Box representations of embedded graphs

Louis Esperet

CNRS, Laboratoire G-SCOP, Grenoble, France

Journées Graphes & Surfaces, Grenoble

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Boxicity

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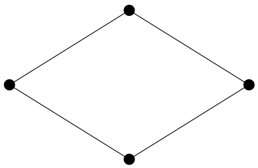

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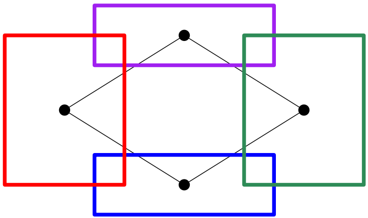

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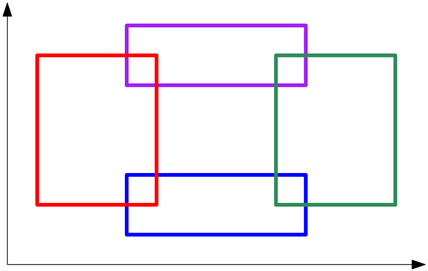

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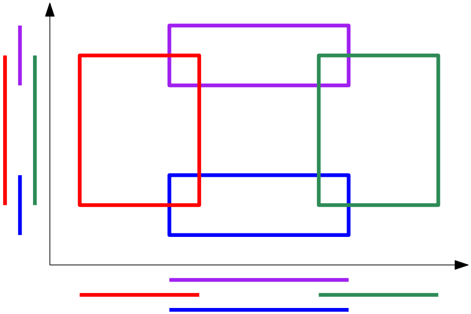

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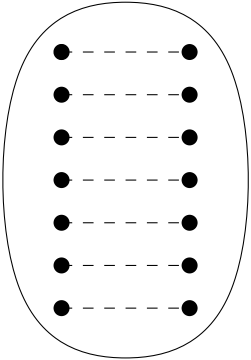
The boxicity of a graph $G = (V, E)$ is the smallest k for which there exist k interval graphs $G_i = (V, E_i)$, $1 \leq i \leq k$, such that $E = E_1 \cap \ldots \cap E_k$.

Graphs with large boxicity



K_n minus a perfect matching

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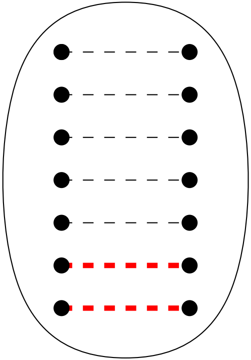

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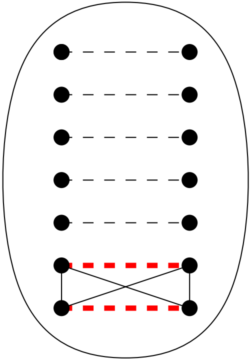

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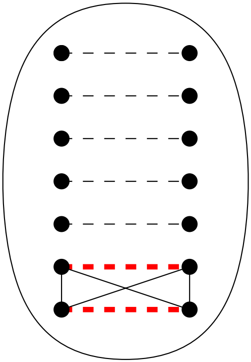

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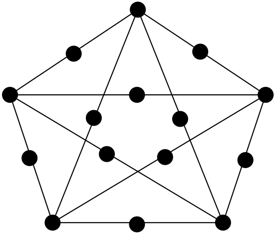
boxicity $n/2$

Graphs with large boxicity



Subdivided K_n

boxicity $\Theta(\log \log n)$

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Graphs with Euler genus g without non-contractible cycles of length at most $40 \cdot 2^g$ have boxicity at most 5.

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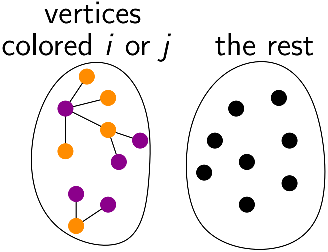

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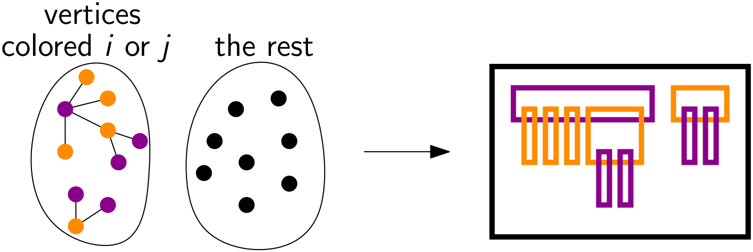

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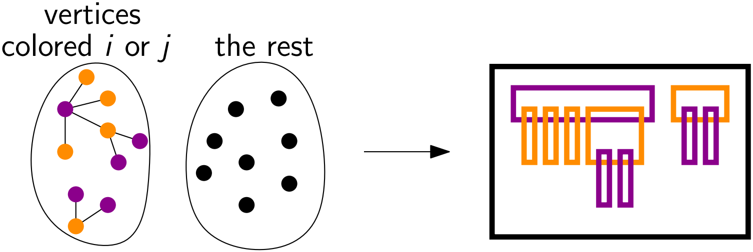

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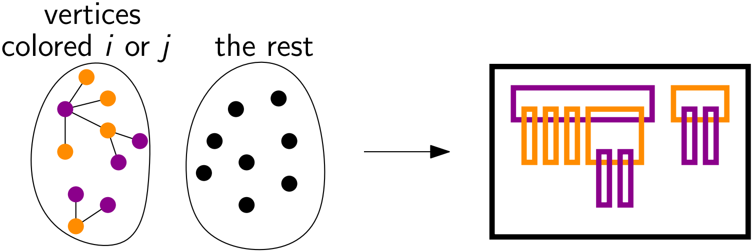

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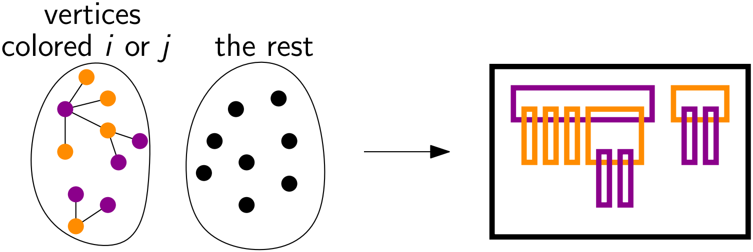

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Boxicity of graphs on surfaces

Theorem (Kawarabayashi, Thomassen 2012)

If a graph G has Euler genus g, then there is a set A of $O(g)$ vertices such that $G - A$ has an acyclic coloring with 7 colors.

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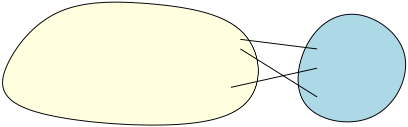

acyclic col. with 7 colors $\quad$ $O(g)$ vertices

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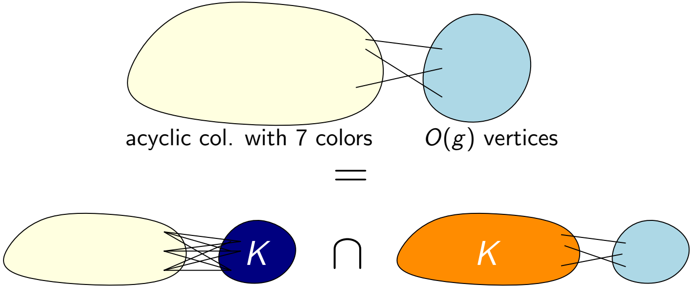

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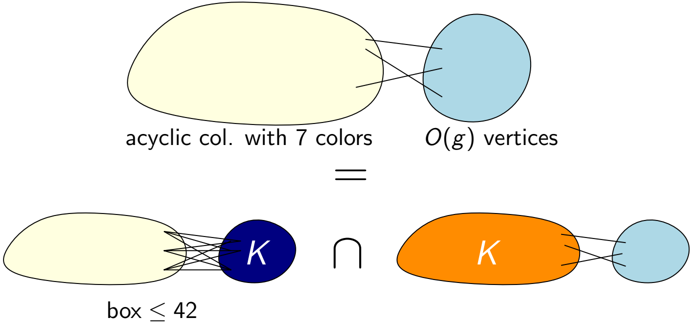

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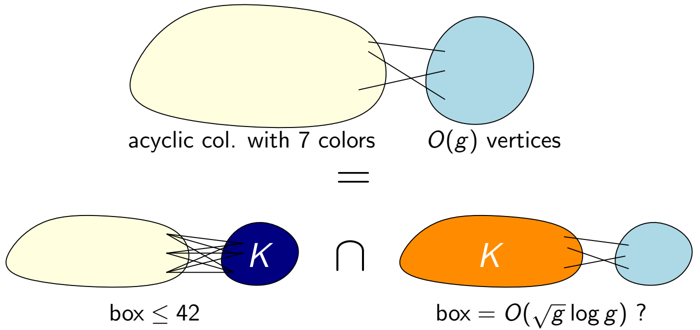

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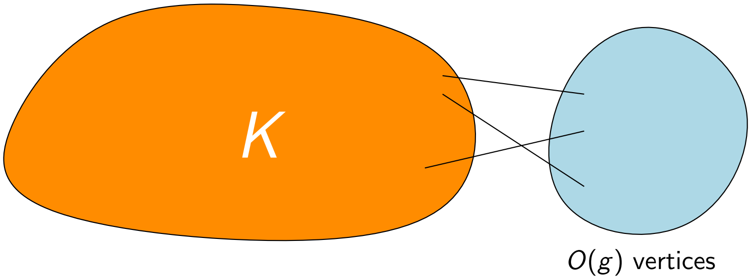

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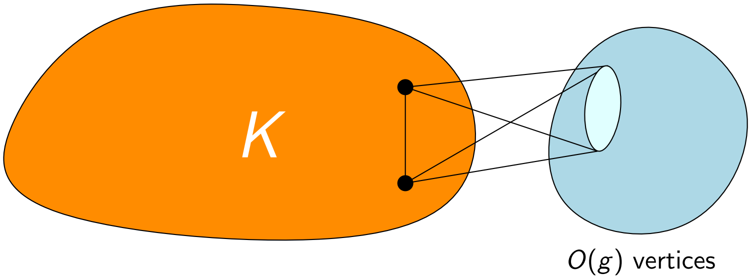

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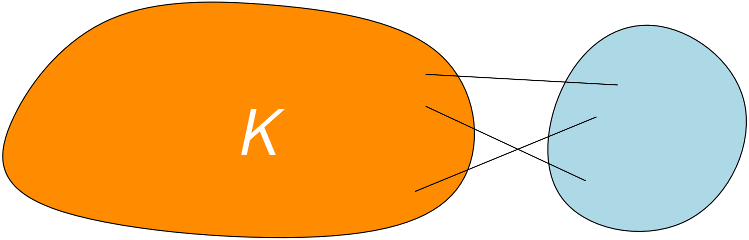

$O(g)$ vertices

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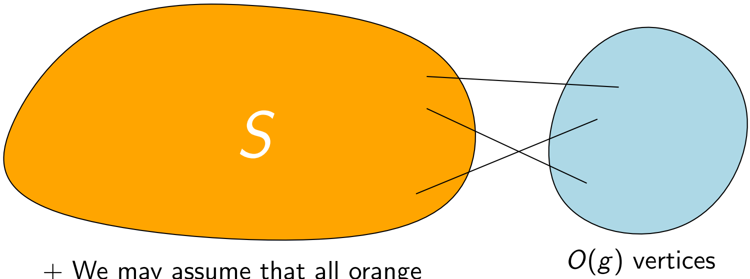

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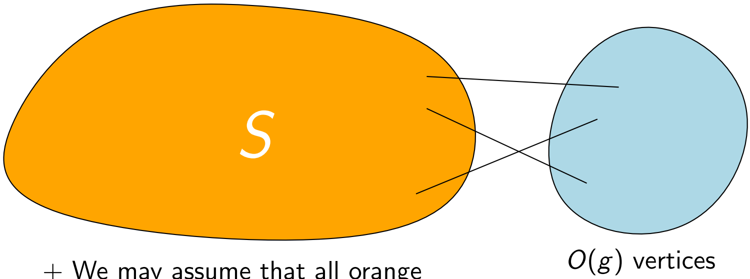

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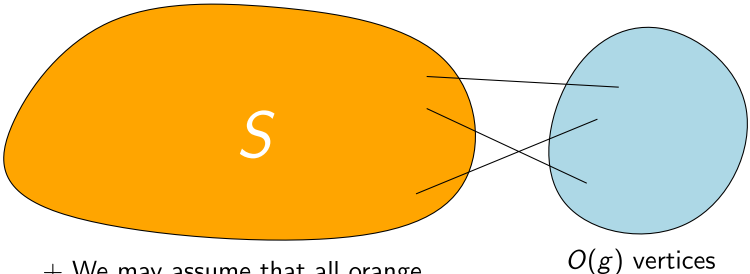

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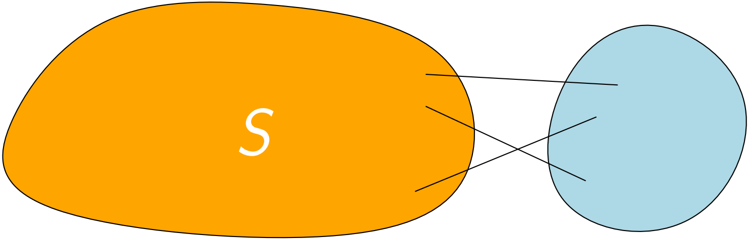

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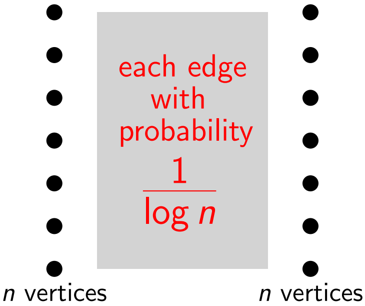
Theorem (Adiga, Chandran, Mathew 2014)

If a graph G with n vertices is k-degenerate, then $\text{box}(G) = O(k \log n)$.

Lower bound

Consider the following random bipartite graph G_n:



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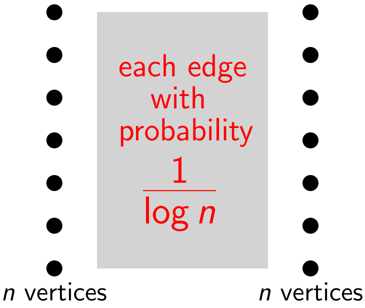

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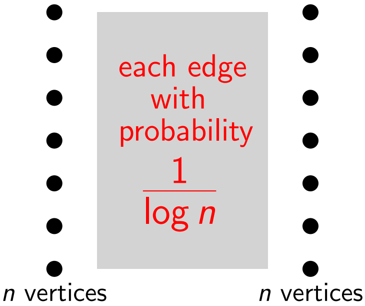

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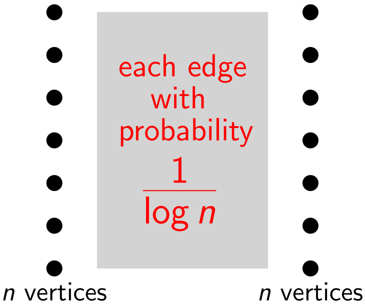

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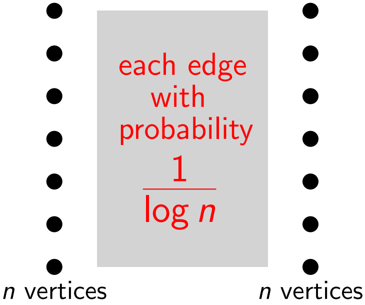

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It follows that $\mathrm{box}(G_n) = \Omega(\sqrt{g \log g})$.

Locally planar graphs

Theorem (E. 2015)

Graphs with genus g, without non-contractible cycles of length at most $40 \cdot 2^g$, have boxicity at most 5.

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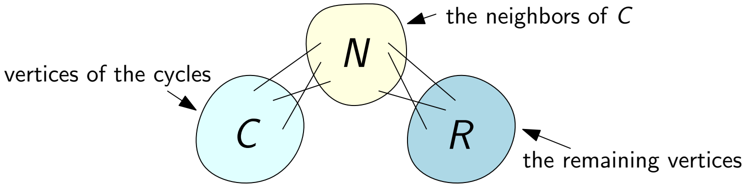

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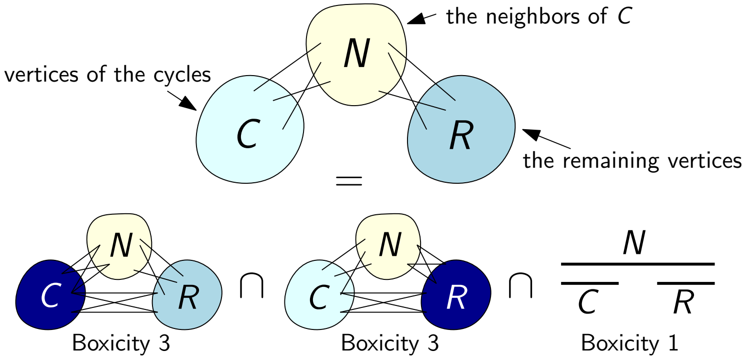

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For any proper minor-closed class $\mathcal{F}$, there is an integer $g = g(\mathcal{F})$ such that any graph of $\mathcal{F}$ of girth at least g has boxicity at most 3.

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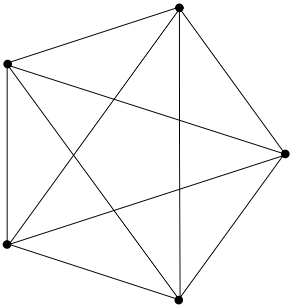

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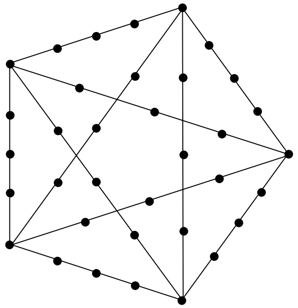

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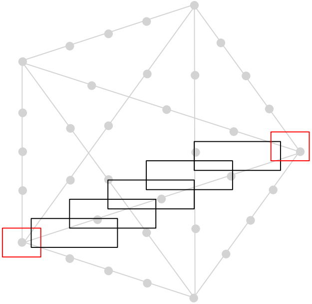

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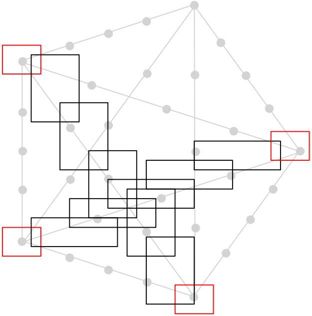

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Theorem (Galluccio Goddyn Hell. 2001)

For any proper minor-closed class $\mathcal{F}$, there is an integer $g = g(\mathcal{F})$ such that any graph of $\mathcal{F}$ of girth at least g has a vertex of degree at most one or a path with 5 internal vertices of degree 2.

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Theorem (E. 2015)

There is a constant c such that any graph of Euler genus g and girth at least $c \log g$ has boxicity at most 3.

Open problems

- What is the boxicity of K_t-minor-free graphs? (somewhere between $\Omega(t\sqrt{\log t})$ and $t^4(\log t)^2$)

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- What is the boxicity of toroidal graphs? (somewhere between 4 and 6)
- Is it true that locally planar graphs have boxicity at most 3?
- Is it true that if G has Euler genus g, then $O(g)$ vertices can be removed from G so that the resulting graph has boxicity at most 3? (it is true with 5 instead of 3)

The Colin de Verdière invariant

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- Outerplanar graphs have boxicity at most 2 (Scheinerman 1984).
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The random graphs seen earlier show that there are infinitely many graphs G with $\mathrm{box}(G) \geq \mu(G)\sqrt{\log \mu(G)}$.