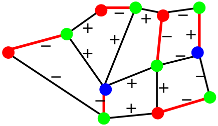

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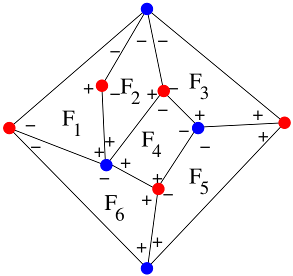

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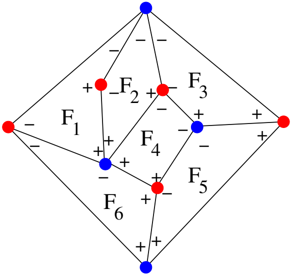

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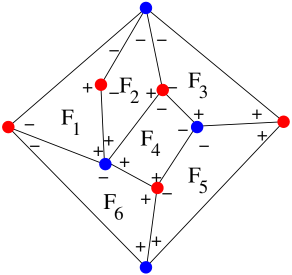

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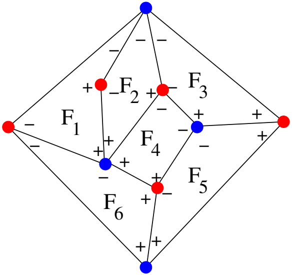

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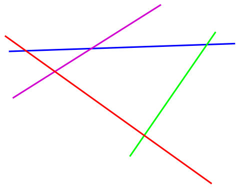
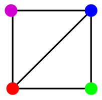
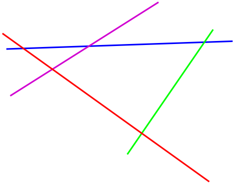
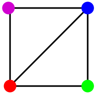


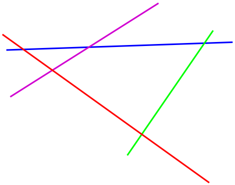


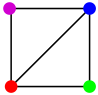




Theorem (Chalopin and G. '09)

Planar graphs are intersection graphs of segments.





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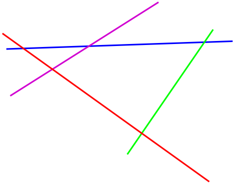
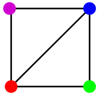
Every planar graph has a model in which there are only four possible slopes.

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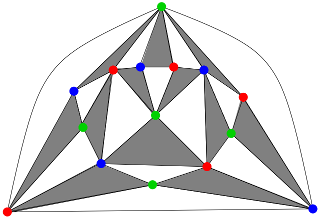
Open Problem (de Fraysseix *et al.*)

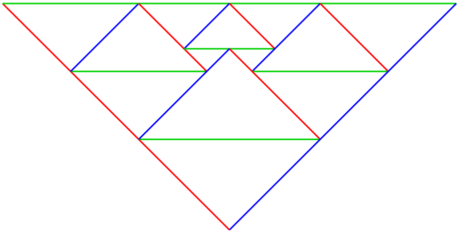
Does every planar graph G have a model in which there are only $\chi(G)$ possible slopes?

Eulerian Triangulations

Theorem (G. and de Visme '14)

Every Eulerian triangulation has a 3-slope contact system, with degeneracies.





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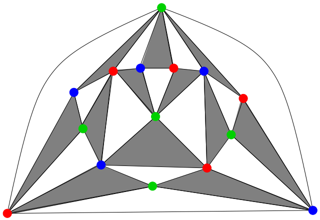

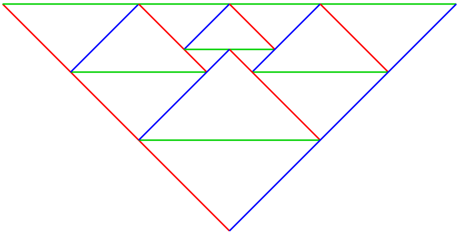

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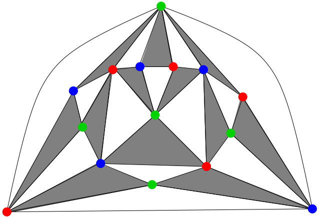

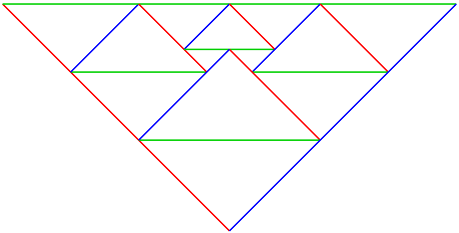

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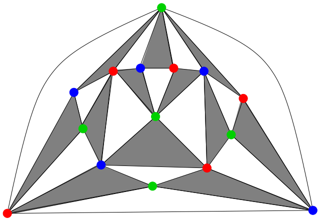

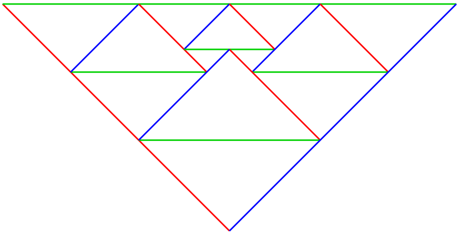

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- Followed by "Disambiguition Step".

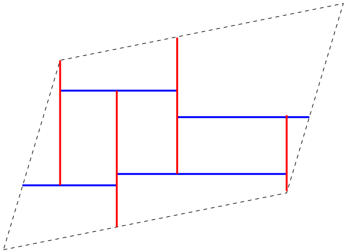
Bipartite Planar Graphs

- 2 Slopes & 2-Orientations
- Prescribing Shapes
- (New) Proof

3-Colorable Planar Graph

Future Work

Contact Systems on the Torus



Bipartite Quadrangulations:

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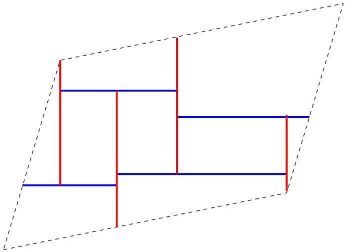

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- n vertices & n faces & $2n$ edges.

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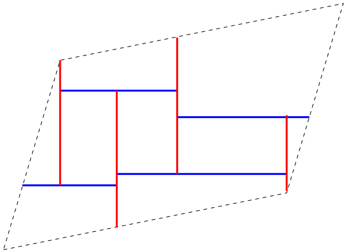

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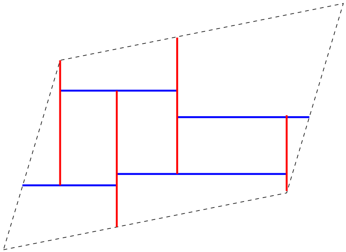

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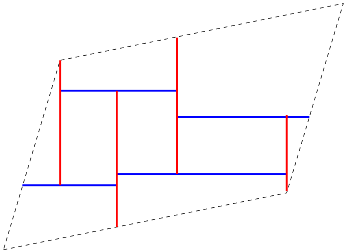

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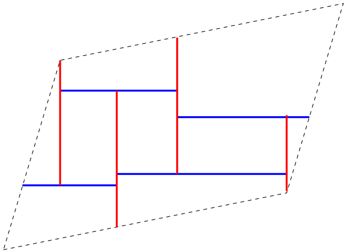

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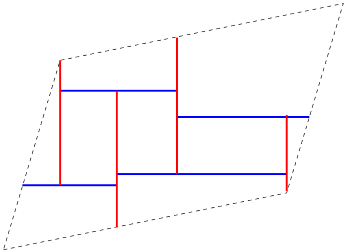

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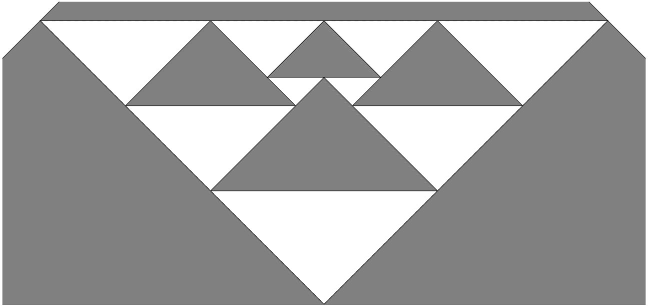
Probably not

Contact Systems of Homothetic Triangles



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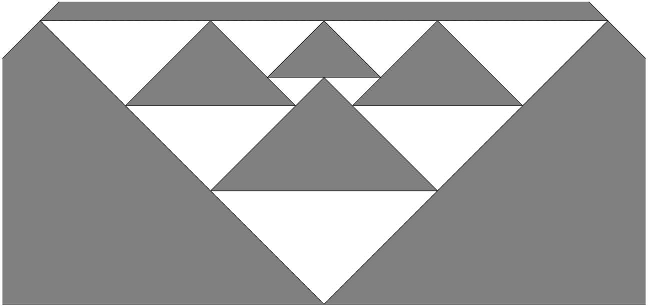
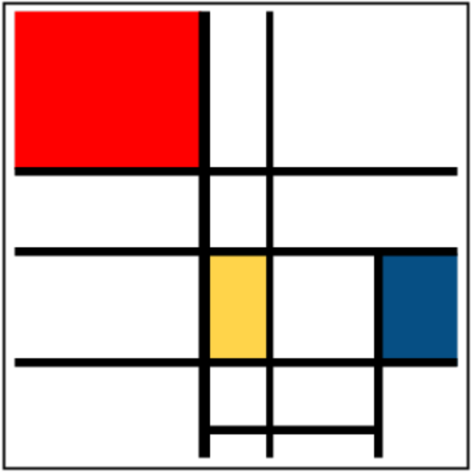


- Pb : 3 sides $\neq$ $+/-$ possibility



Thanks !