Blossoming trees and planar maps

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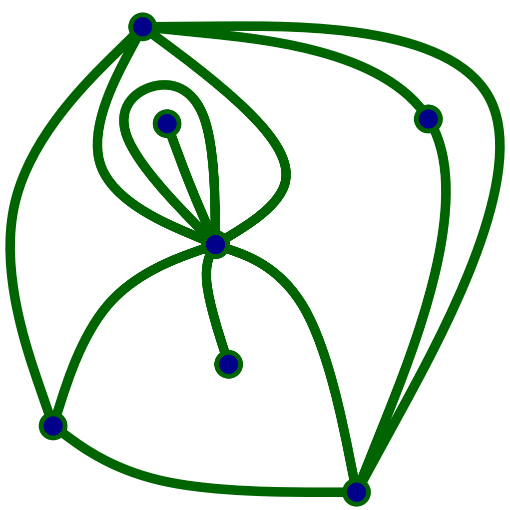
and Dominique Poulalhon (LIAFA)

Journées EGOS,
February, 2nd 2016

Plane Maps – Definition.

A **plane map** is the proper embedding of a finite connected graph in the plane seen up to continuous deformations.



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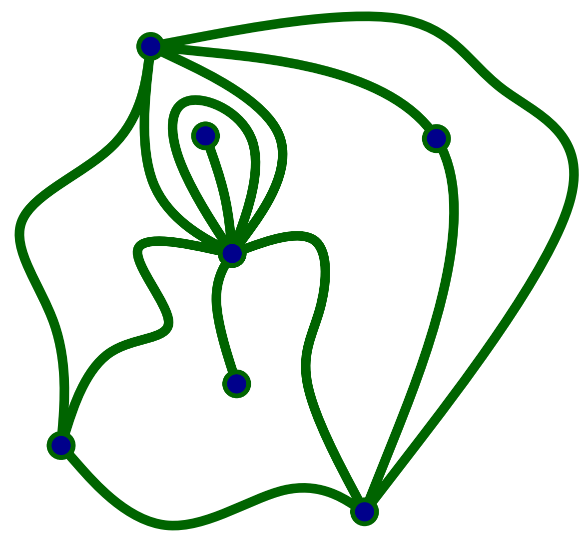

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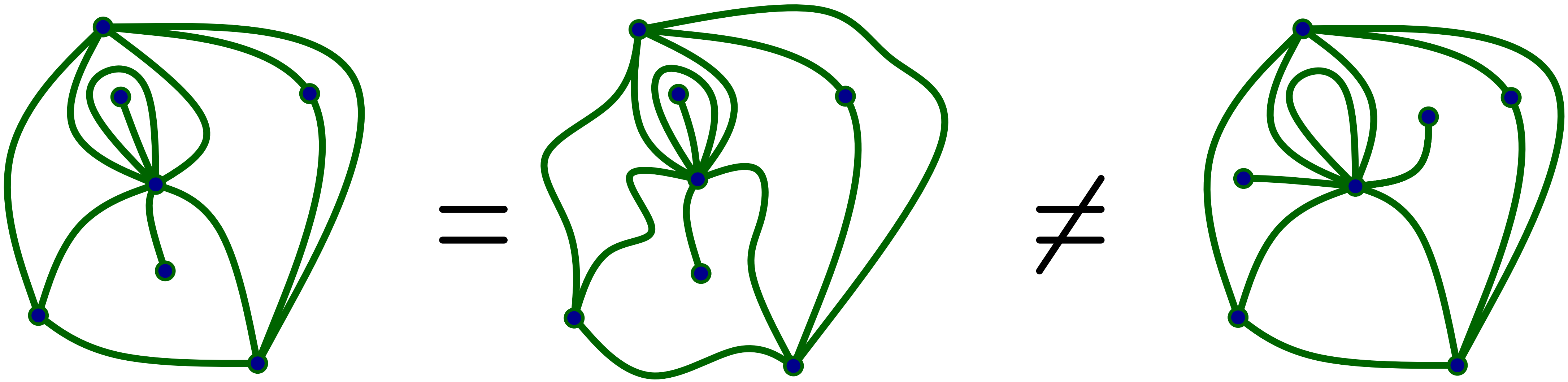

plane map = planar graph + cyclic order of neigbours around each vertex + choice of an outer face

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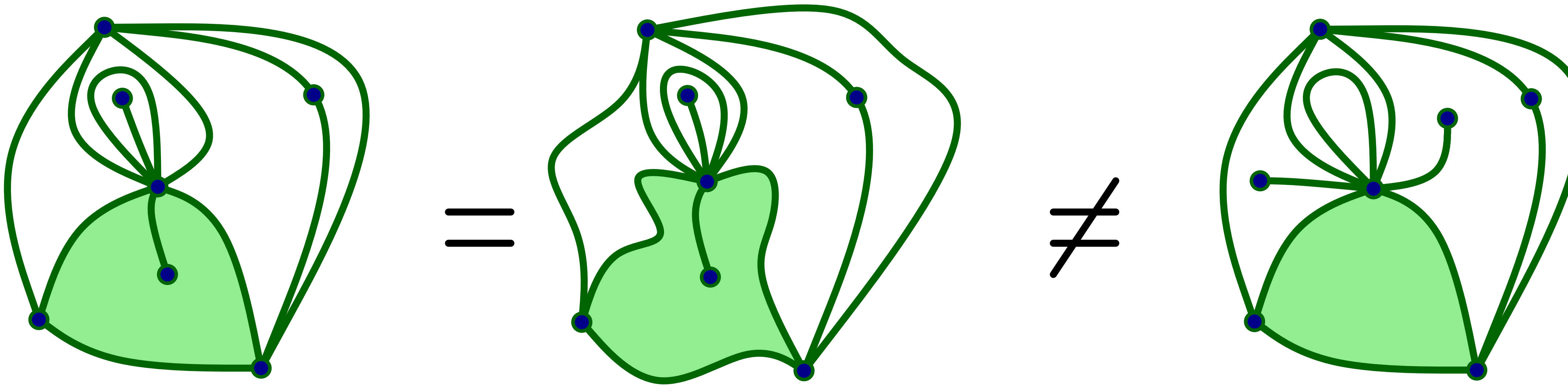

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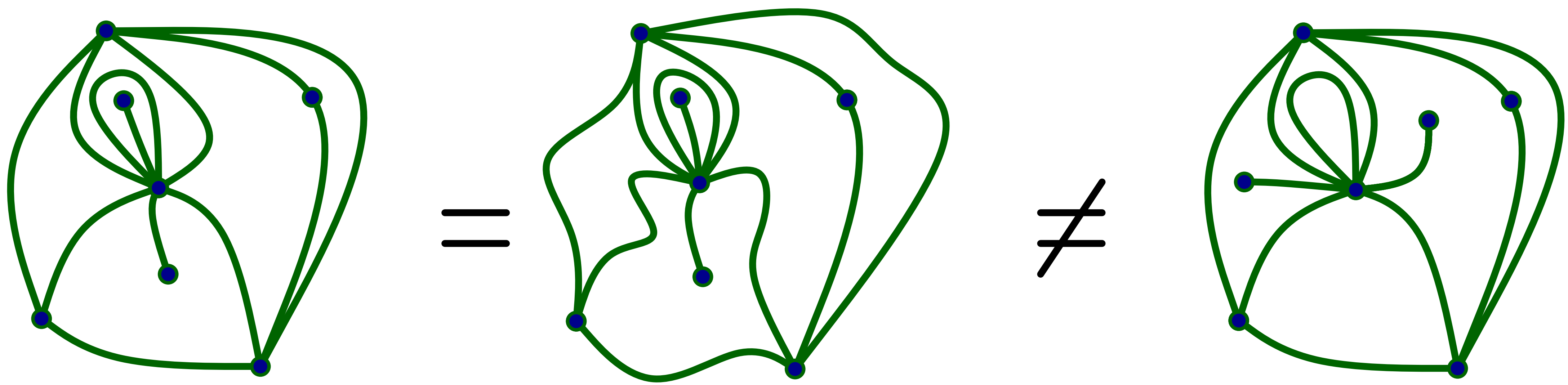

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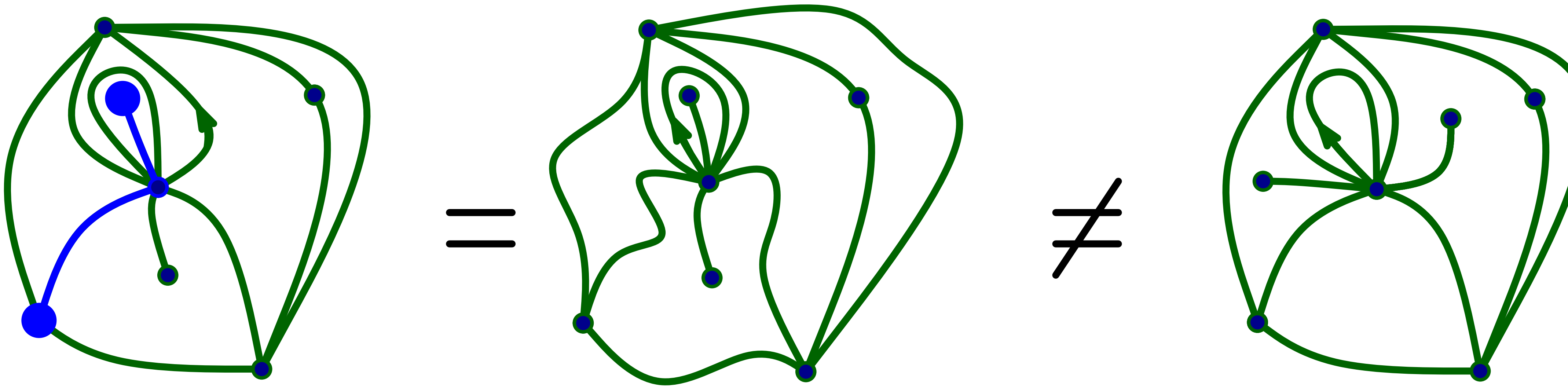
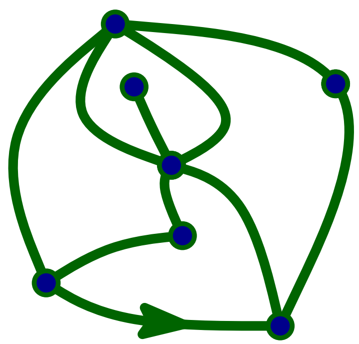


plane map = planar graph + cyclic order of neigbours around each vertex + choice of an outer face

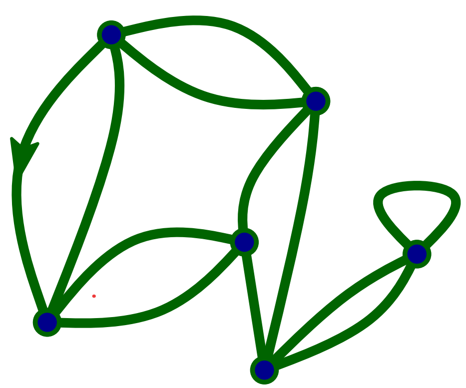
Plane maps are **rooted** : by orienting an edge.

Distance between two vertices = number of edges between them.
Plane map = **Metric space**

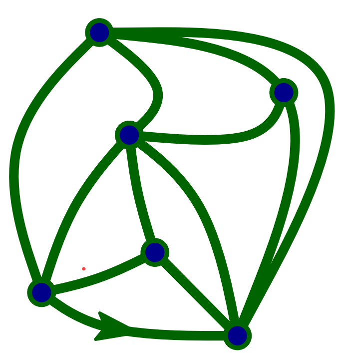
Which maps ?



Quadrangulations



4-regular maps



Simple triangulations (no loops nor multiple edges)

Maps enumeration 101

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Euler Formula : $\#\,\text{vertices} + \#\,\text{faces} = \mathbf{2} + \#\,\text{edges}$

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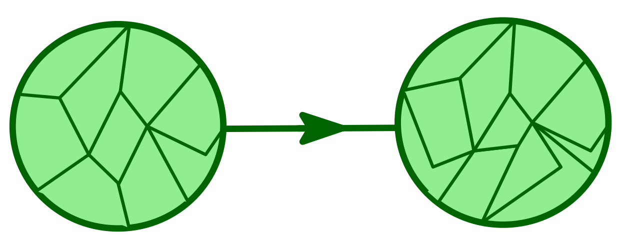
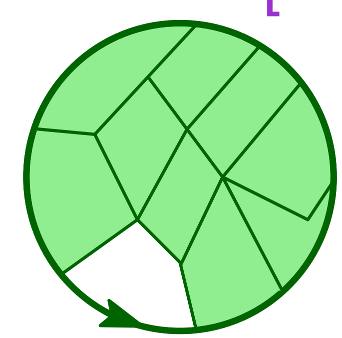
Structure allows recursive decomposition $\Rightarrow$ enumeration [Tutte, '60s].

Two possibilities:



The root edge is an isthmus

OR



The root edge is NOT an isthmus

$$q_n = \text{number of quadrangulations with } n \text{ faces} = \frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}$$

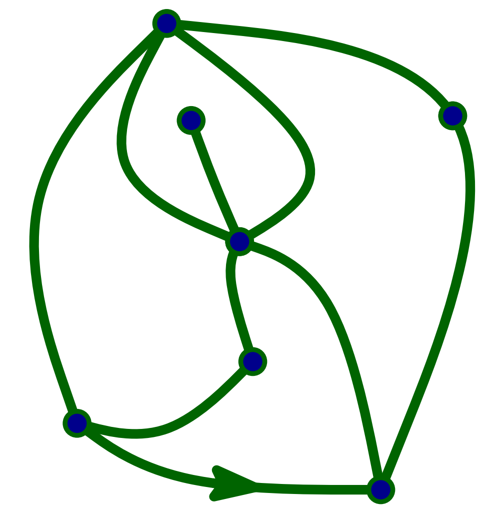
Random maps

$\mathcal{Q}_n = \{\text{Quadrangulations of size } n\}$

$= n+2$ vertices, n faces, $2n$ edges

$Q_n =$ Random element of $\mathcal{Q}_n$

$(V(Q_n), d_{gr})$ is a random compact metric space



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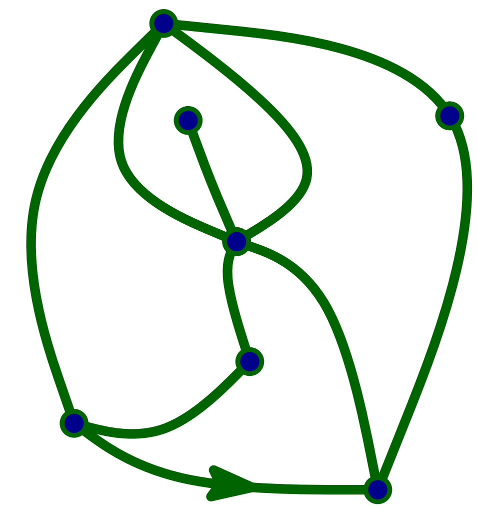
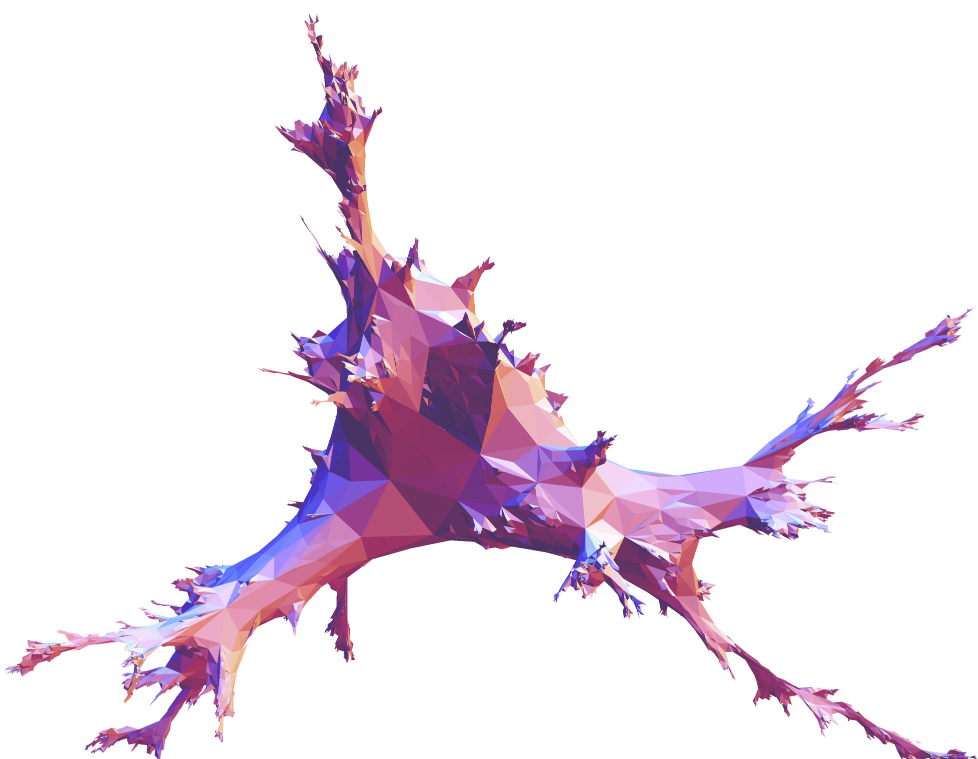
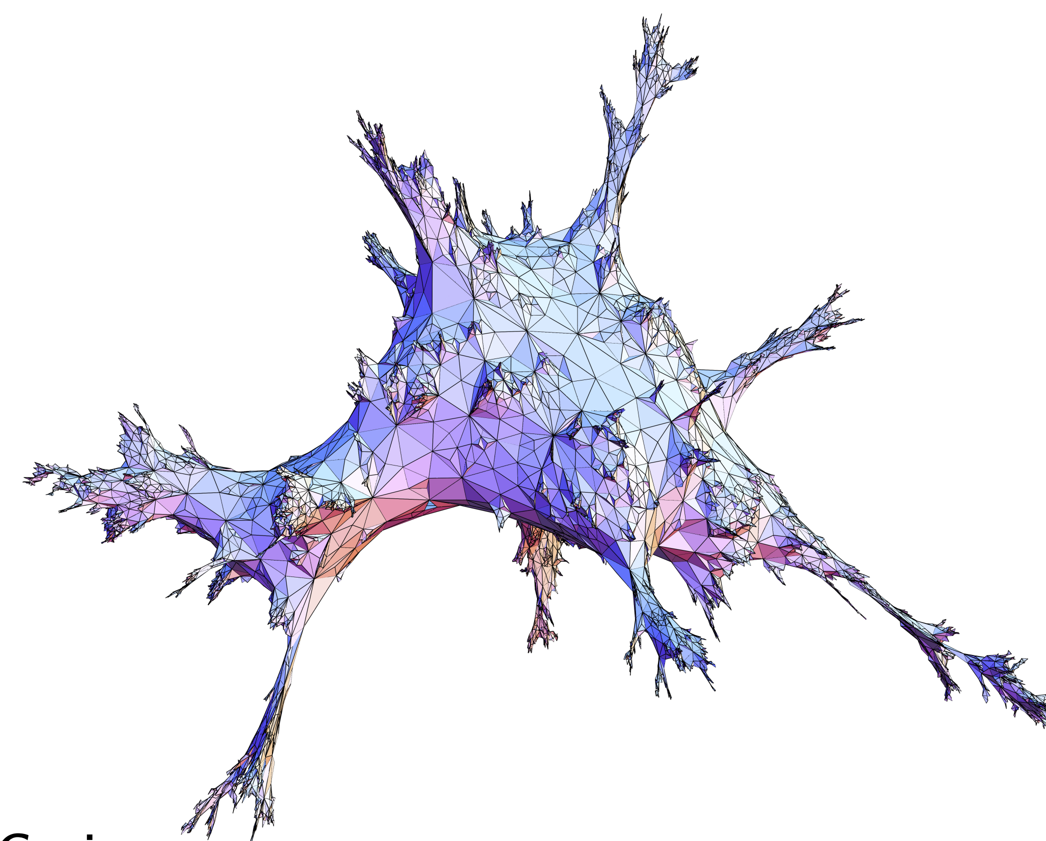


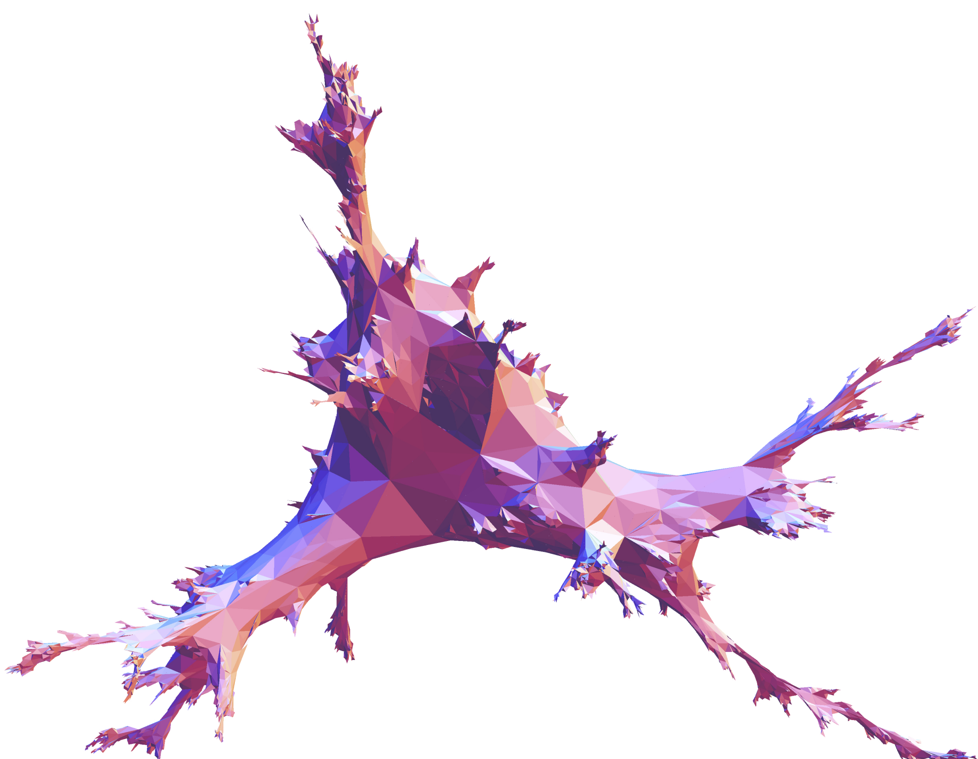


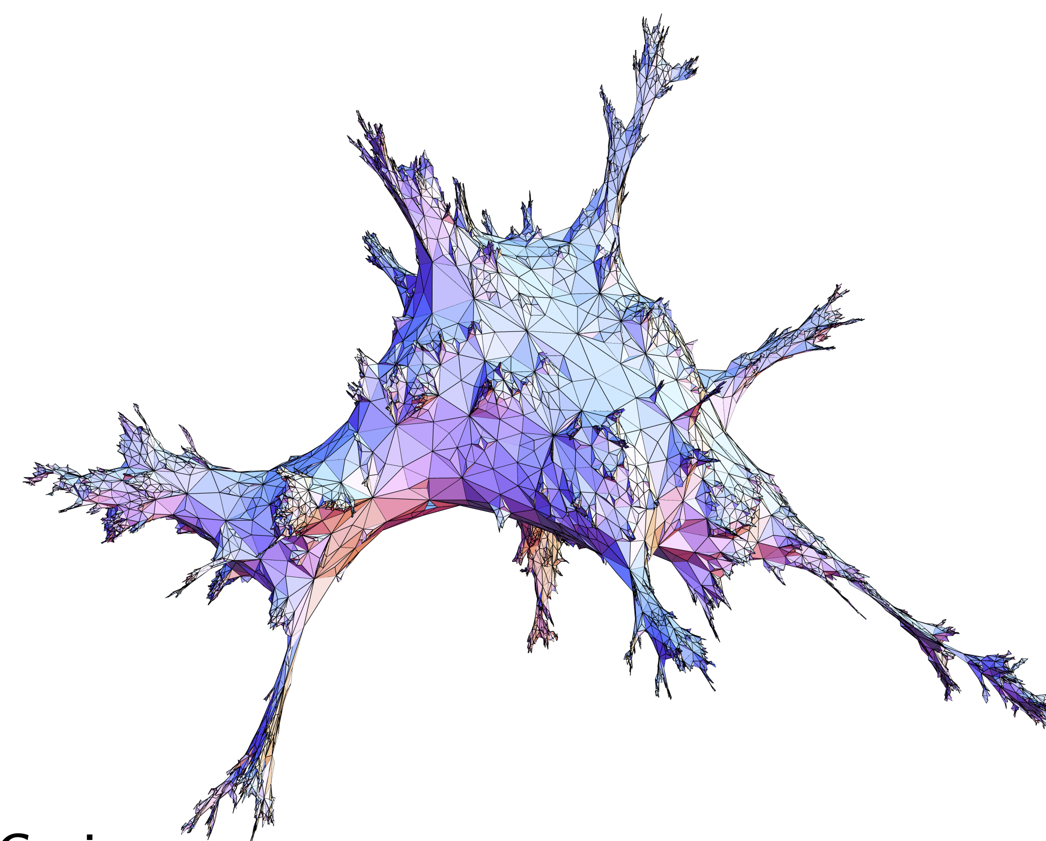


Simulations by N.Curien

Random maps

What is the behavior of Q_n when n goes to infinity ?
typical distances?
convergence towards a continuous object ?





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well understood:

• Schaeffer's bijection : quadrangulations $\leftrightarrow$ labeled trees.
Labels in the trees $=$ distances in the map.

• distance between two random points $\sim n^{1/4}$ + law of the distance
[Chassaing-Schaeffer '04]

• cvgence of normalized quadrangulations + limiting object: Brownian map.
[Marckert-Mokkadem '06], [Le Gall '07], [Miermont '08],
[Miermont 13], [Le Gall 13]

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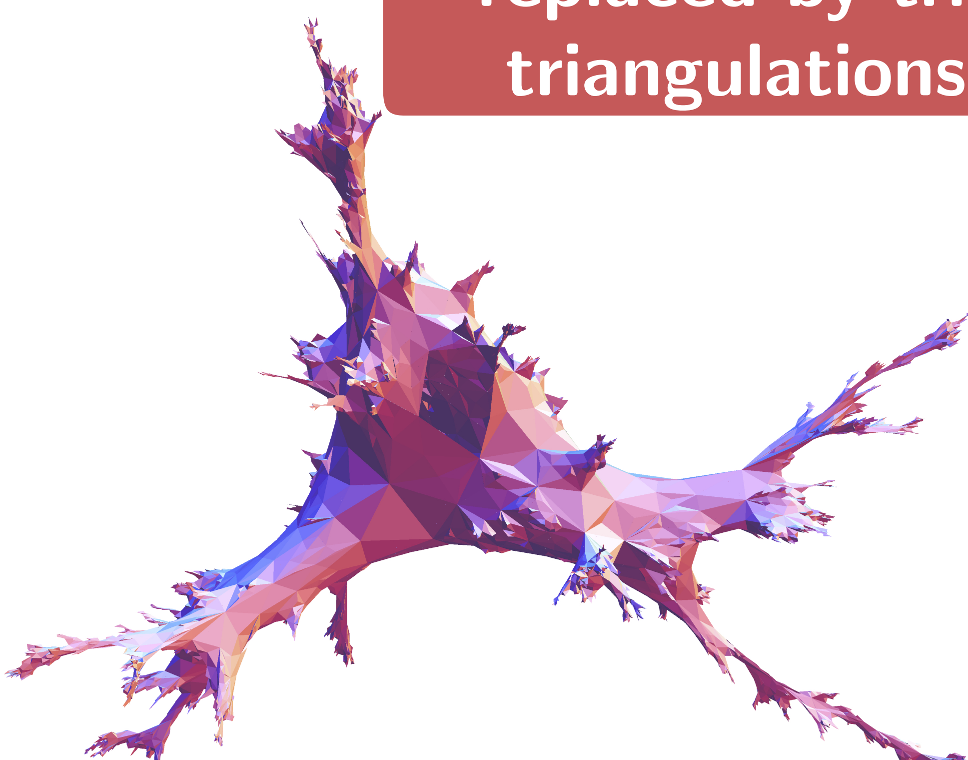
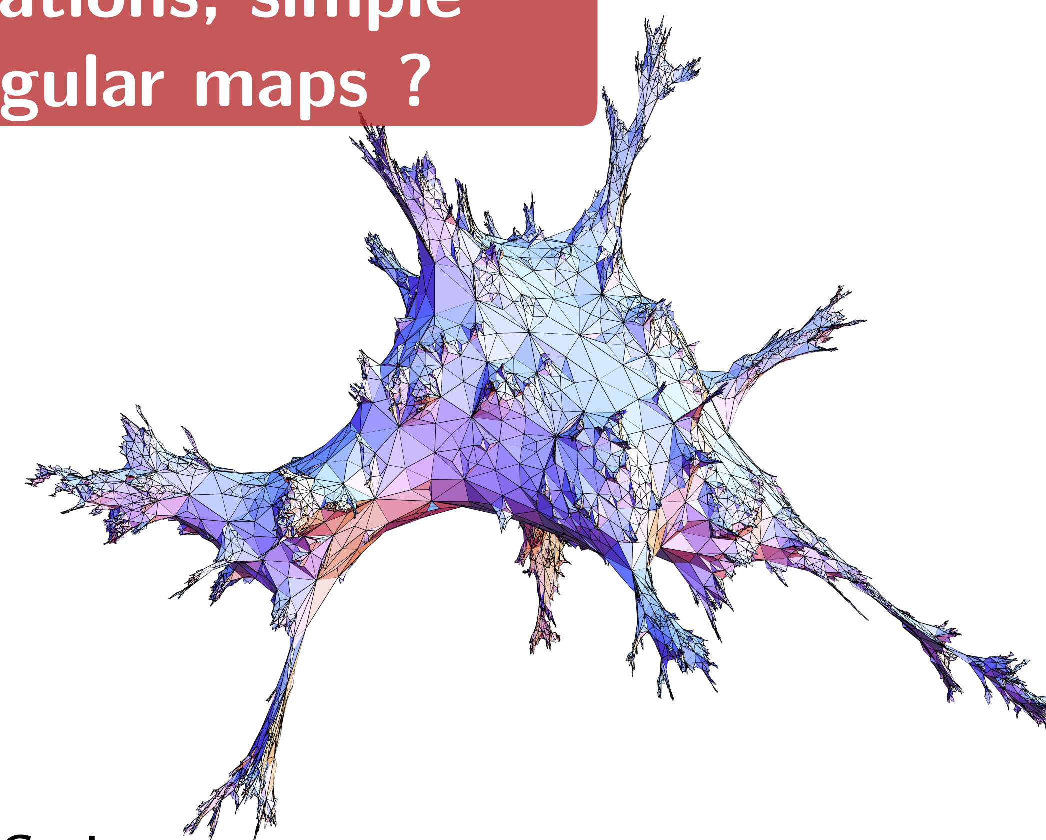
+ what if quadrangulations are replaced by triangulations, simple triangulations, 4-regular maps ?





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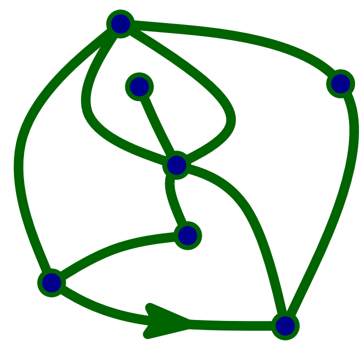
Idea : The Brownian map is a **universal** limiting object.
All "reasonable models" of maps (properly rescaled) are expected to converge towards it.

Problem : These results relie on nice bijections between maps and labeled trees [Schaeffer '98], [Bouttier-Di Francesco-Guitter '04].

Which maps ?

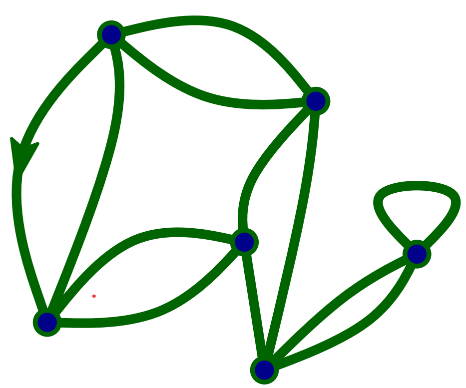


Quadrangulations

Number of quadrangulations with n faces:

$$q_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n}$$

[Tutte, 60], [Cori-Vauquelin '81], [Schaeffer '98]



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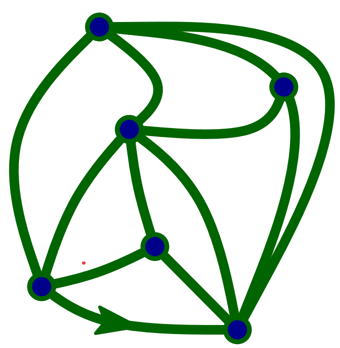

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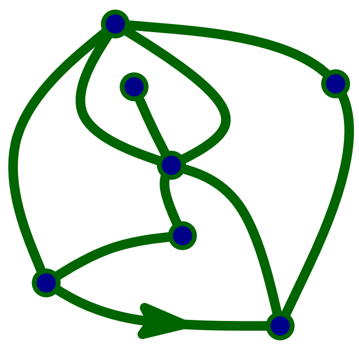

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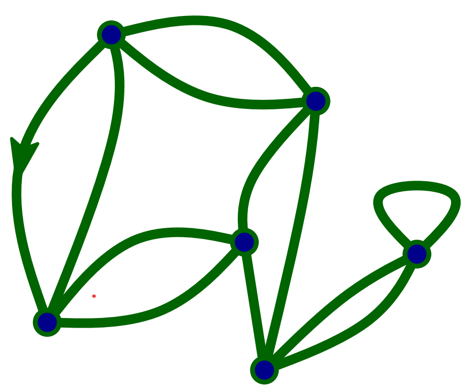
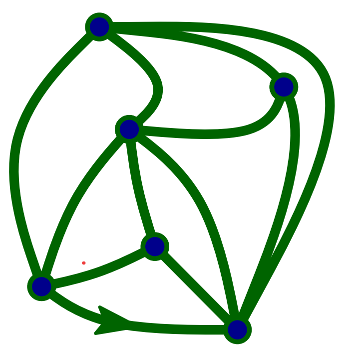


4-regular maps

Number of rooted 4-regular maps with n vertices:

$$R_n = \frac{2 \cdot 3^n}{n+1} \binom{2n}{n}$$

[Tutte, 62], [Schaeffer '97]



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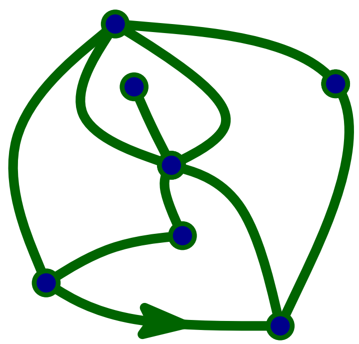

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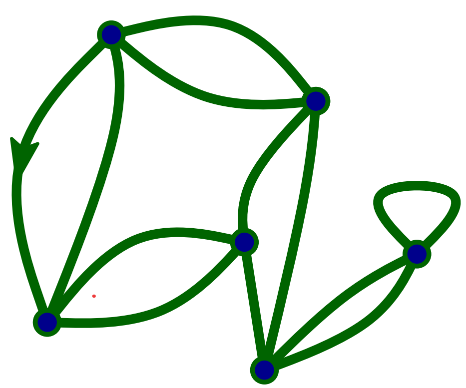

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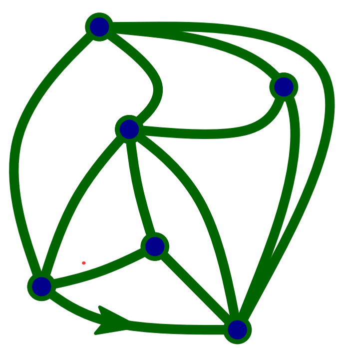

Simple triangulations (no loops nor multiple edges)

Number of simple triangulations with $n+2$ vertices:

$$\Delta_n = \frac{2 \cdot (4n-3)!}{n!(3n-1)!}$$

[Tutte, 62], [Poulalhon-Schaeffer '05]

History : what questions about maps ?

- **Enumerate them** : a lot of different techniques

Recursive decomposition: [Tutte, '60]

Matrix integrals: [t'Hooft, '74], [Brézin, Itzykson, Parisi and Zuber '78]

Representation of the symmetric group: [Goulden and Jackson '87].

Bijective approach with labeled trees: [Cori-Vauquelin '81], [Schaeffer '98], [Bouttier, Di Francesco and Guitter '04], [Bernardi and Fusy], ...
Bijective approach with blossoming trees: [Schaeffer '98], [Schaeffer and Bousquet-Mélou '00], [Poulalhon and Schaeffer '05], [Fusy, Poulalhon and Schaeffer '06], [Bernardi and Fusy]

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Bijective approach with **blossoming trees**.

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Today: what's the plan ?

What is a blossoming tree ?

Can we unify the constructions involving blossoming trees ?

Can we prove some convergence results to the Brownian map using blossoming trees ?
i.e. can we put "distances" on trees ?

Today: what's the plan ?

What is a blossoming tree ? **Wait a second**

Can we unify the constructions involving blossoming trees ?

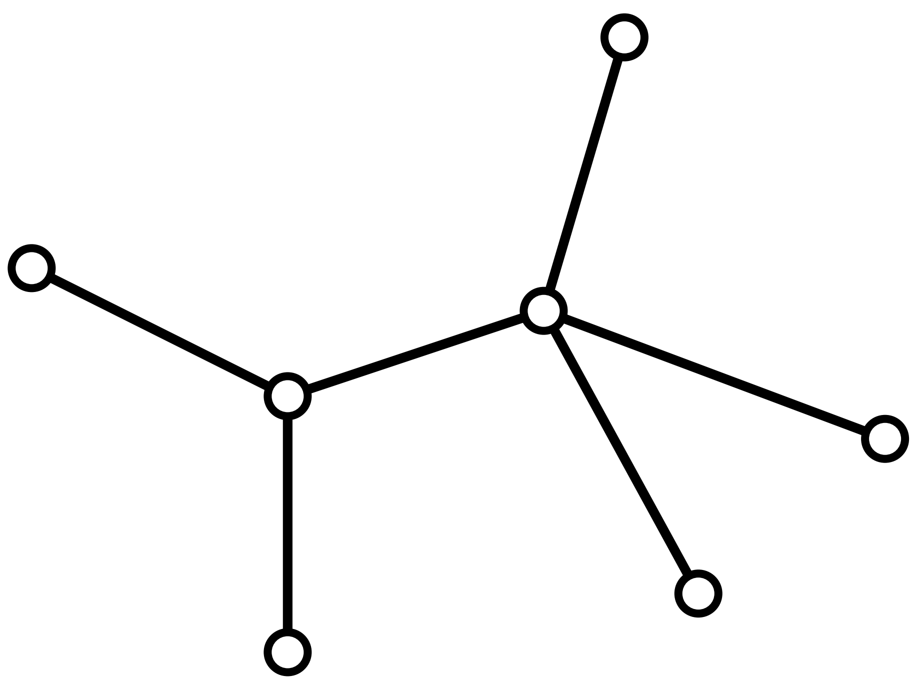
Yes, cf also [Bernardi,Fusy]

Can we prove some convergence results to the Brownian map using blossoming trees ?
i.e. can we put "distances" on trees ? **Yes ... for some models**

What is a blossoming tree ?

A **blossoming tree** is a plane tree where vertices can carry **opening stems** or **closing stems**, such that :

$$\# \text{ closing stems} = \# \text{ opening stems}$$



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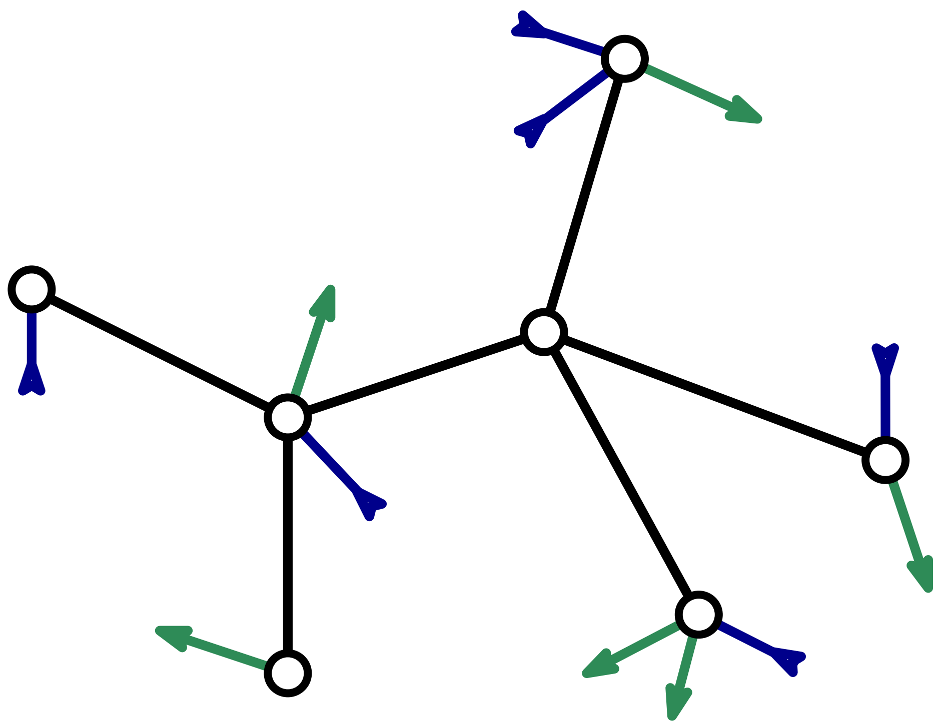

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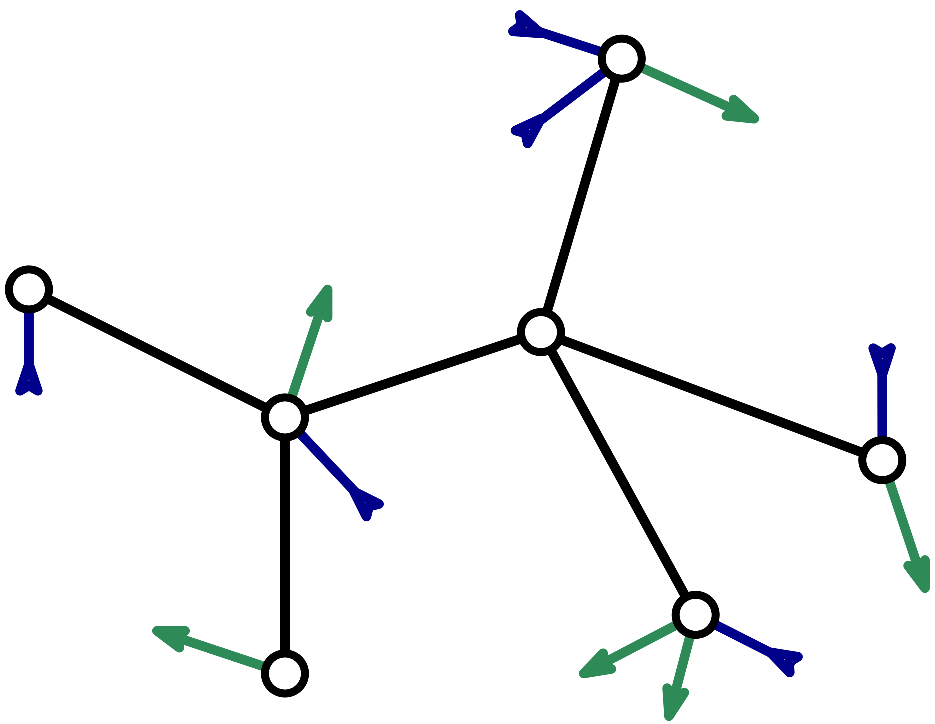

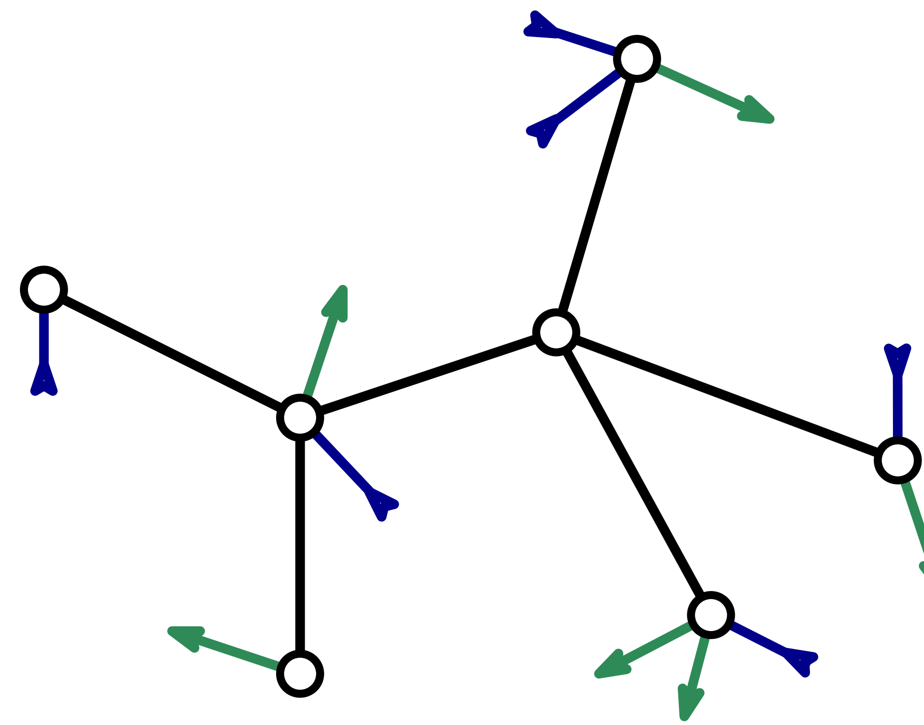

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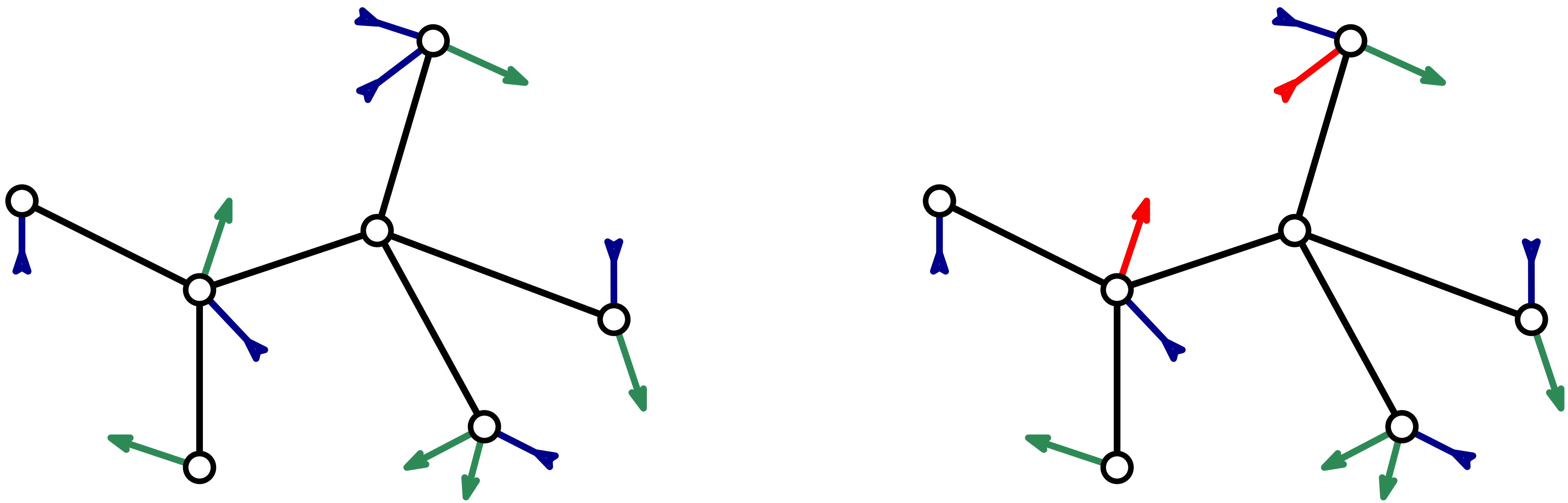

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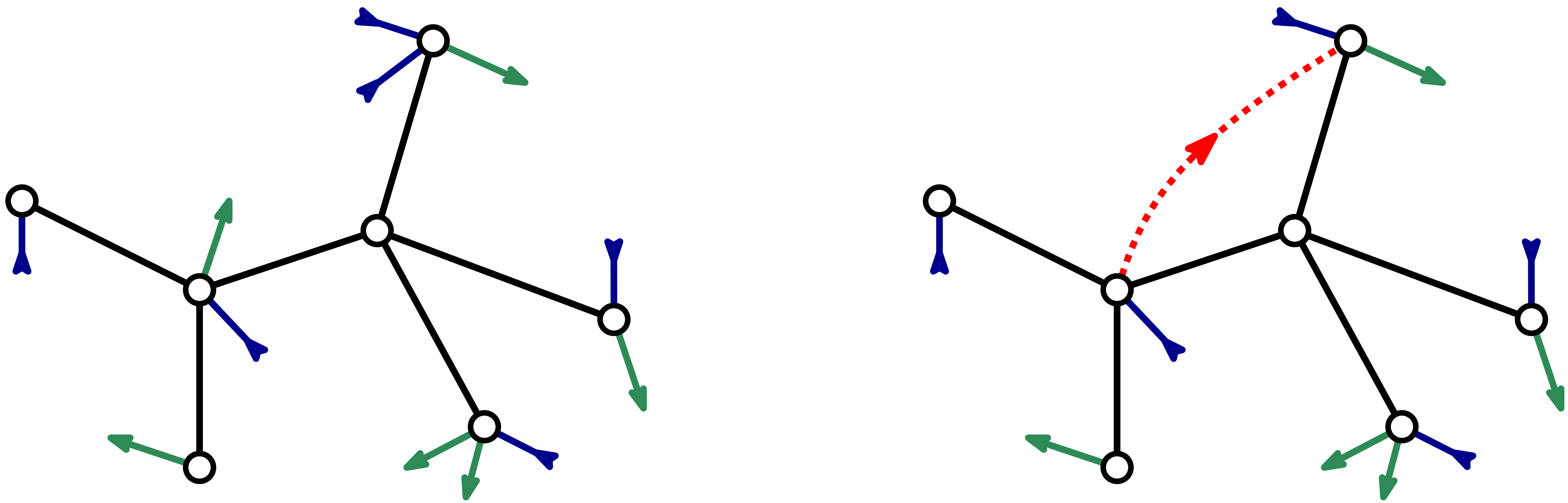

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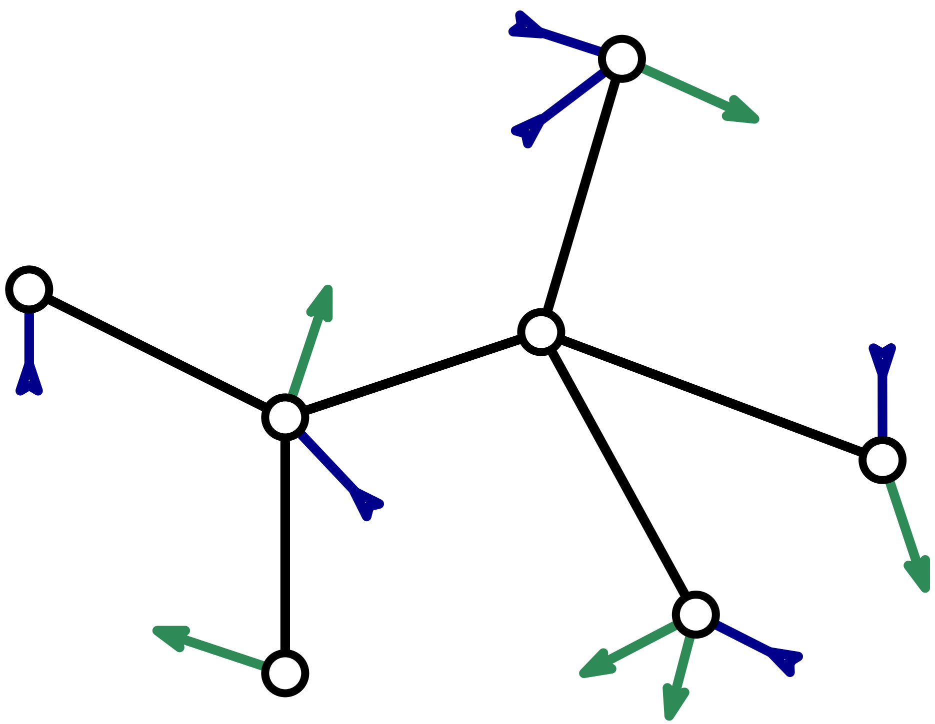

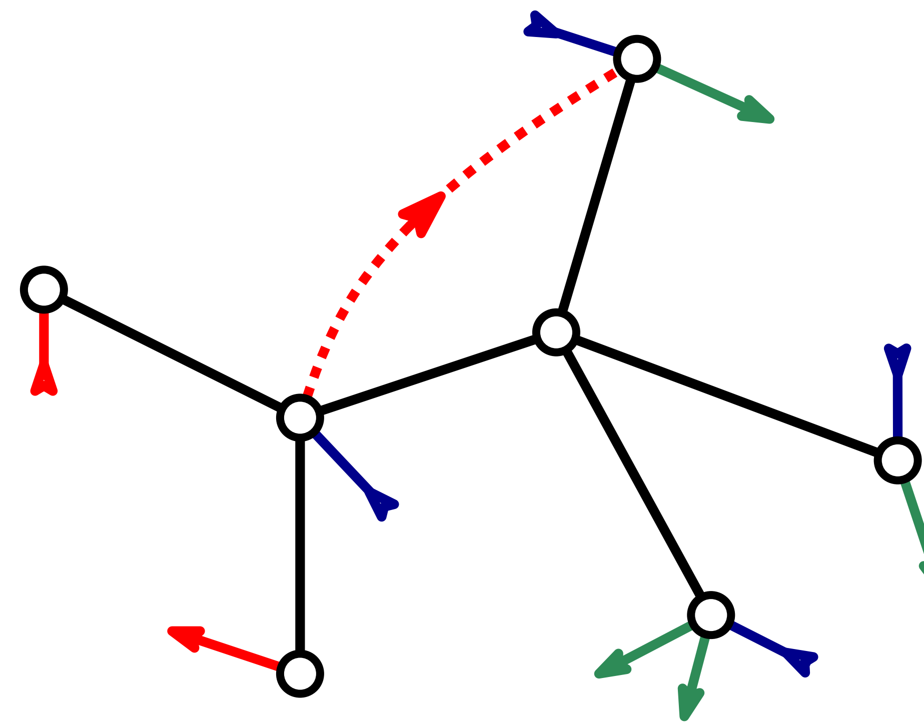

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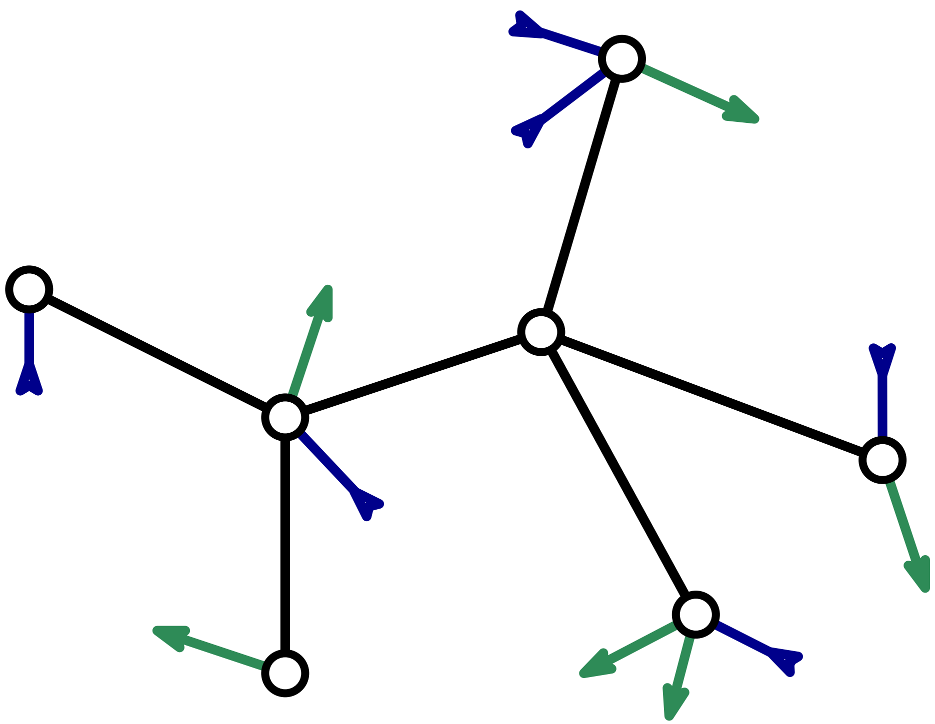

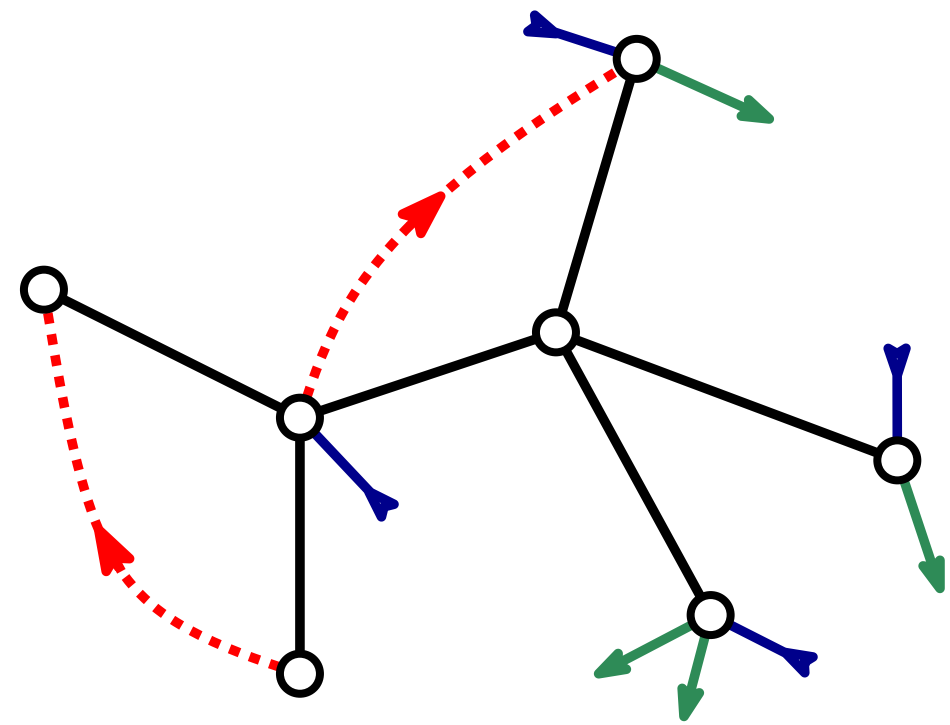

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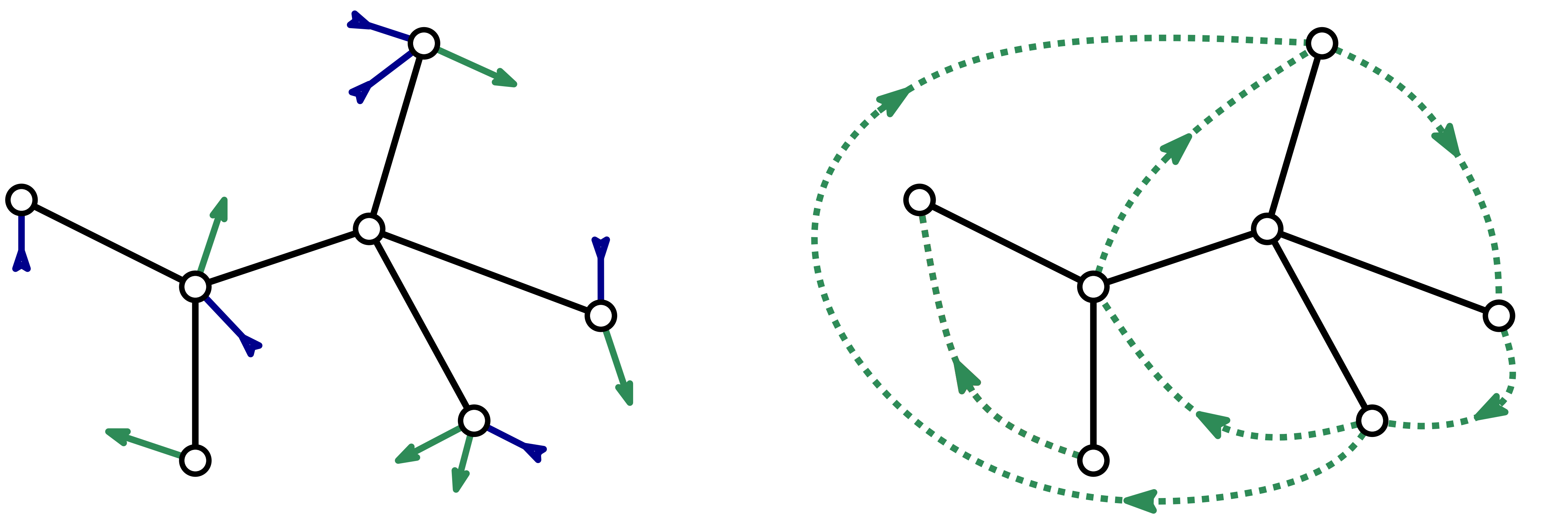

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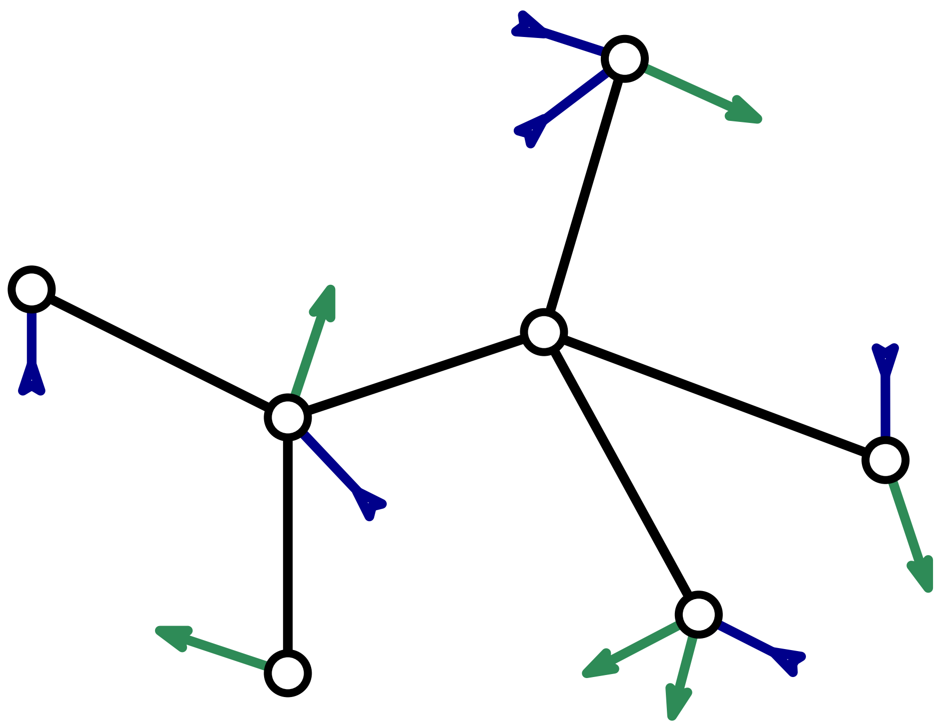

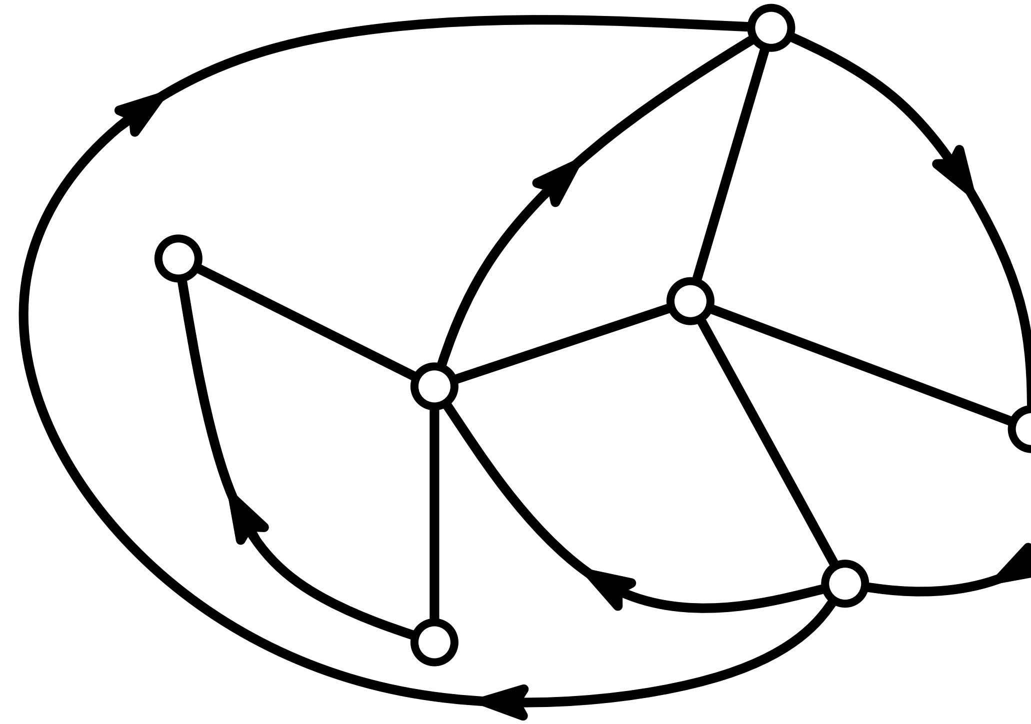

A plane map can be canonically associated to any blossoming tree by making all closures clockwise.

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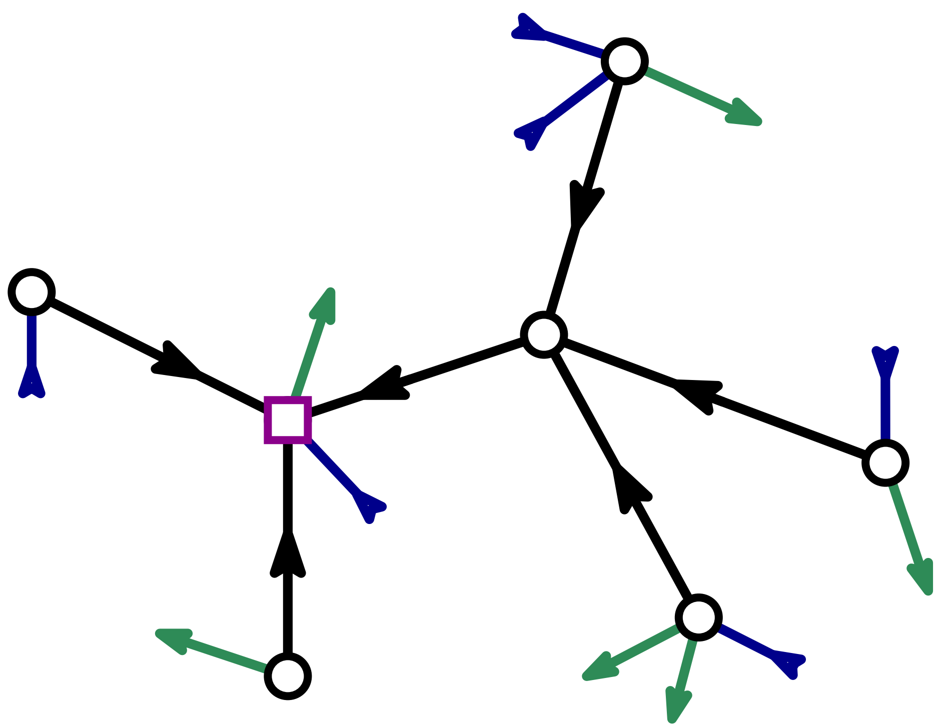

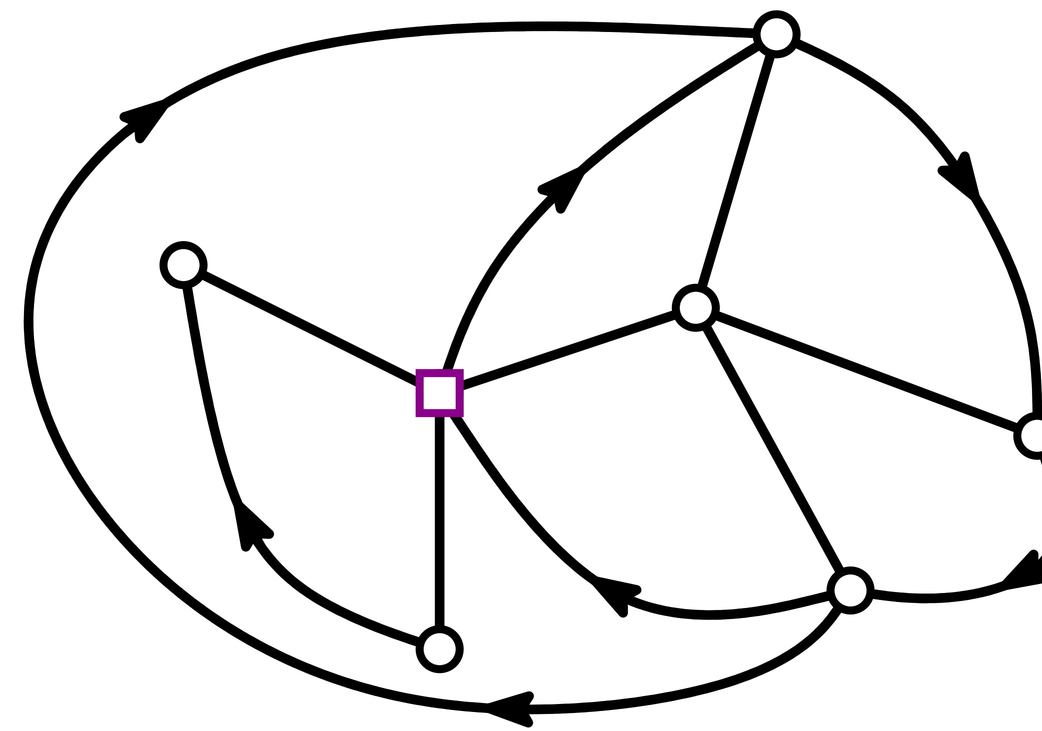

A plane map can be canonically associated to any blossoming tree by making all closures clockwise.

If the tree is rooted and its edges oriented towards the root + closure edges oriented naturally

$\Rightarrow$ Accessible orientation of the map without ccw cycles.

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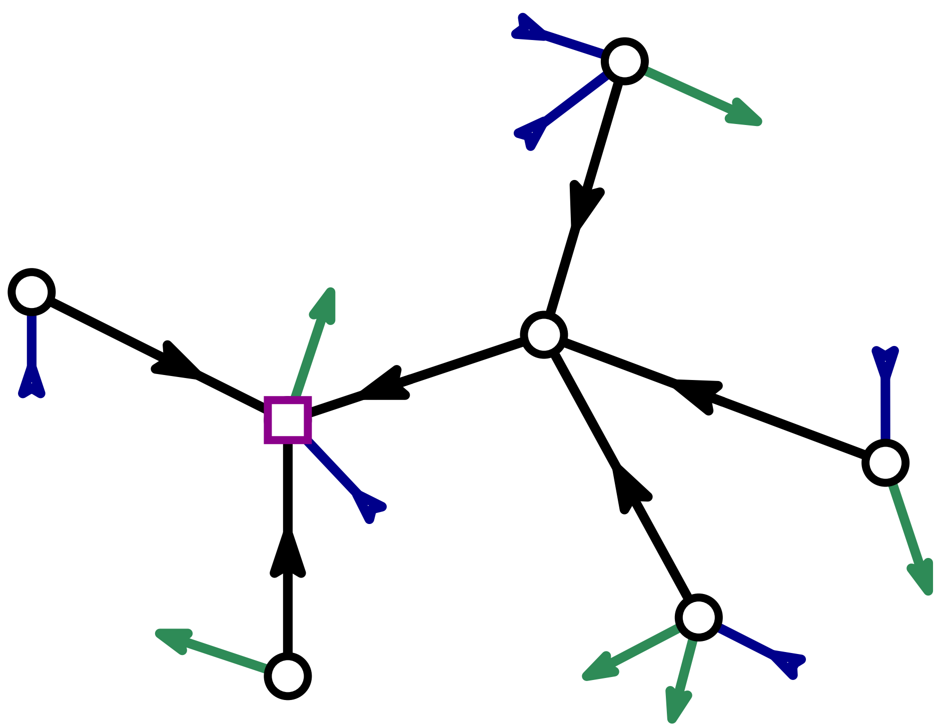

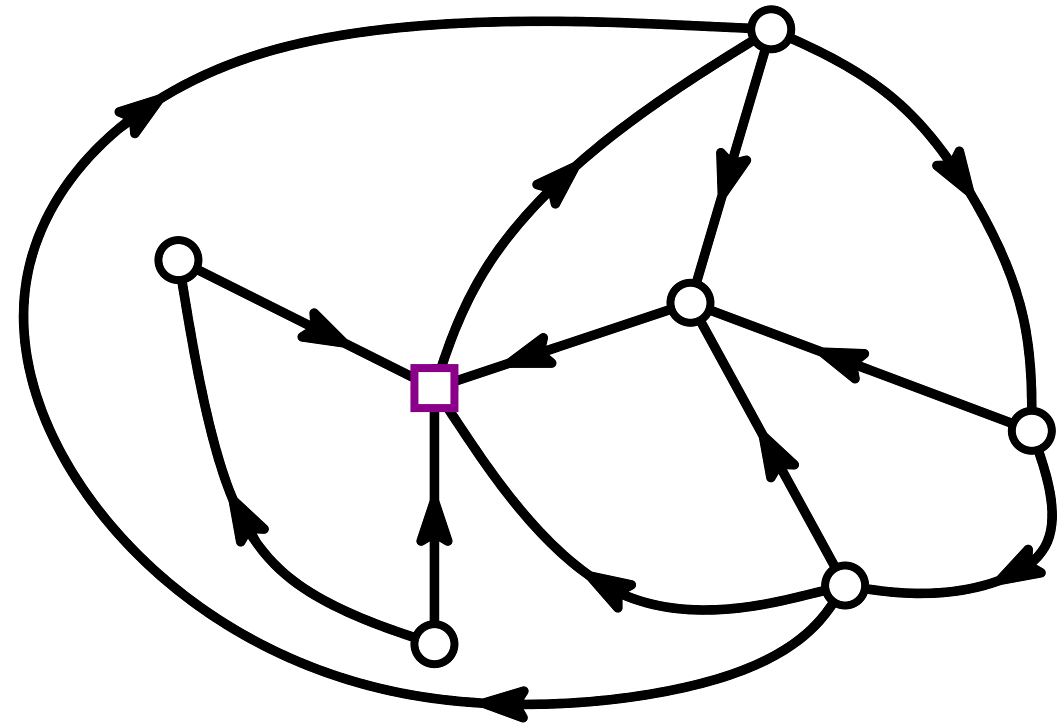

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Theorem : [Bernardi '07], [A., Poulalhon 14+]
If a plane map M has a marked vertex v is endowed with an orientation such that :

- there exists a directed path from any vertex to v,
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then there exists a **unique** blossoming tree rooted at v whose closure is M endowed with the same orientation.

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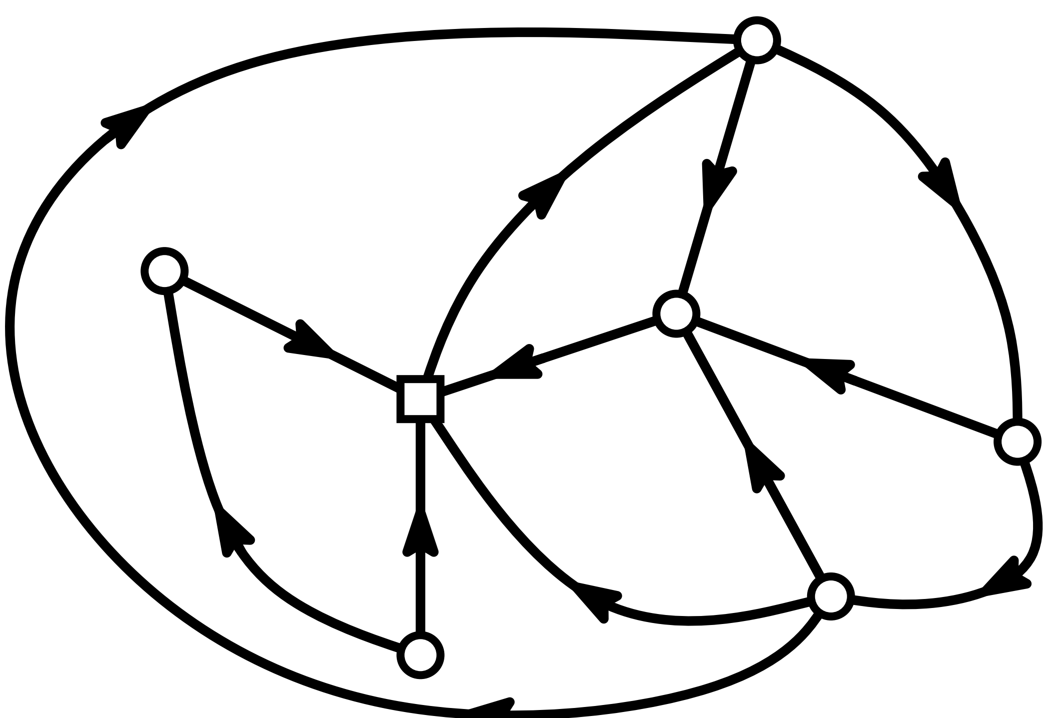

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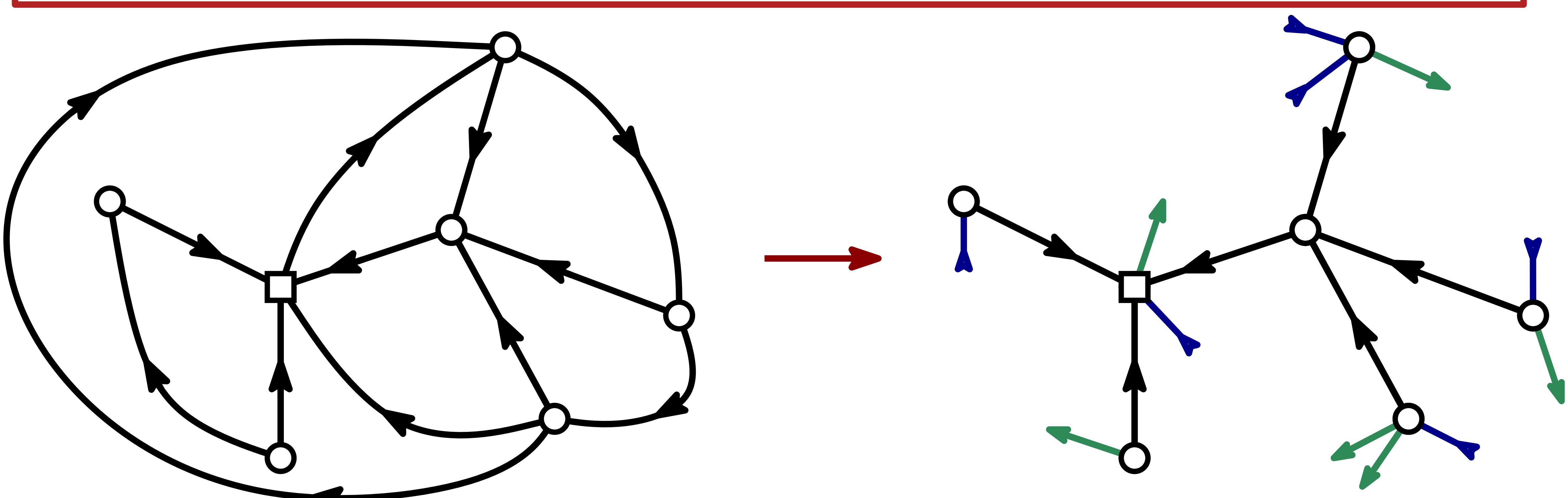

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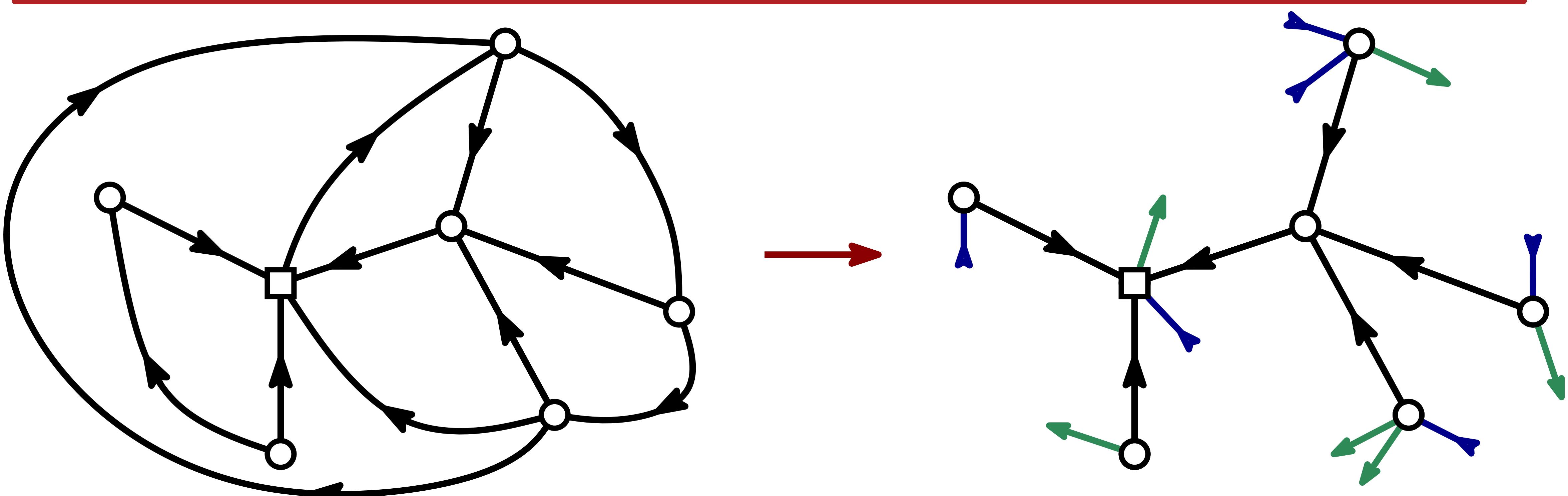
Proof by induction on the number of faces + identification of closure edges ....



Orientations

Orientation = orientation of the edges of the map.

To apply the construction: need to find **canonical orientations**

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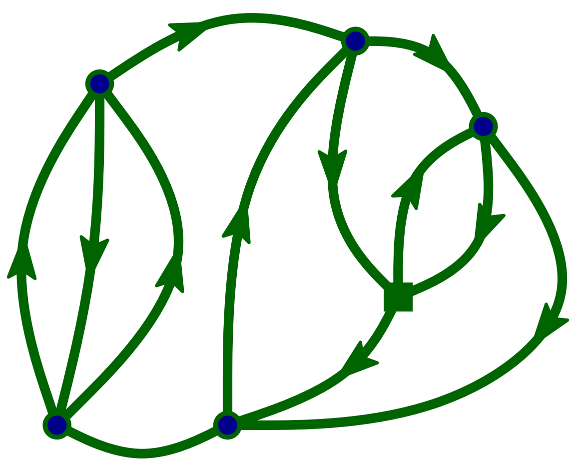
4-regular maps



2 outgoing edges/vertex
2 ingoing edges/vertex

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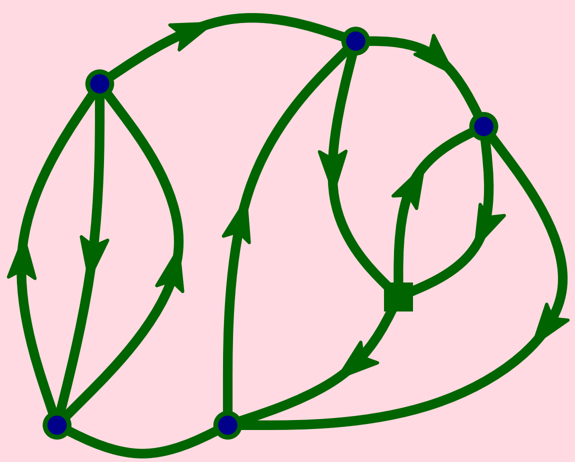

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A map is 4-regular iff it admits an orientation with indegree 2 and outdegree 2 for each vertex.

Orientations

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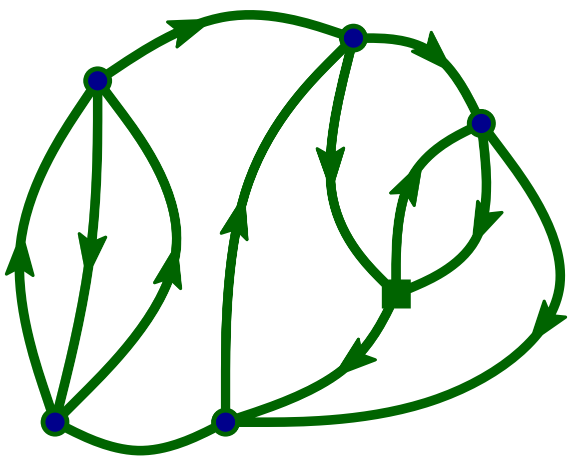
4-regular maps



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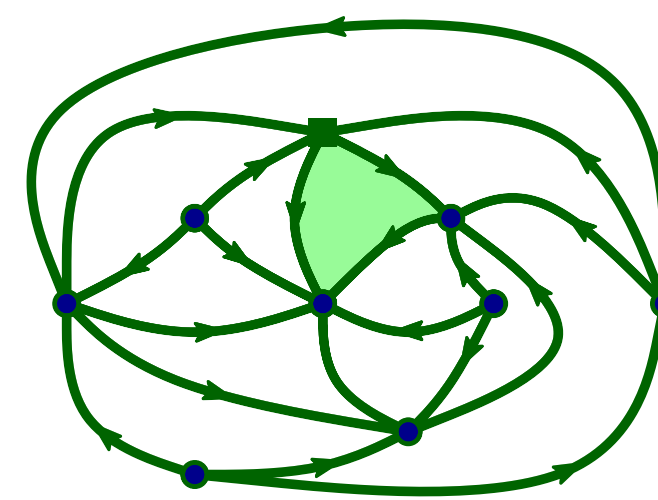
Simple triangulations



3 outgoing edges / non-root vertex
1 outgoing edge / root vertex

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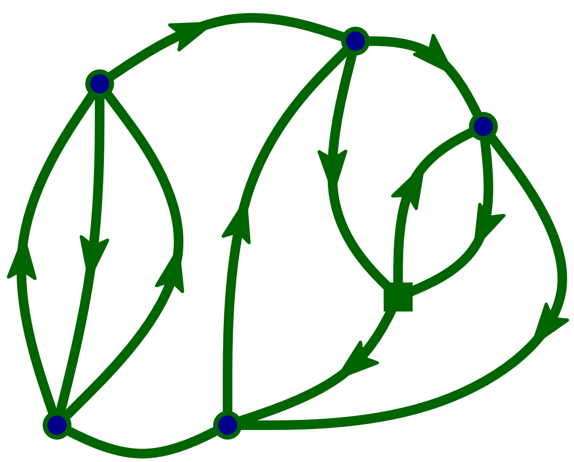

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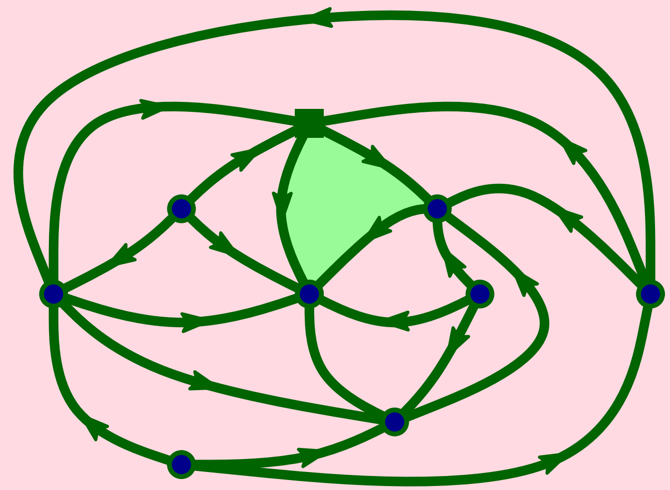

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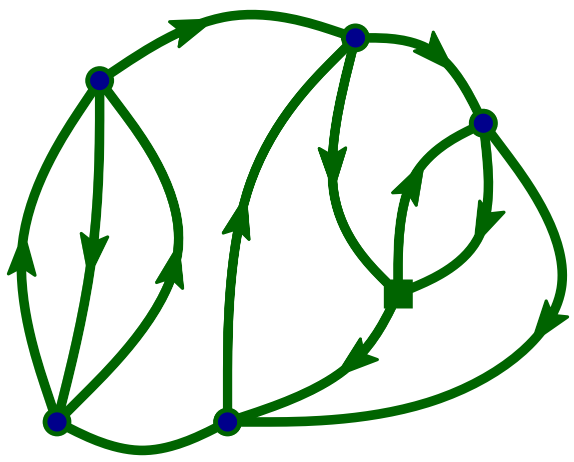
A triangulation is simple iff it admits an orientation with:
outdegree 3 for each non-root vertex
outdegree 1 for each vertex on the root face.

Orientations

4-regular maps



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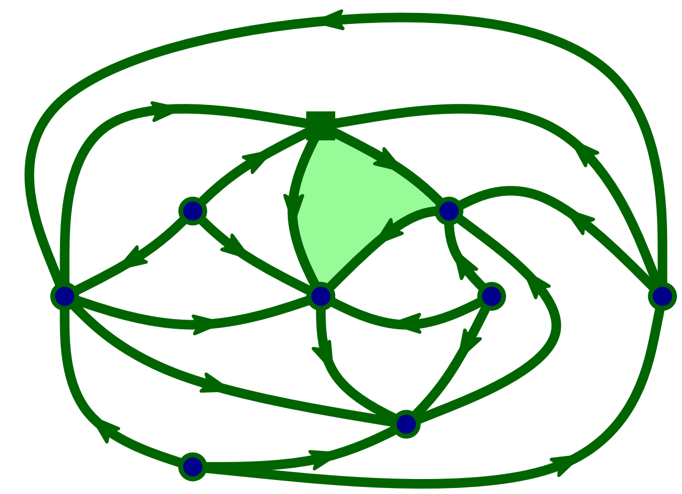

3 outgoing edges / non-root vertex
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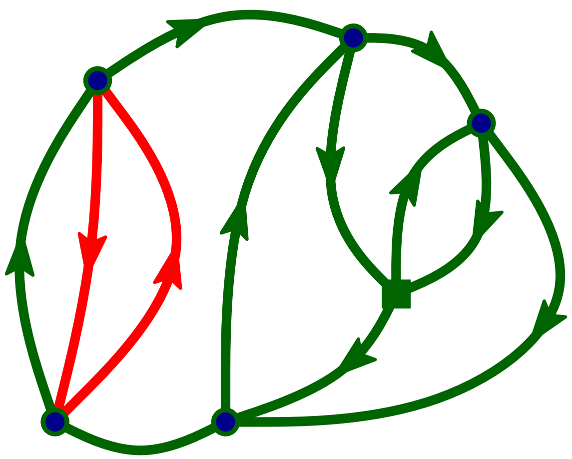
Many families admit a caracterization via orientations

(description of the orientation = outdegree for each vertex is prescribed)

Orientations

4-regular maps



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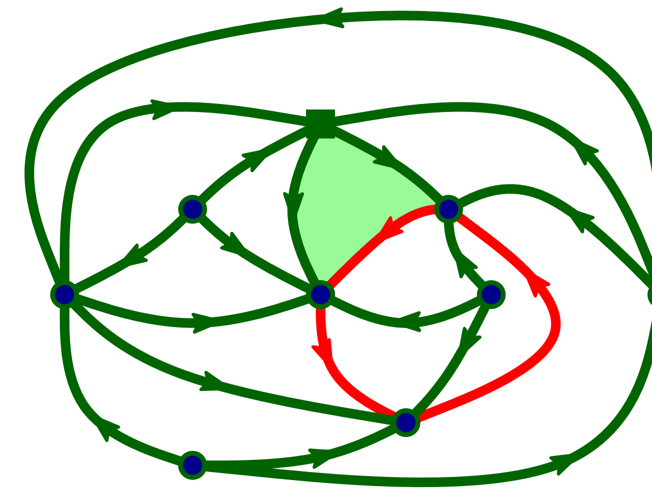

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Theorem requires accessible orientation without ccw cycles:

Too much too ask ?

Orientations

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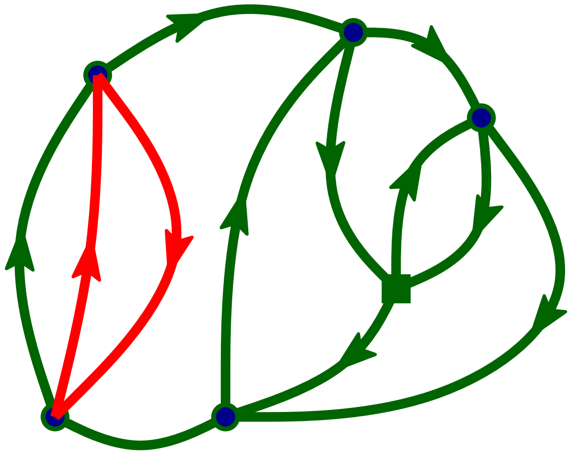

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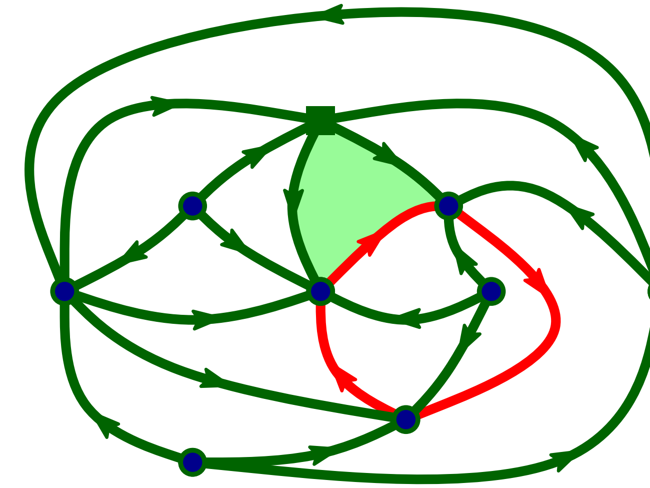


3 outgoing edges / non-root vertex
1 outgoing edge / root vertex

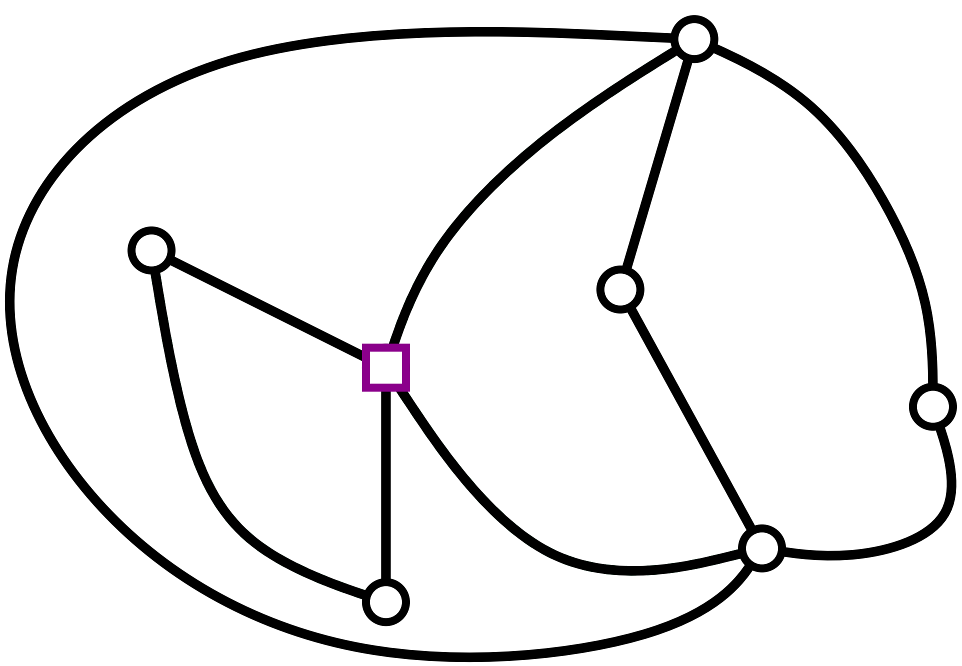
Theorem requires accessible orientation without ccw cycles: **NO !**

Too much too ask ?

Proposition: [Felsner '04]
For a given map and orientation, there exists a unique orientation with the same outdegrees and without ccw cycles.
If there exists one accessible such orientation, all of them are accessible.

Summary

- Take a family of maps,



Maps with even degrees.

Summary

- Take a family of maps,
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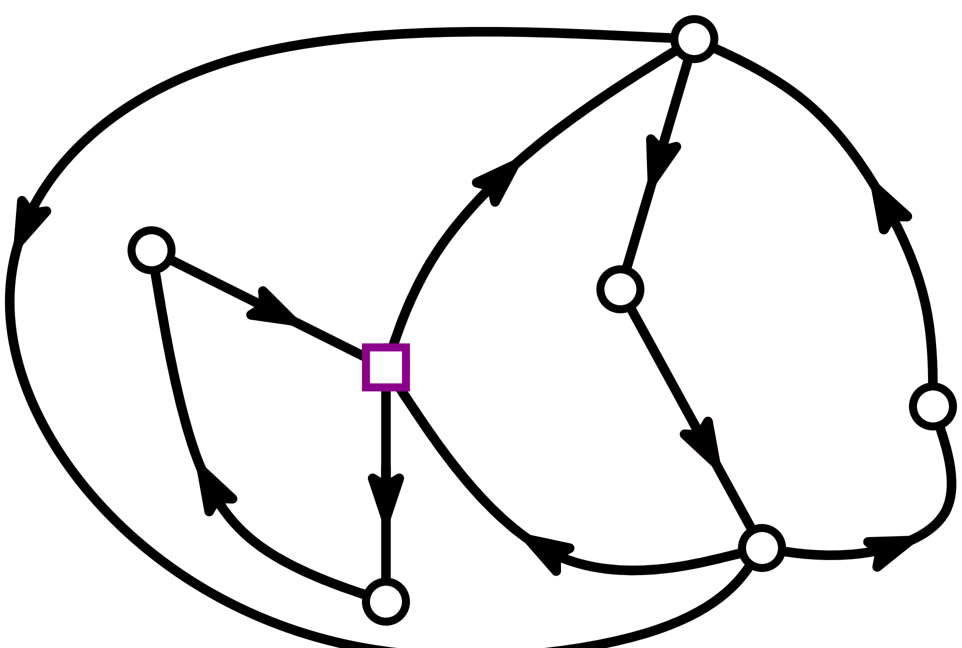

Maps with even degrees.
Orientations with same out/in degrees

Summary

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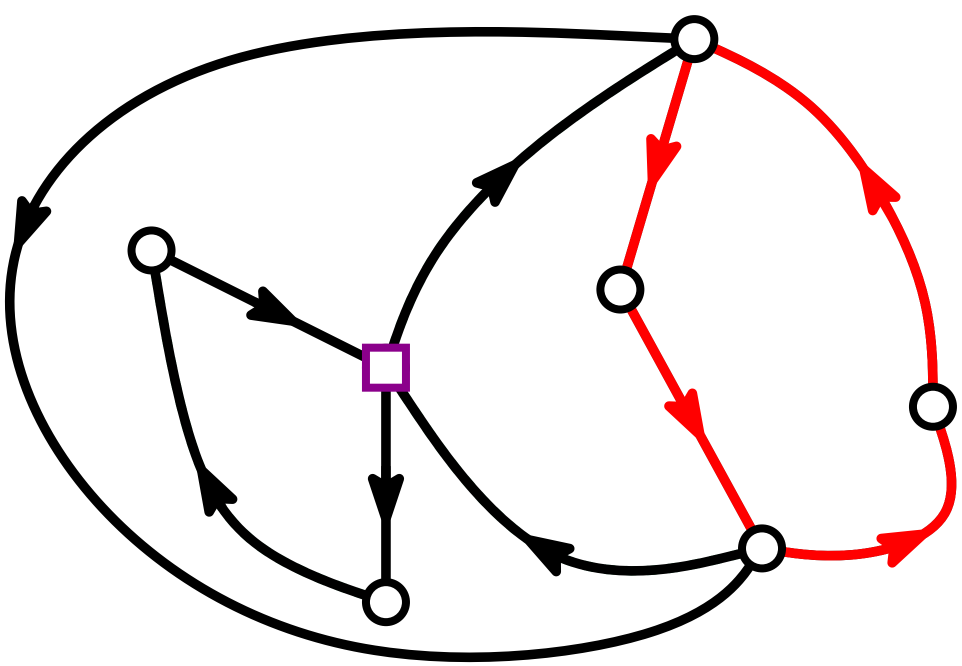

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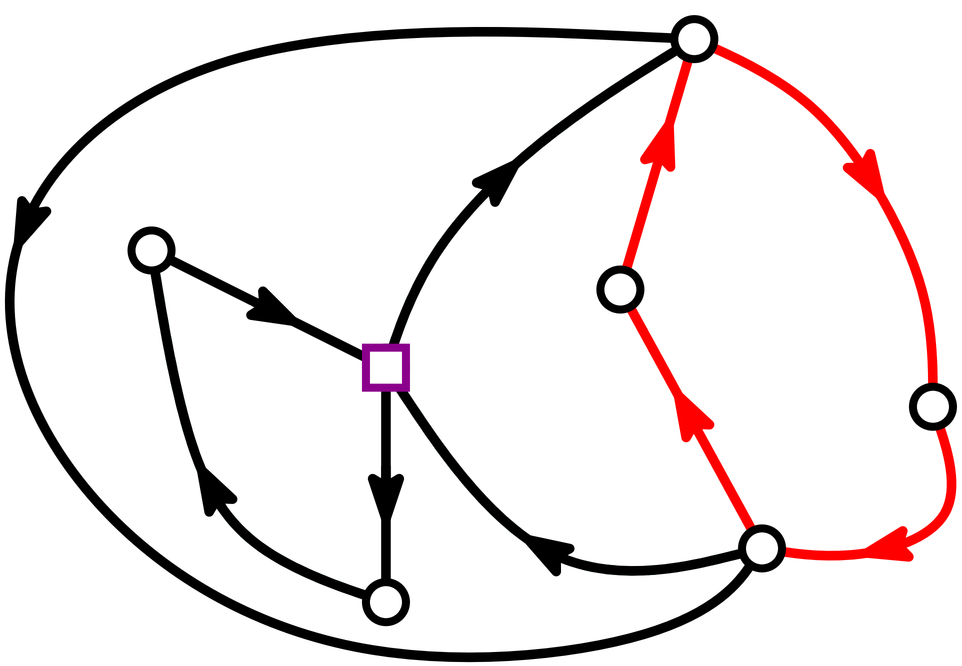

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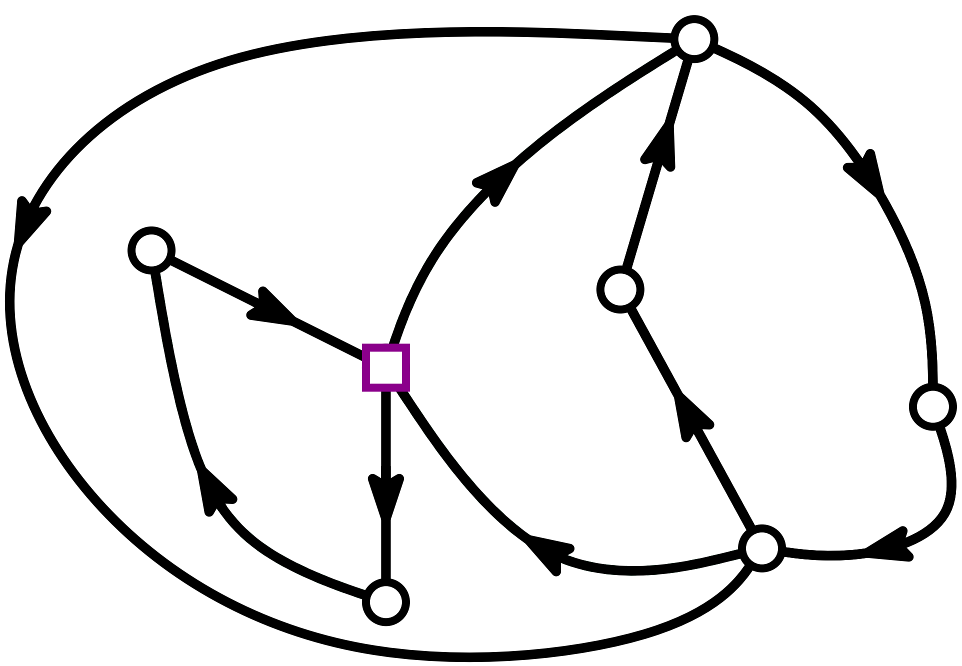

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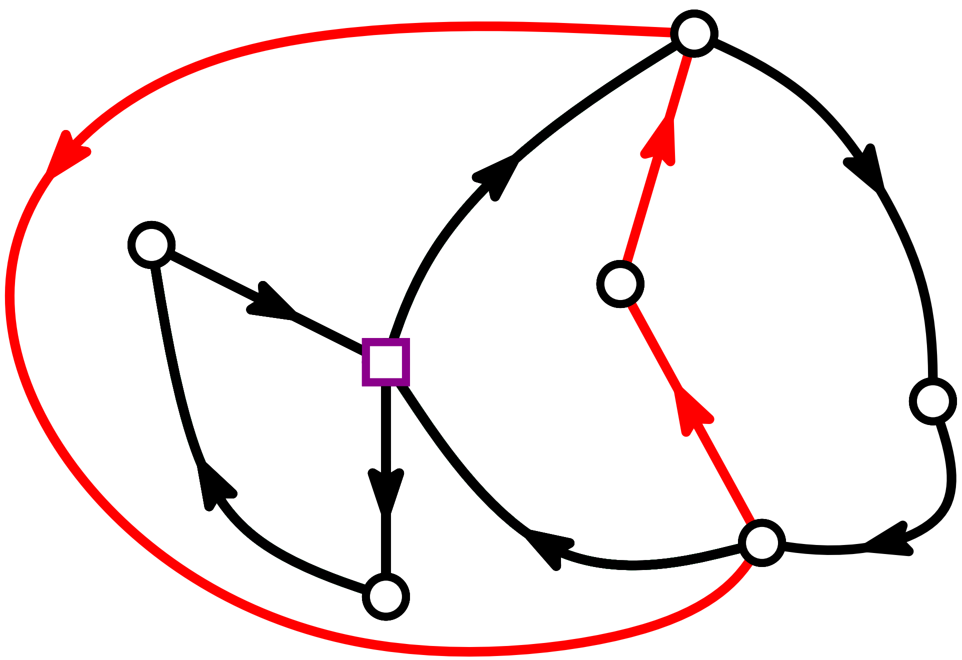

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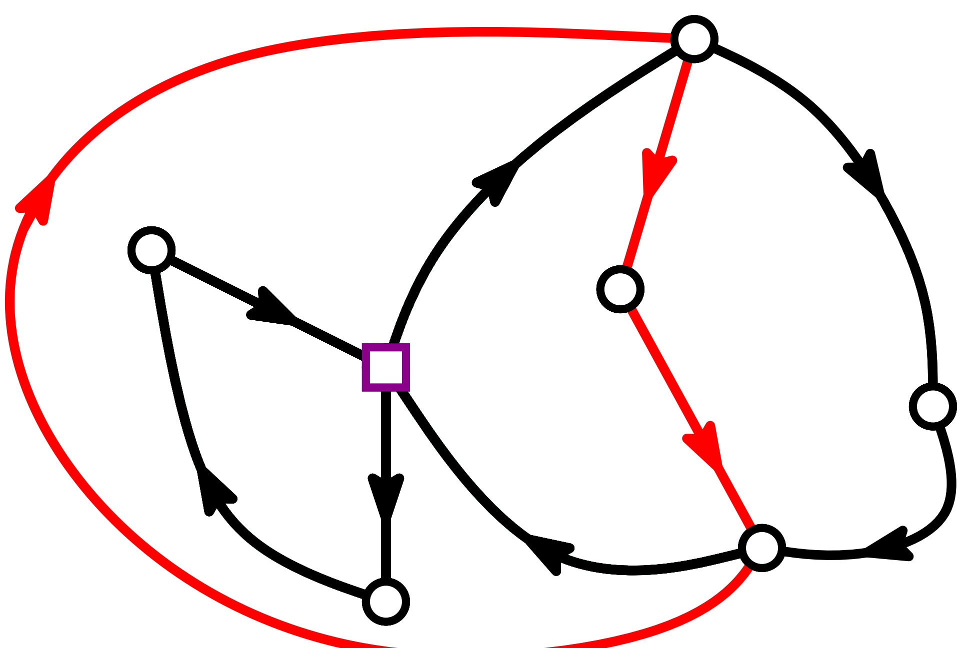

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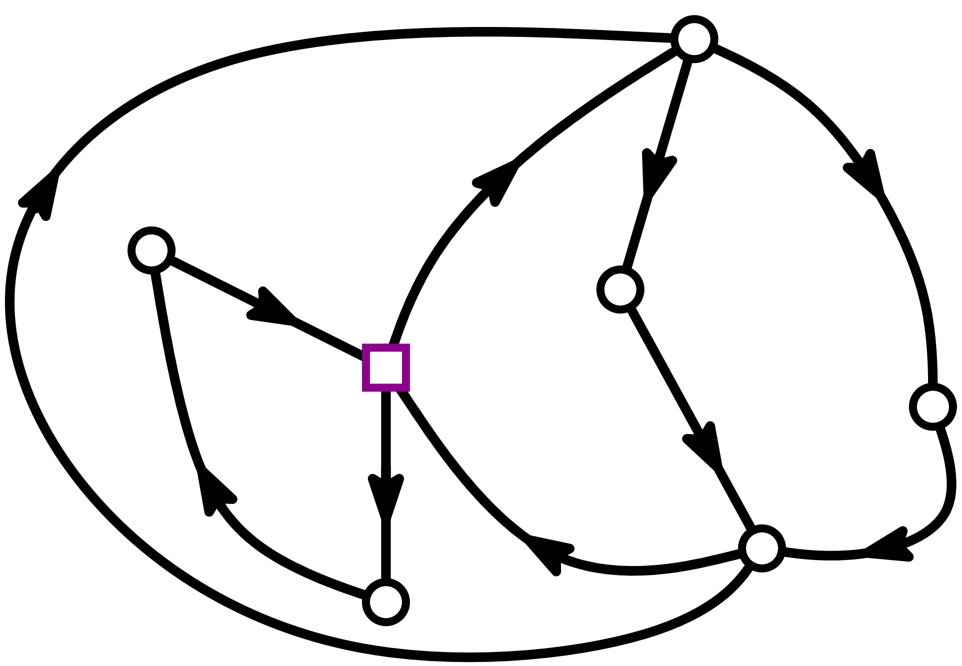

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Summary

- Take a family of maps,
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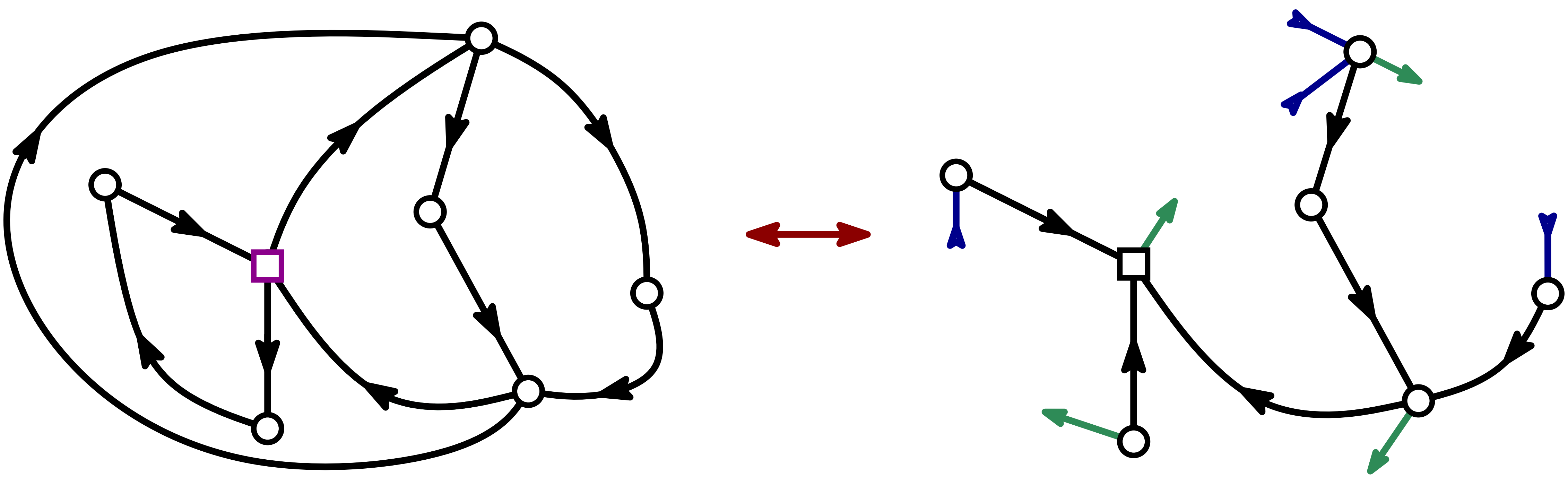
- Apply the bijection,



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Summary

- Take a family of maps,
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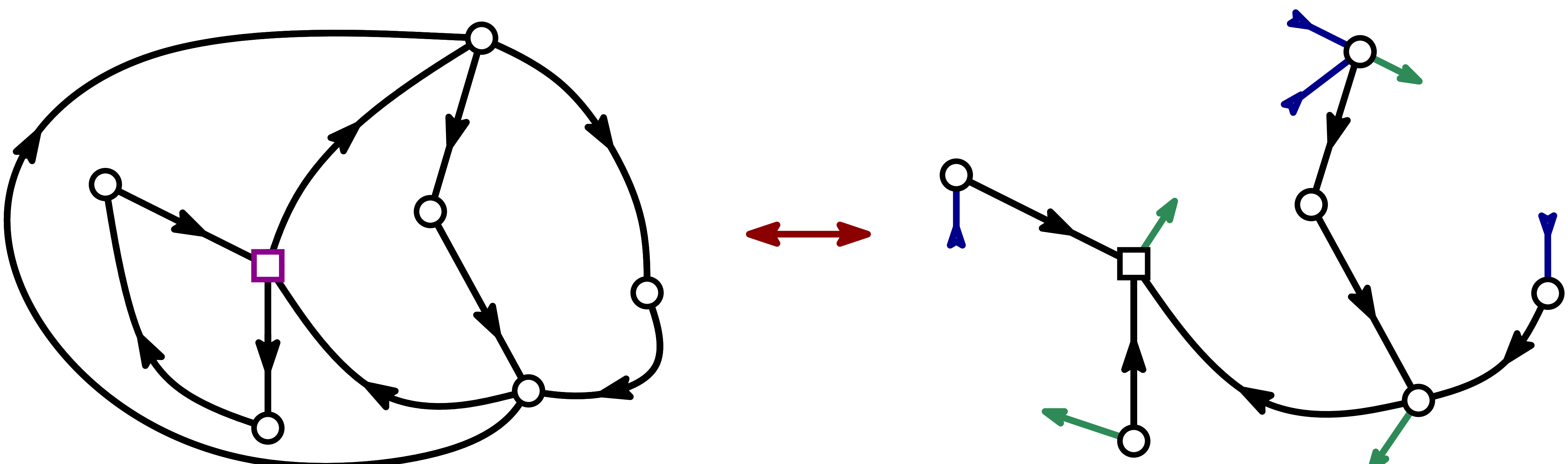
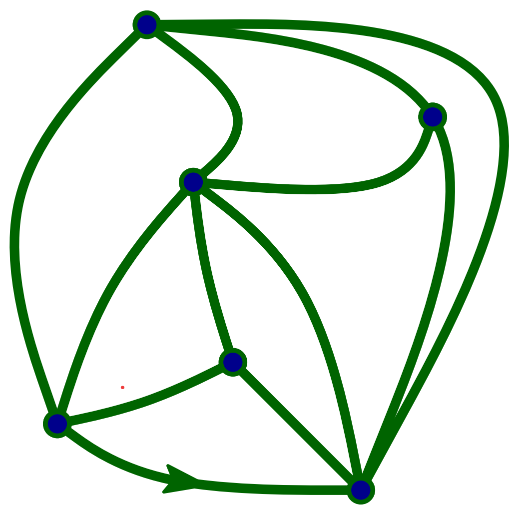
- Consider the unique orientation without counterclockwise cycles,
- Apply the bijection,
- Study the family of blossoming trees.



Maps with even degrees.
Orientations with same out/in degrees

Trees with same out/in degrees

Distances in blossoming trees: simple triangulations



Simple Triangulation :
no multiple edges
no loops

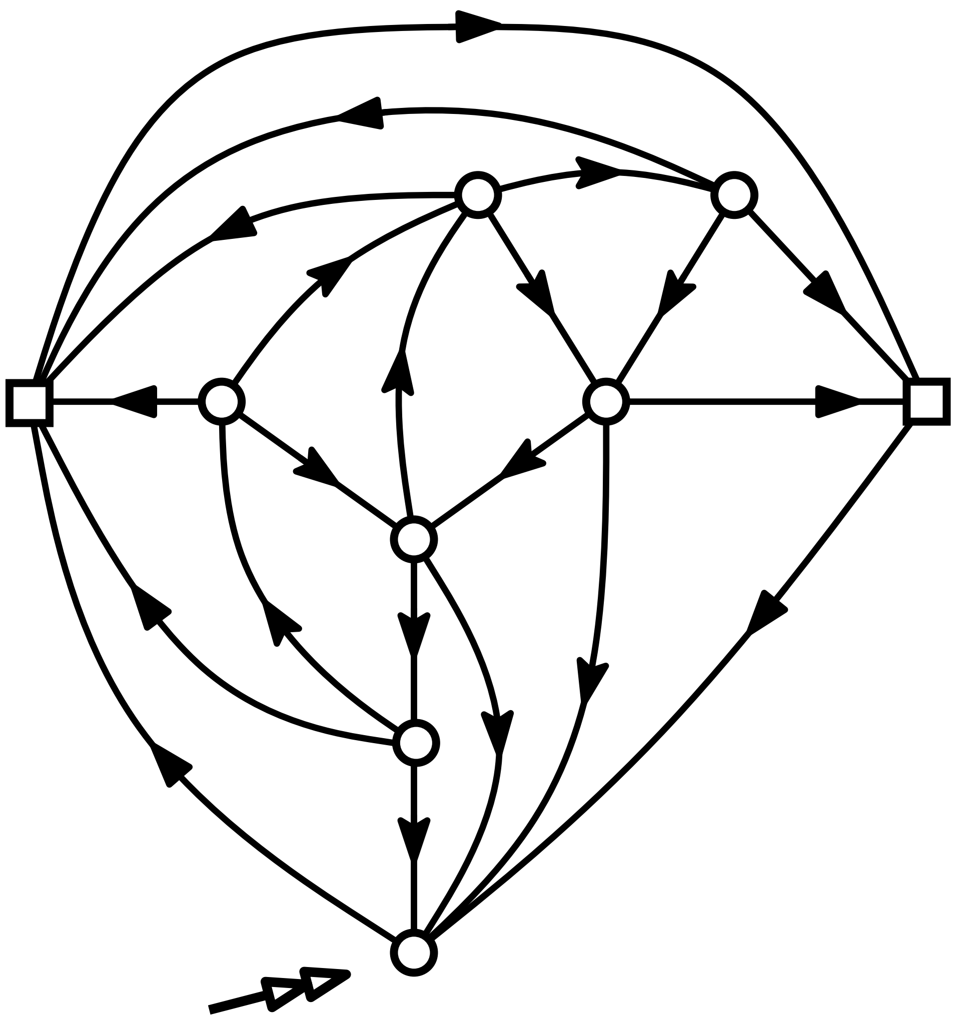
Euler Formula : $v + f = 2 + e$
Triangulation : $2e = 3f$

$\mathcal{M}_n = \{$Simple triangulations of size $n\}$
$= n + 2$ vertices, $2n$ faces, $3n$ edges

$M_n =$ Random element of $\mathcal{M}_n$

What is the behavior of M_n when n goes to infinity ?
typical distances ? Scaling limit of M_n ?

From simple triangulations to blossoming trees



Simple triangulation endowed with its unique orientation such that :

- no counterclockwise cycle
- $\mathrm{out}(v) = 3$ for v an inner vertex
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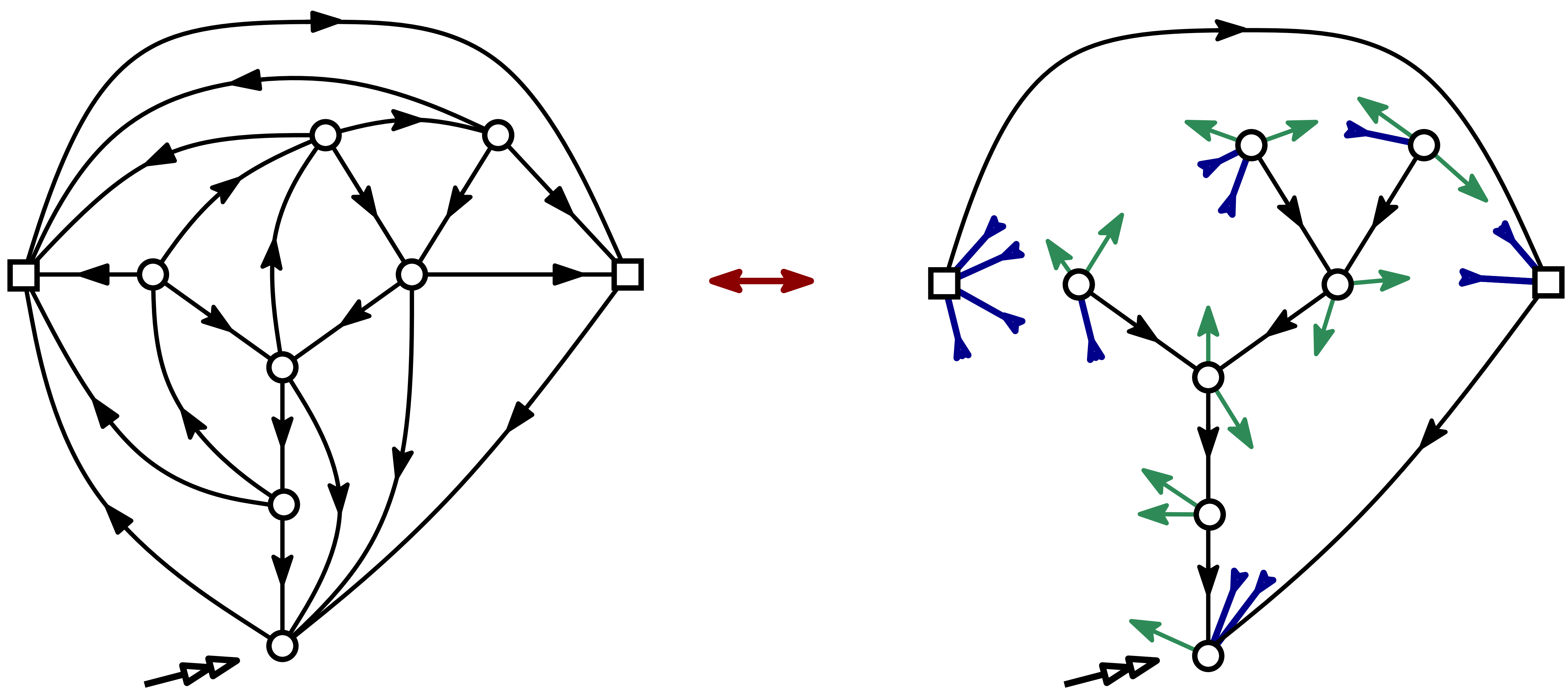
From simple triangulations to blossoming trees



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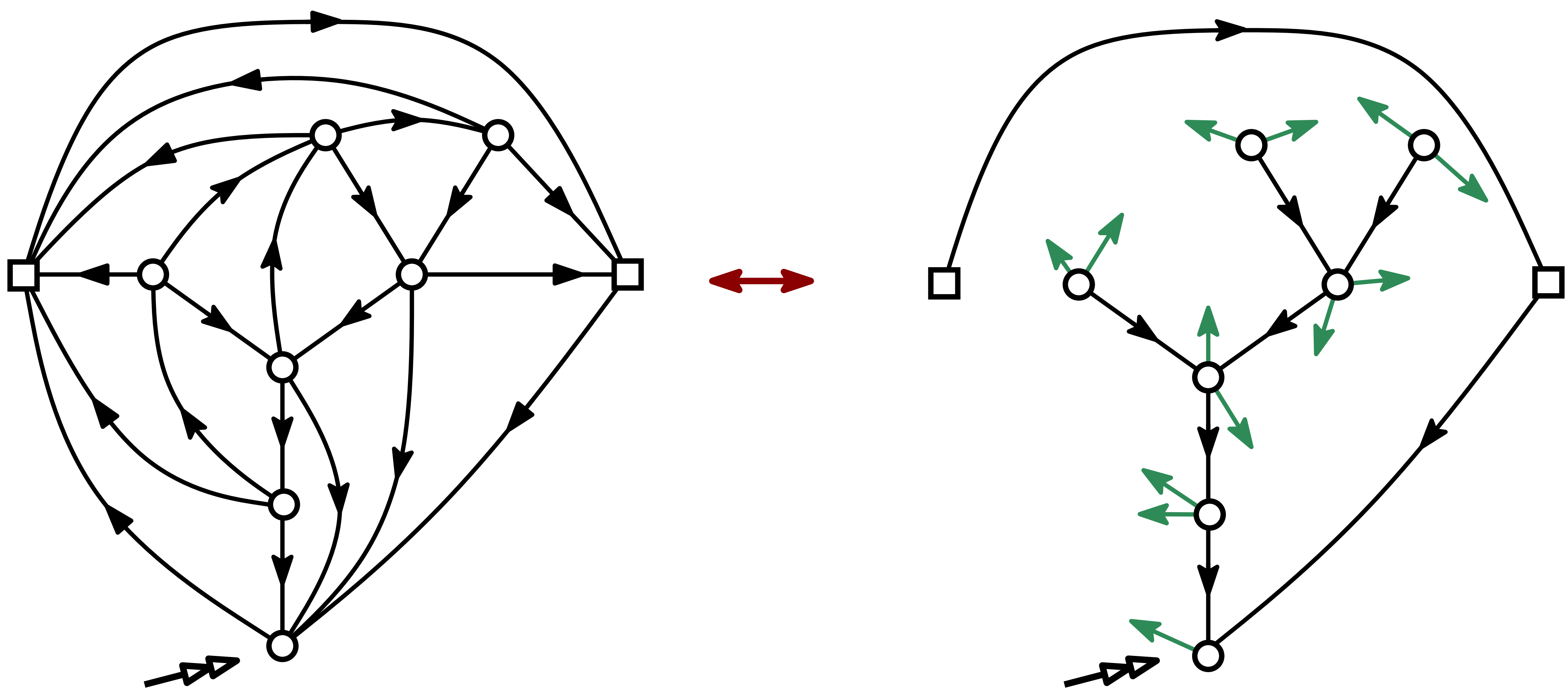

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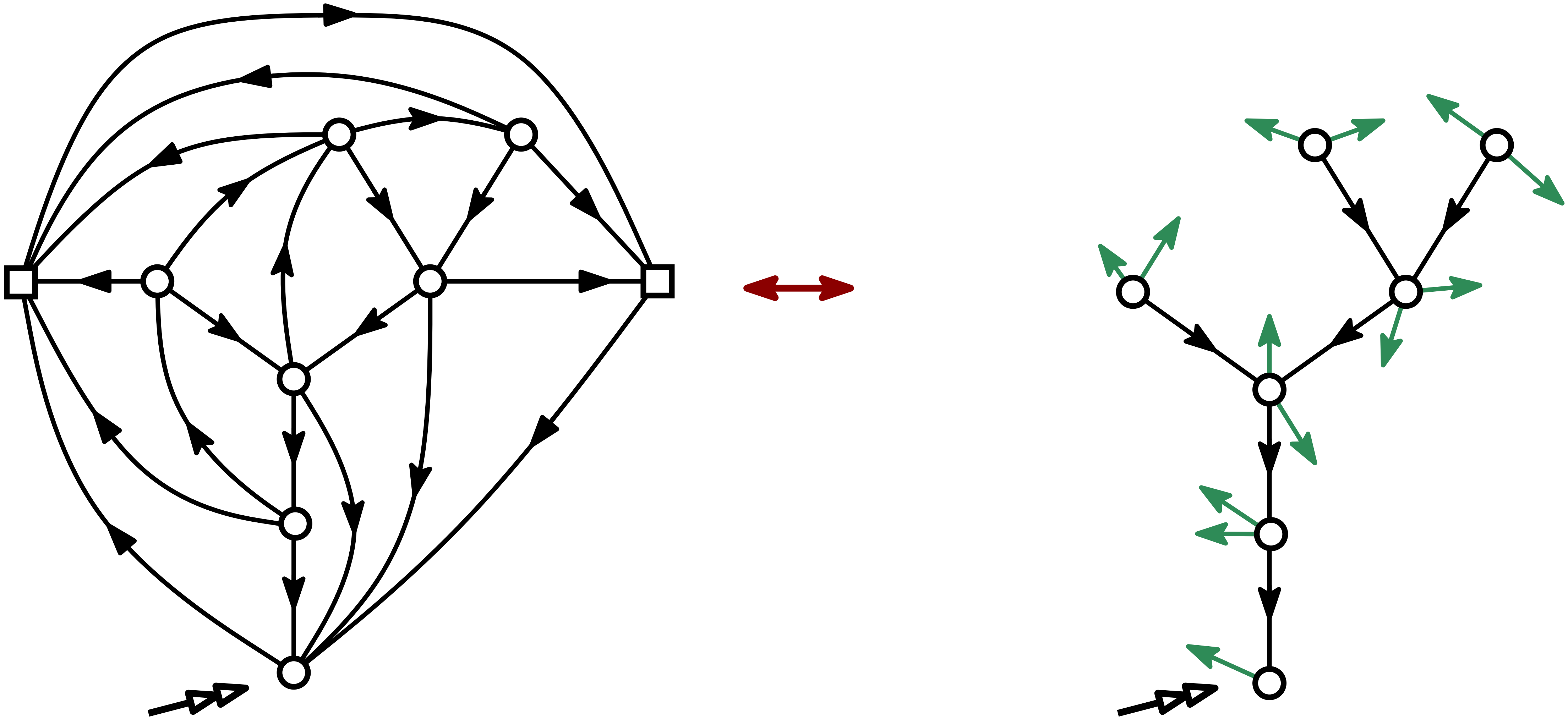

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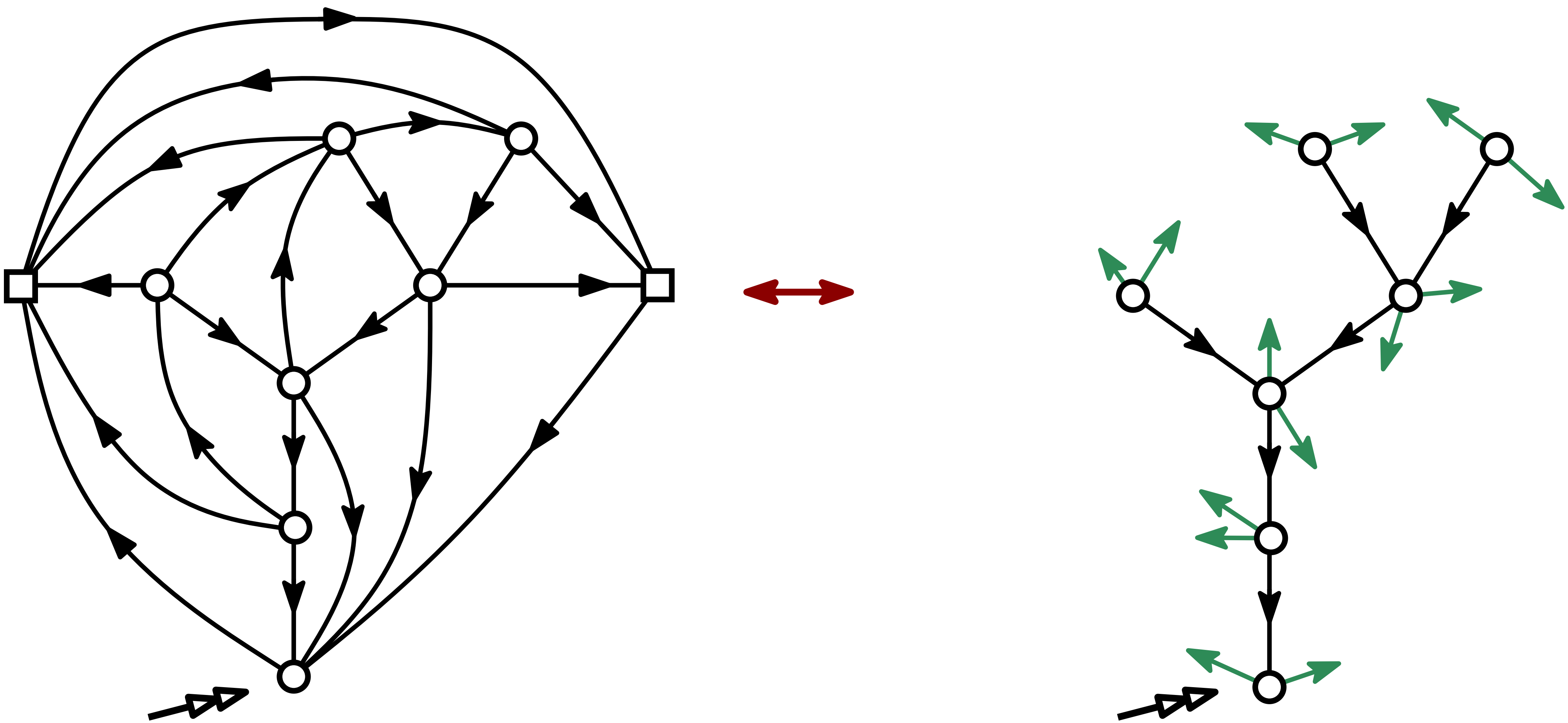
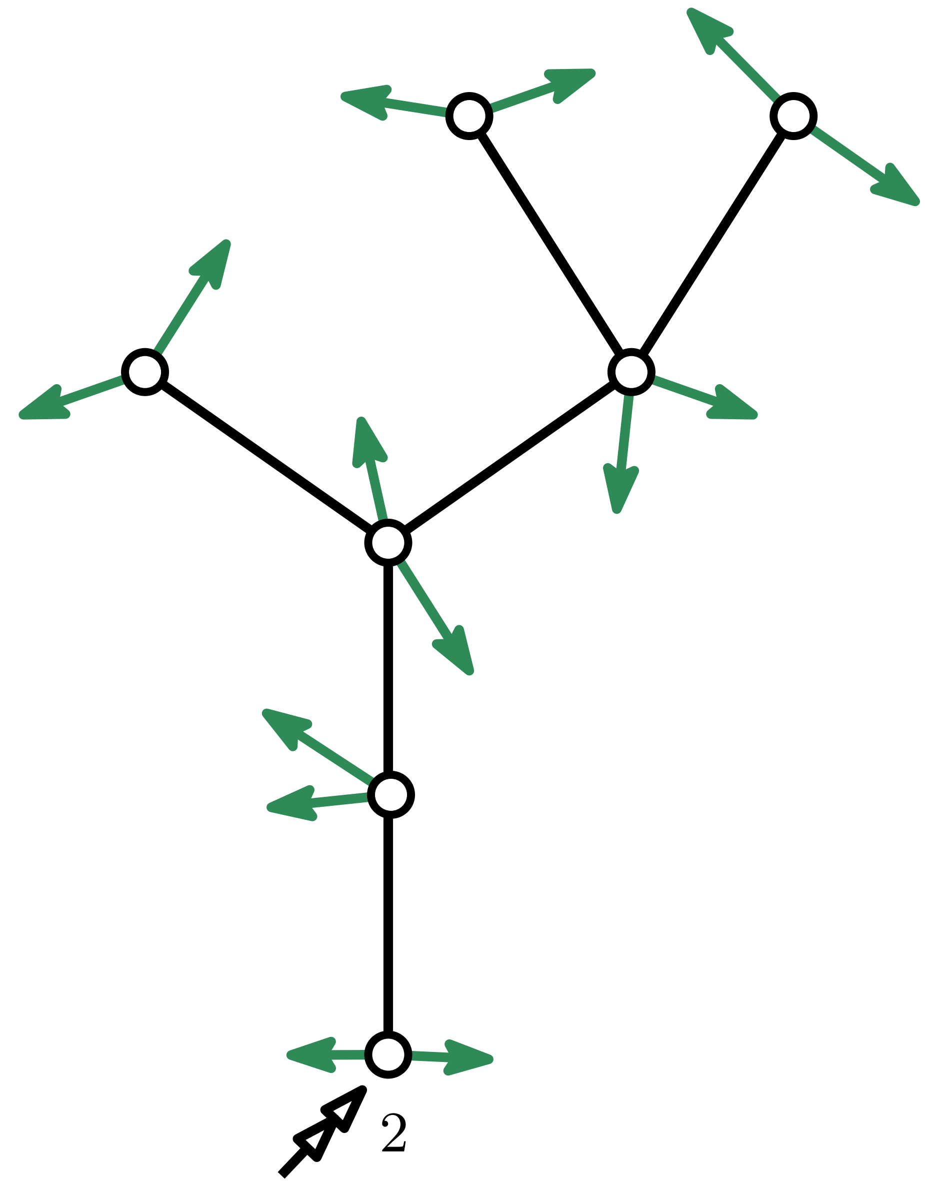


Theorem: [Poulalhon, Schaeffer '05]

The closure operation is a bijection between balanced 2-blossoming trees and simple triangulations.

Same bijection with corner labels

- Start with a planted 2-blossoming tree.
- Give the root corner label 2.



Same bijection with corner labels

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In contour order, apply the following rules:

- Non-leaf to leaf, label does not change.
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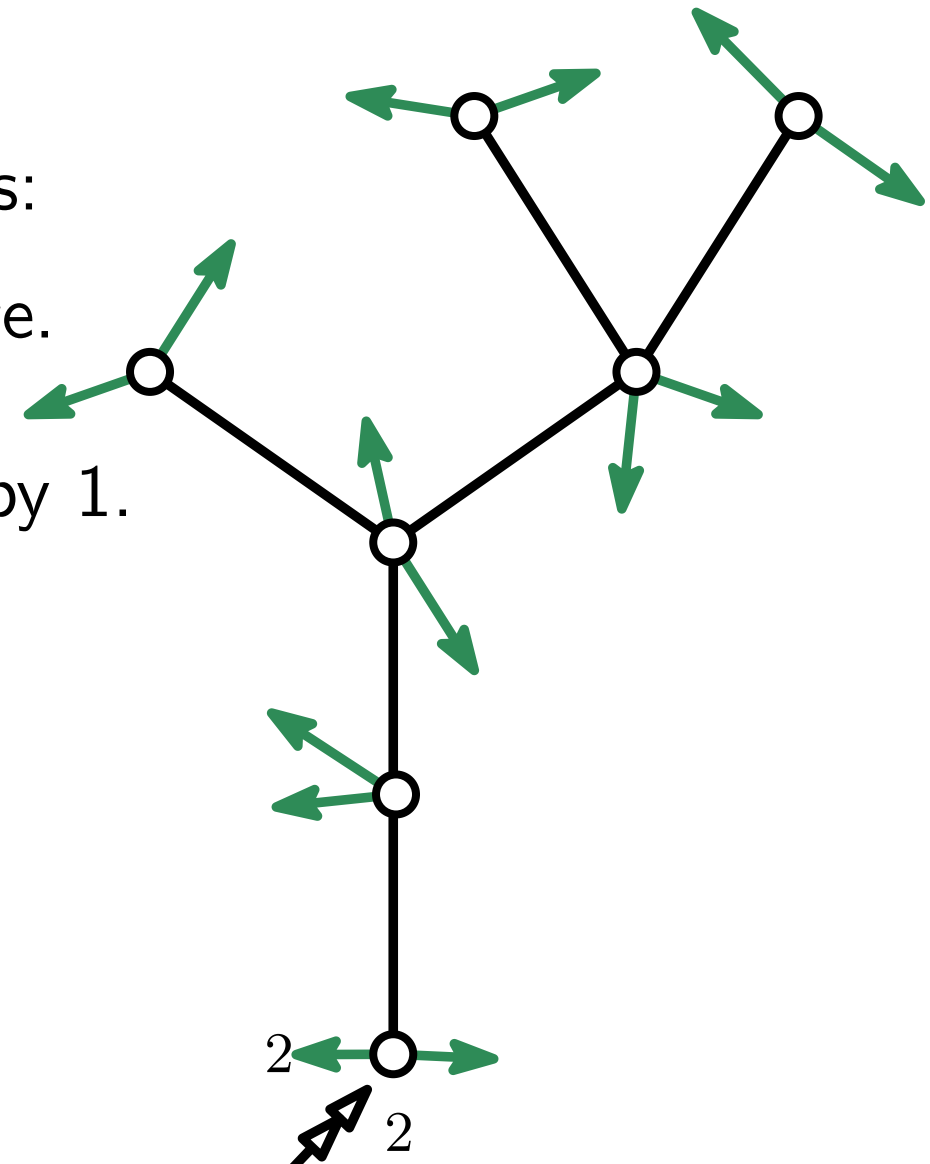

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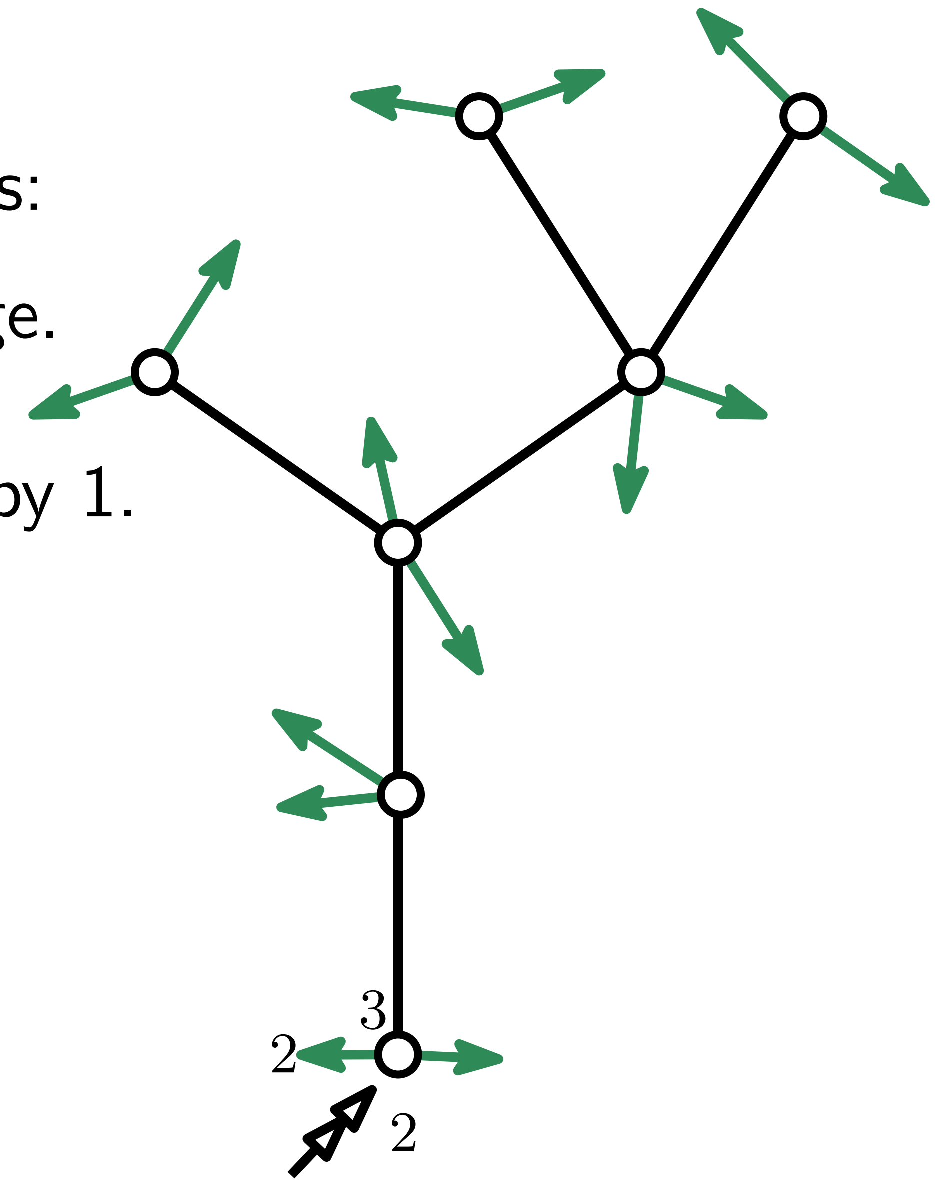

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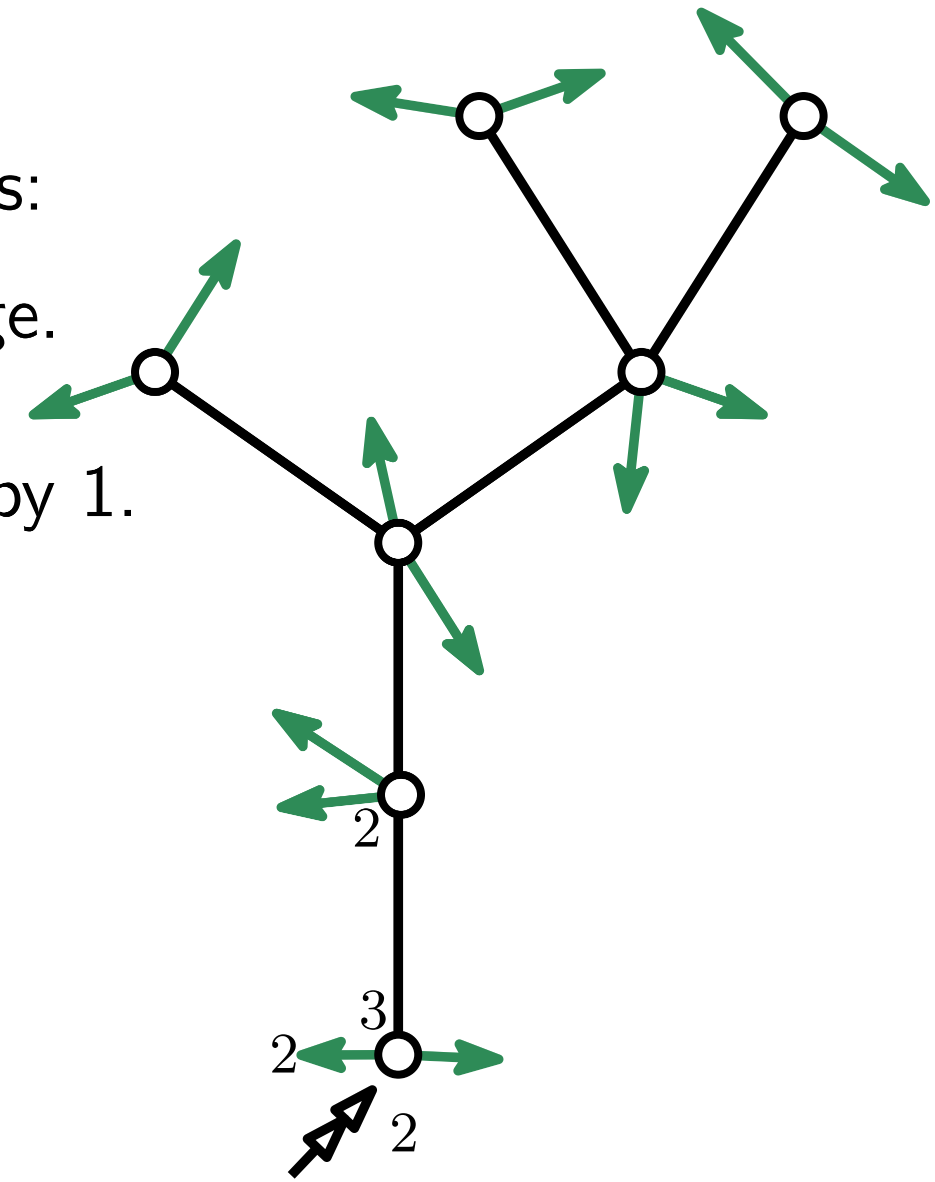

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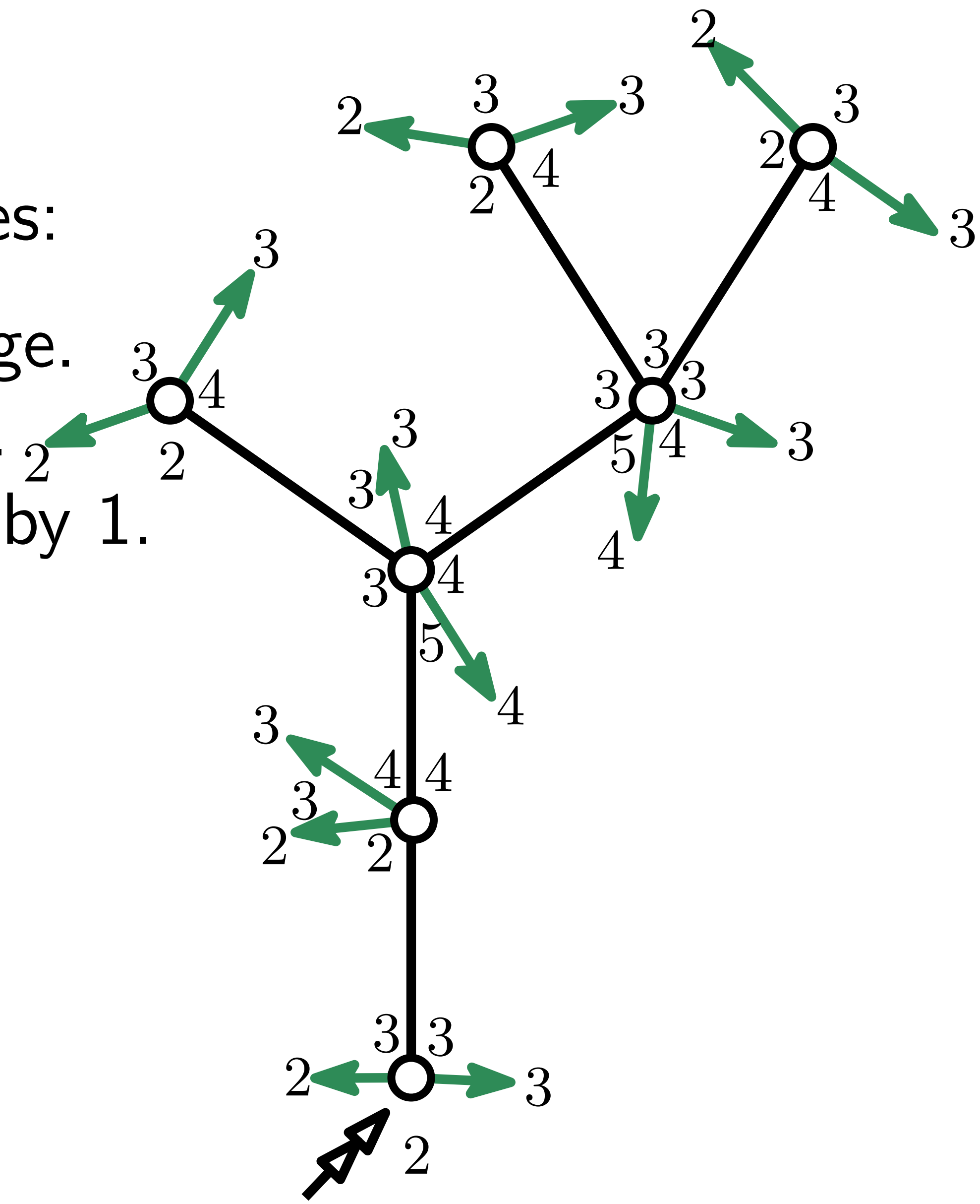

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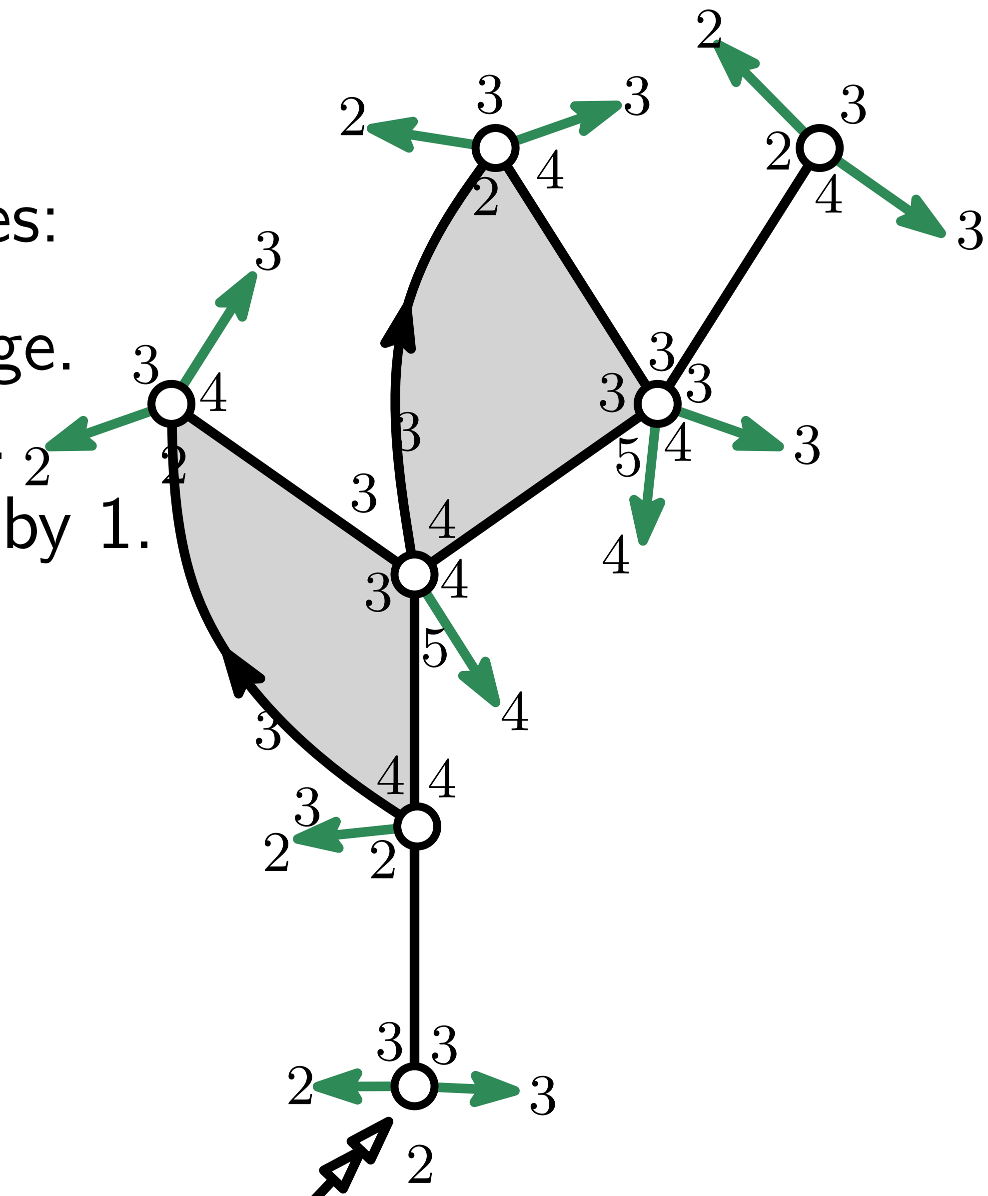

Aside: Tree is balanced $\Leftrightarrow$
all labels ≥ 2
$+$root corner incident to two stems

Closure: Merge each leaf with the first subsequent corner with a smaller label.

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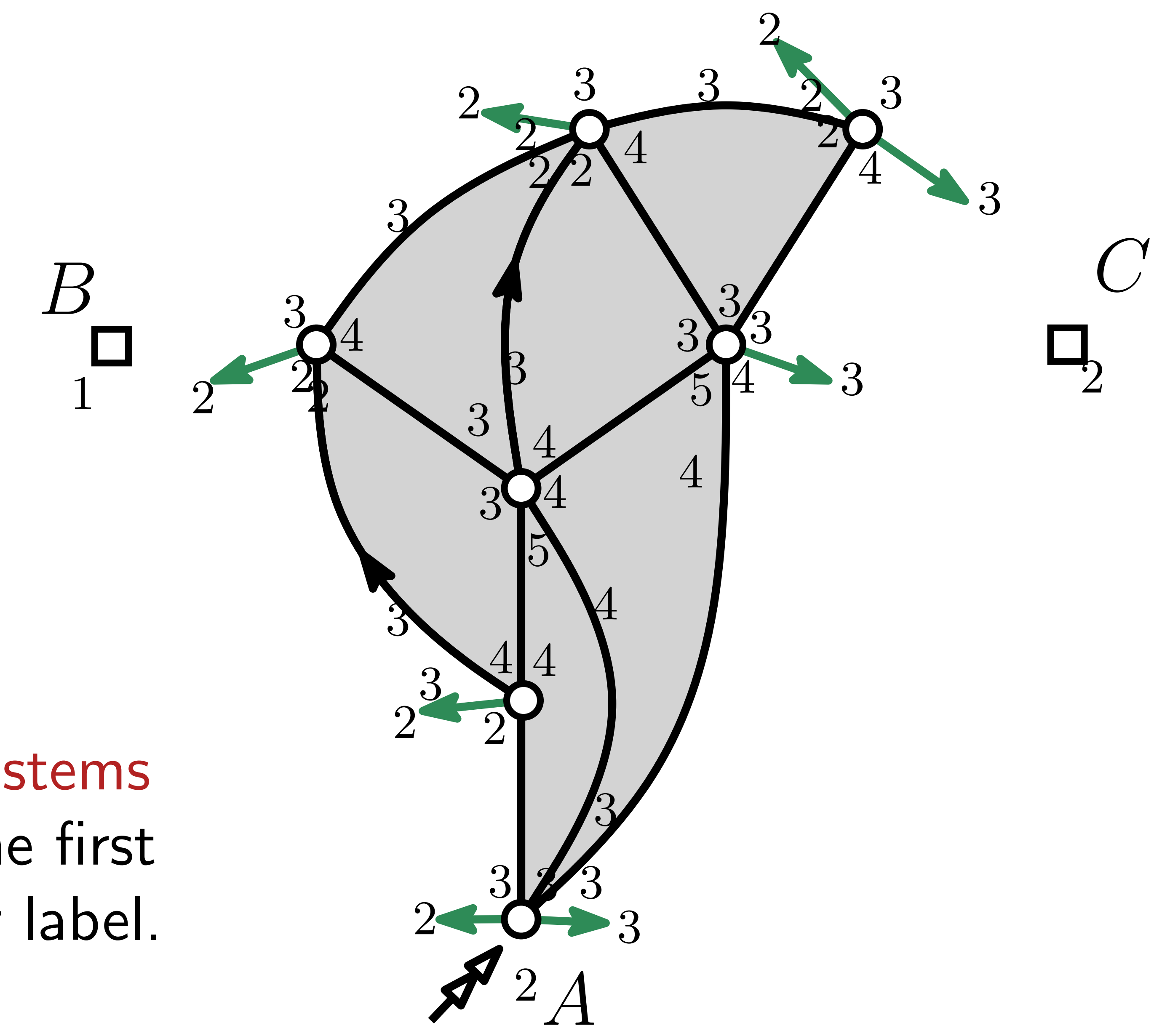

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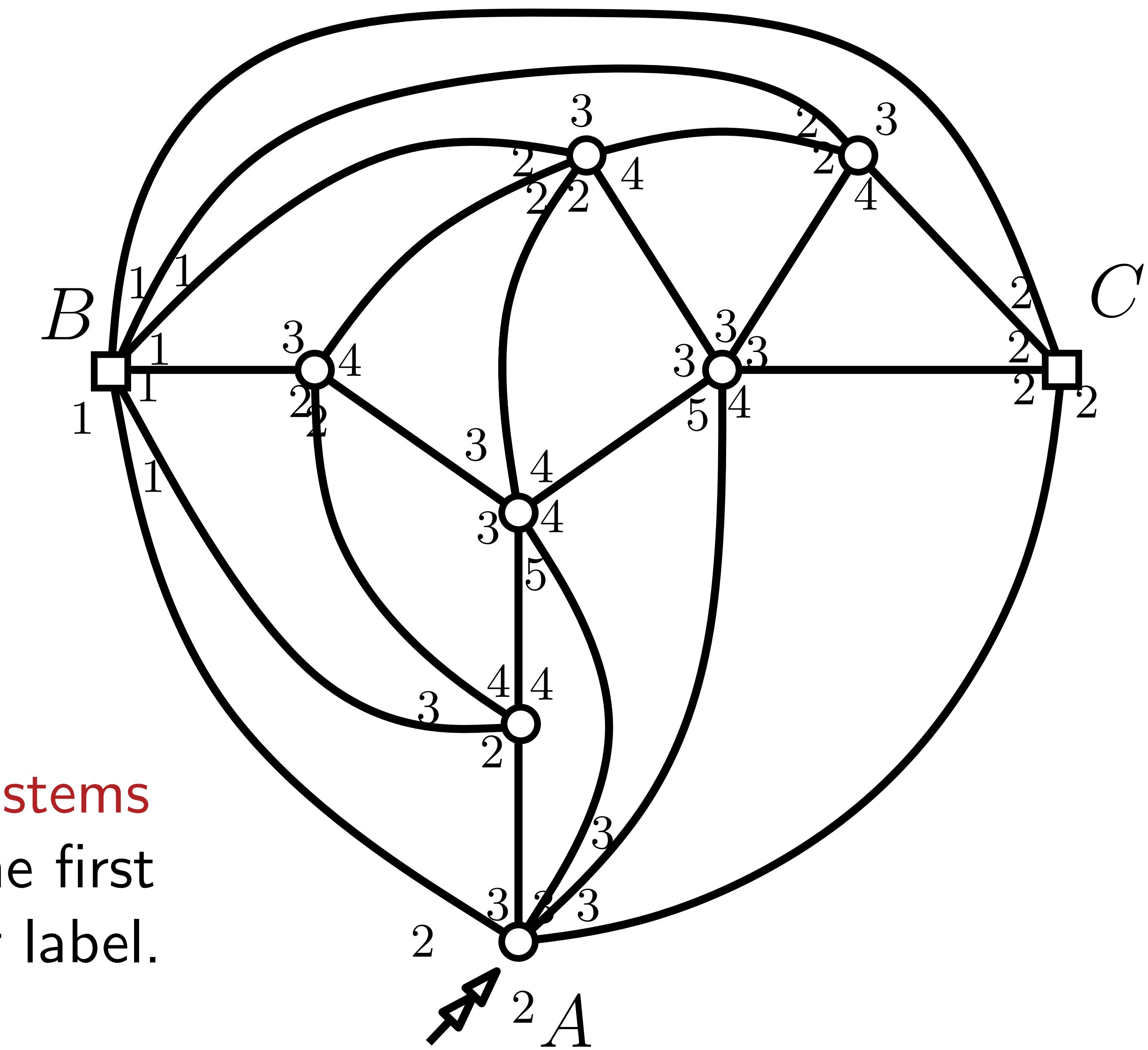

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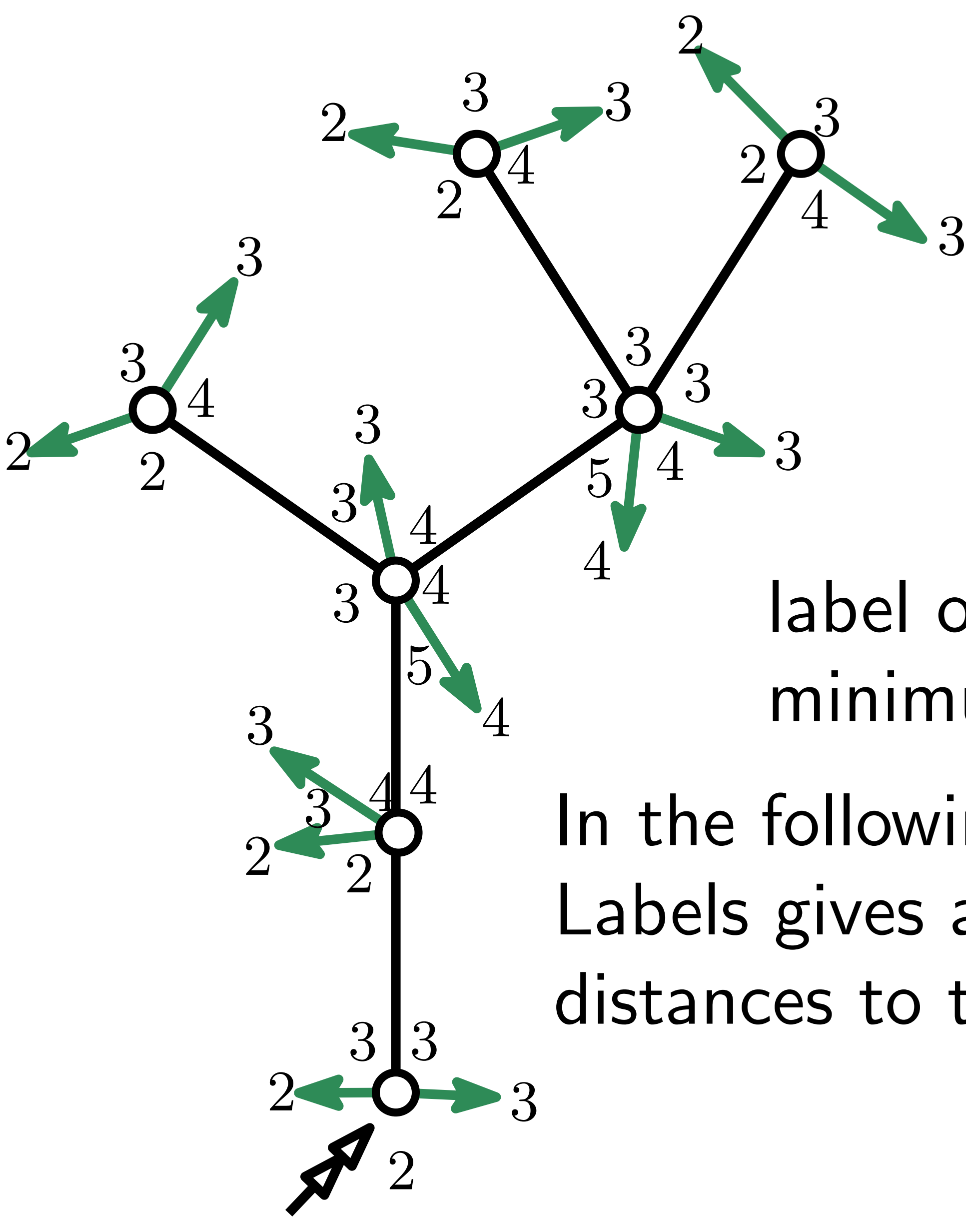
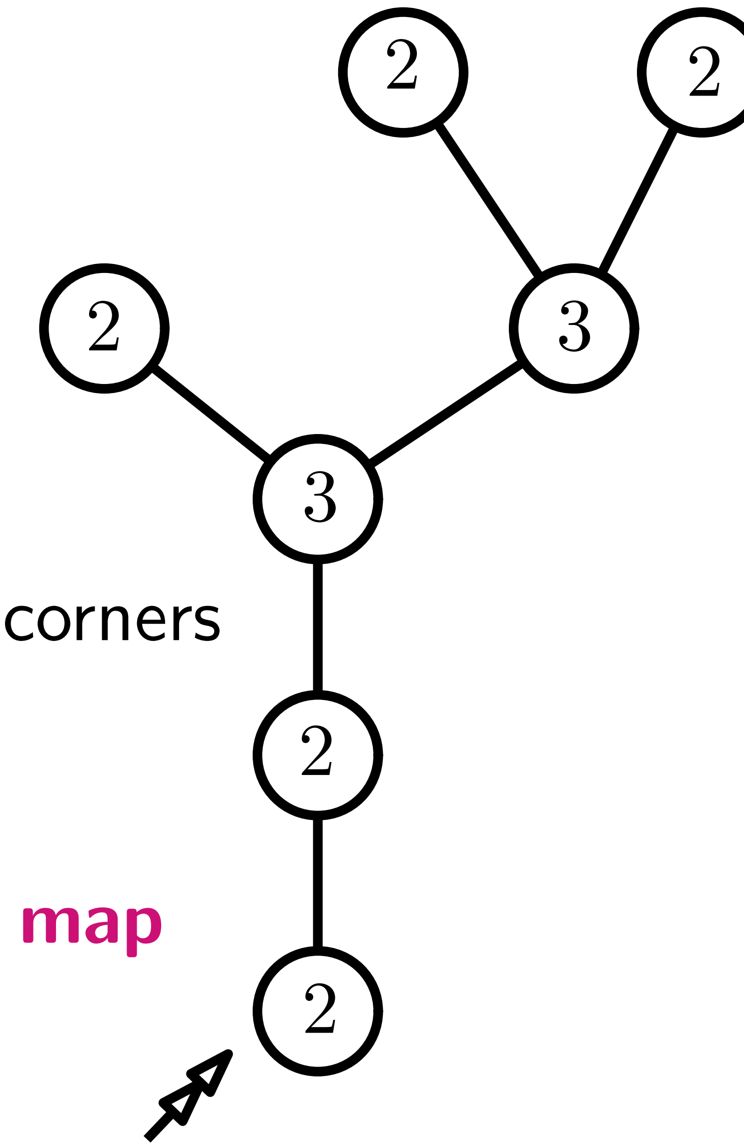
From blossoming trees to labeled trees



label of a vertex =
minimum label of its corners

In the following:
Labels gives approximate
distances to the root **in the map**



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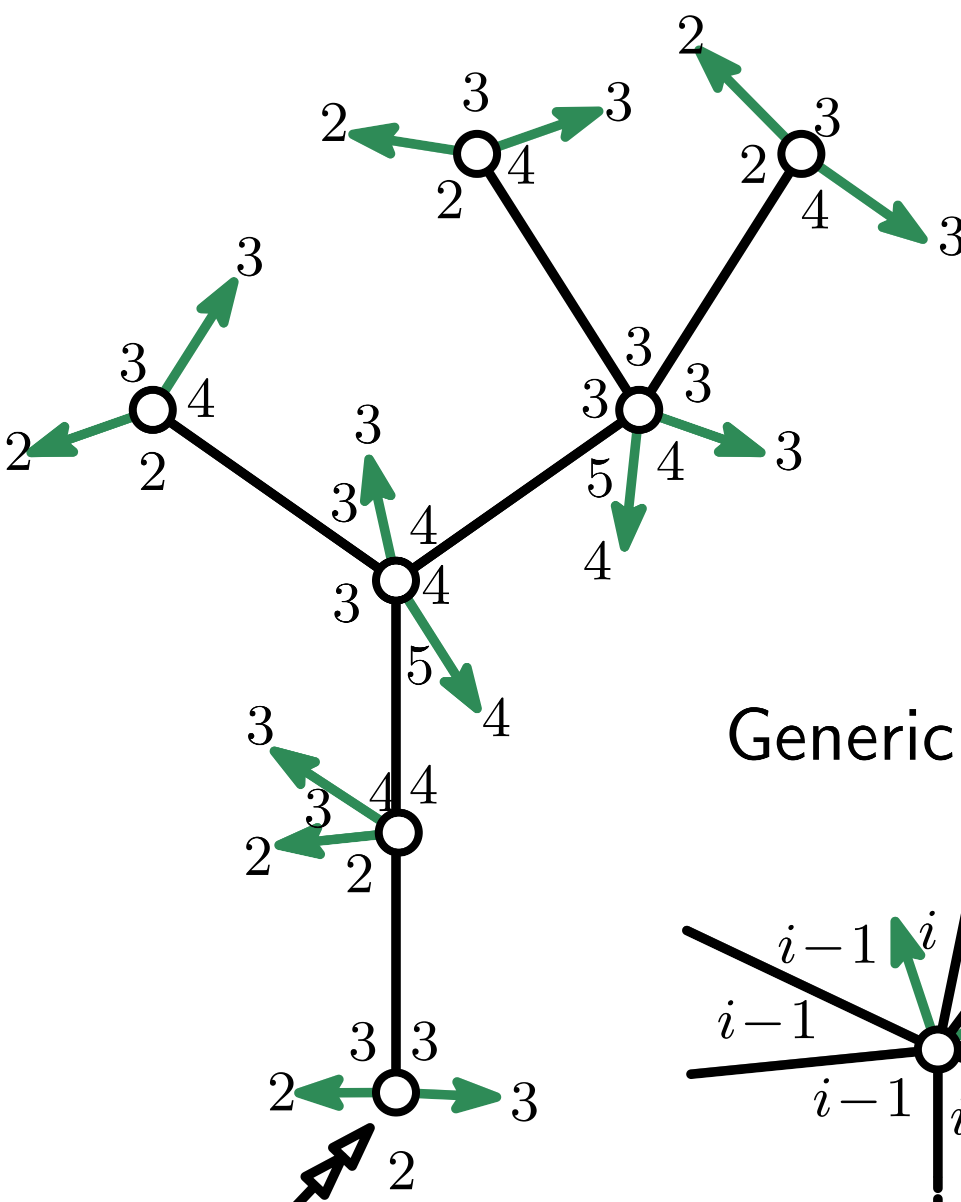
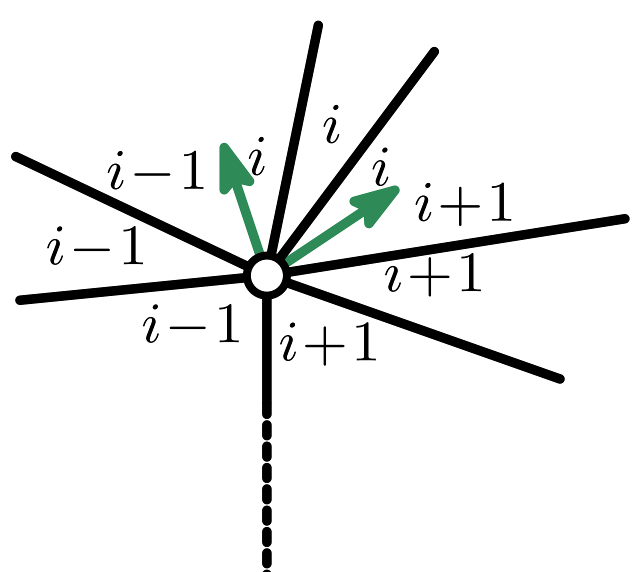
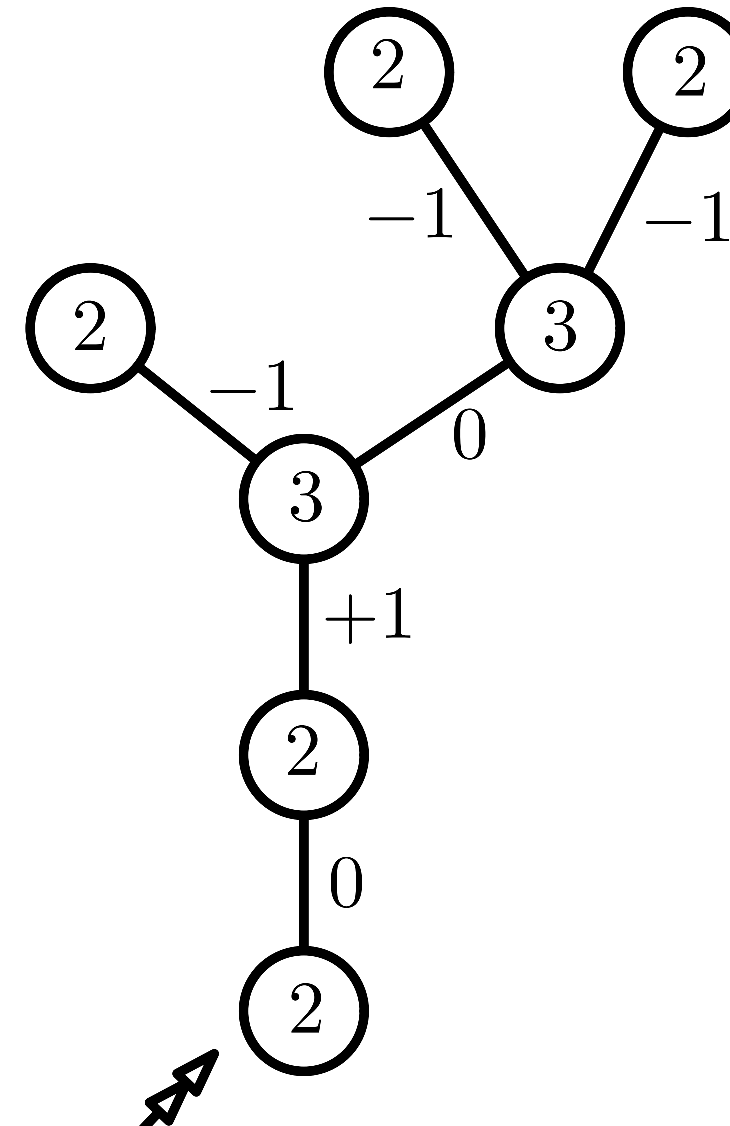


Generic vertex :





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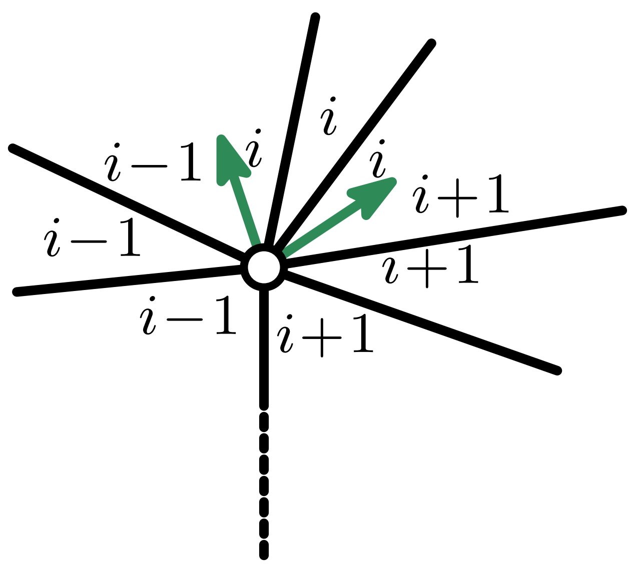

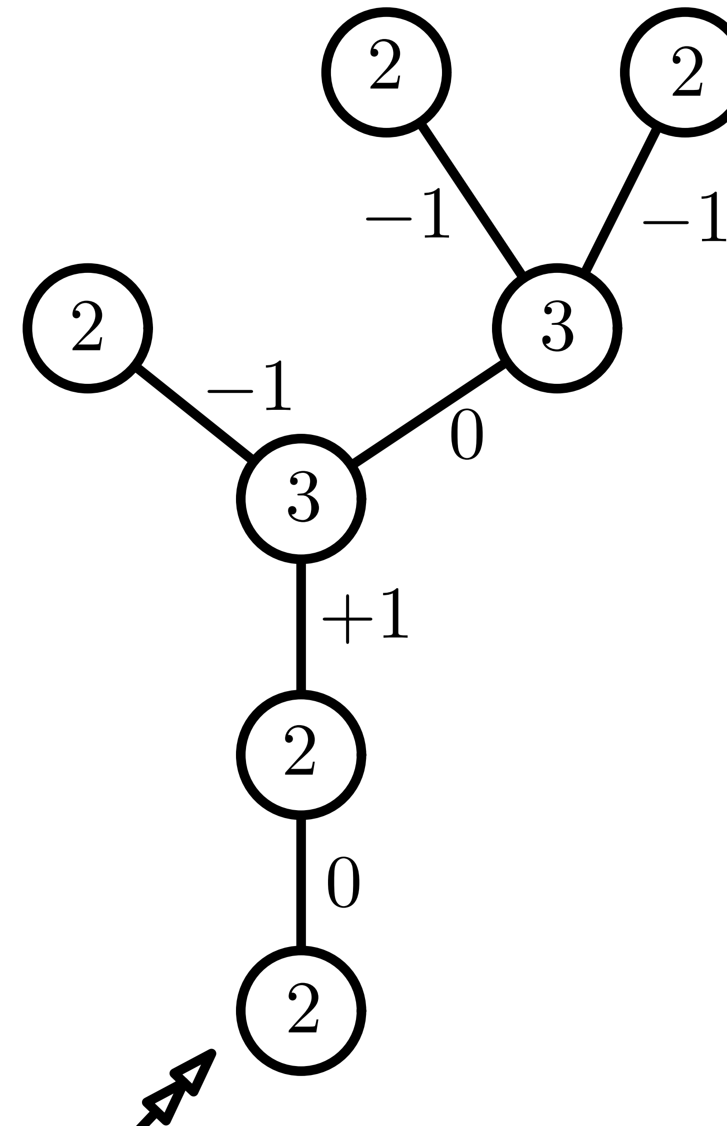

- Can retrieve the blossoming tree from the labeled tree.
- Labeled tree = GW trees + random displacements on edges uniform on

$$\{(-1, -1, \ldots, -1, 0, 0, \ldots, 0, 1, 1 \ldots, 1)\}.$$

almost the setting of [Janson-Marckert] and [Marckert-Miermont] but r.v are not "locally centered" $\Rightarrow$ symmetrization required

Distances in simple triangulations

Claim 1: $3d_M(root, u) \geq$ Label of u

First observation : In the tree, the labels of two adjacent vertices differ by at most 1. **What can go wrong with closures ?**

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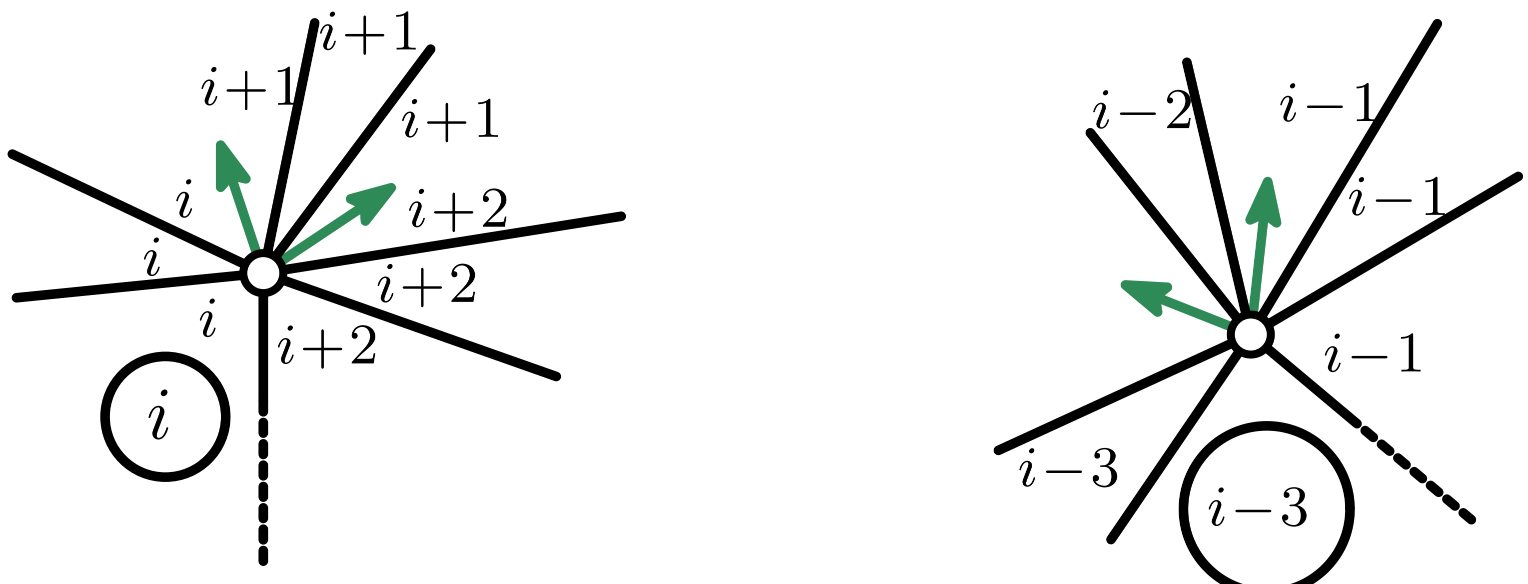

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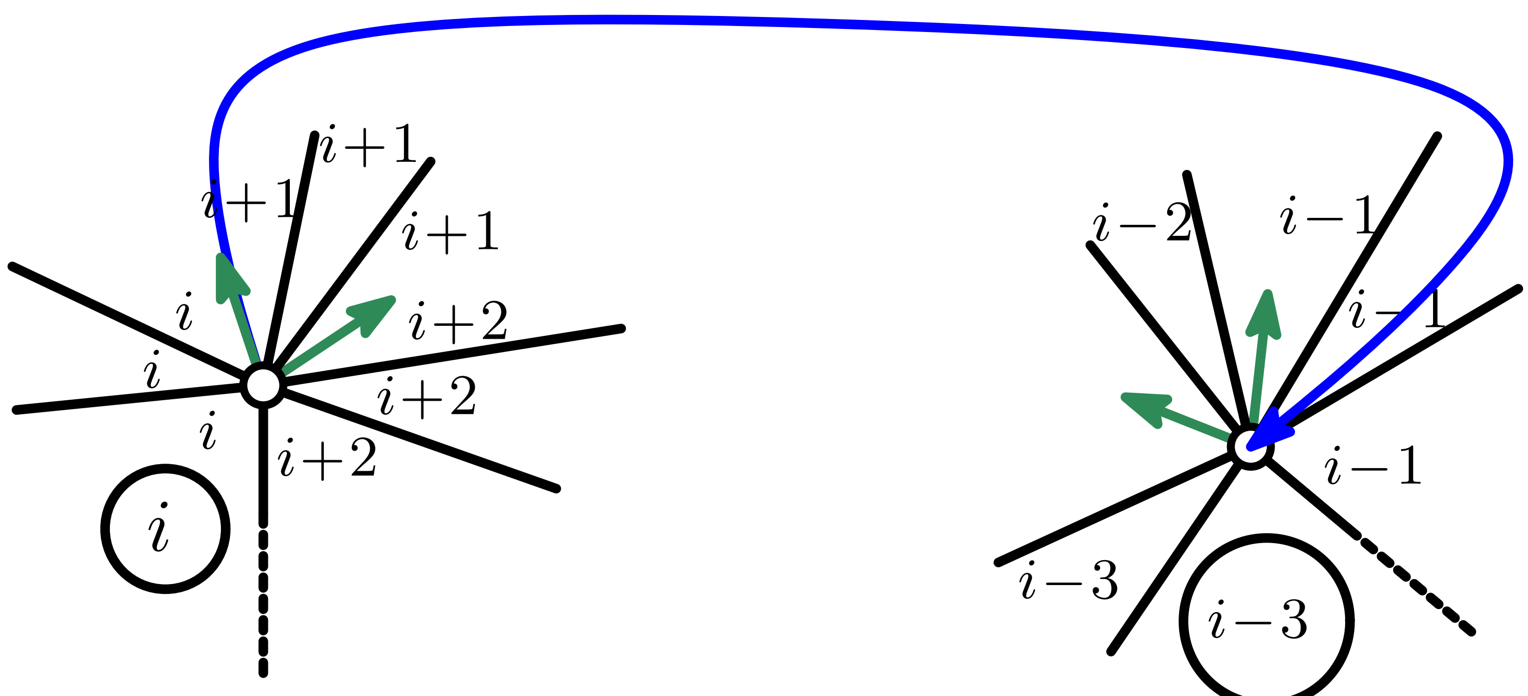

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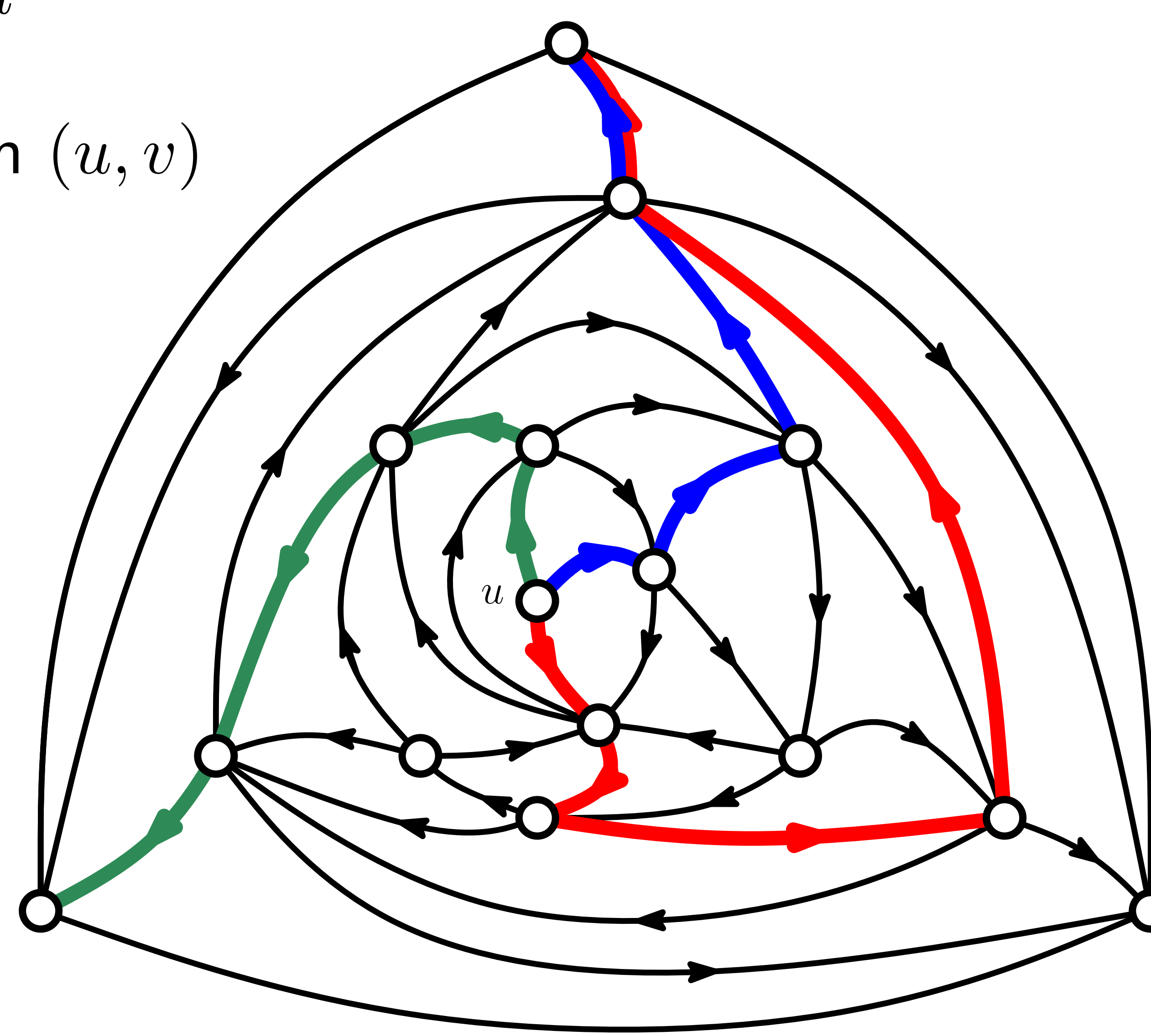

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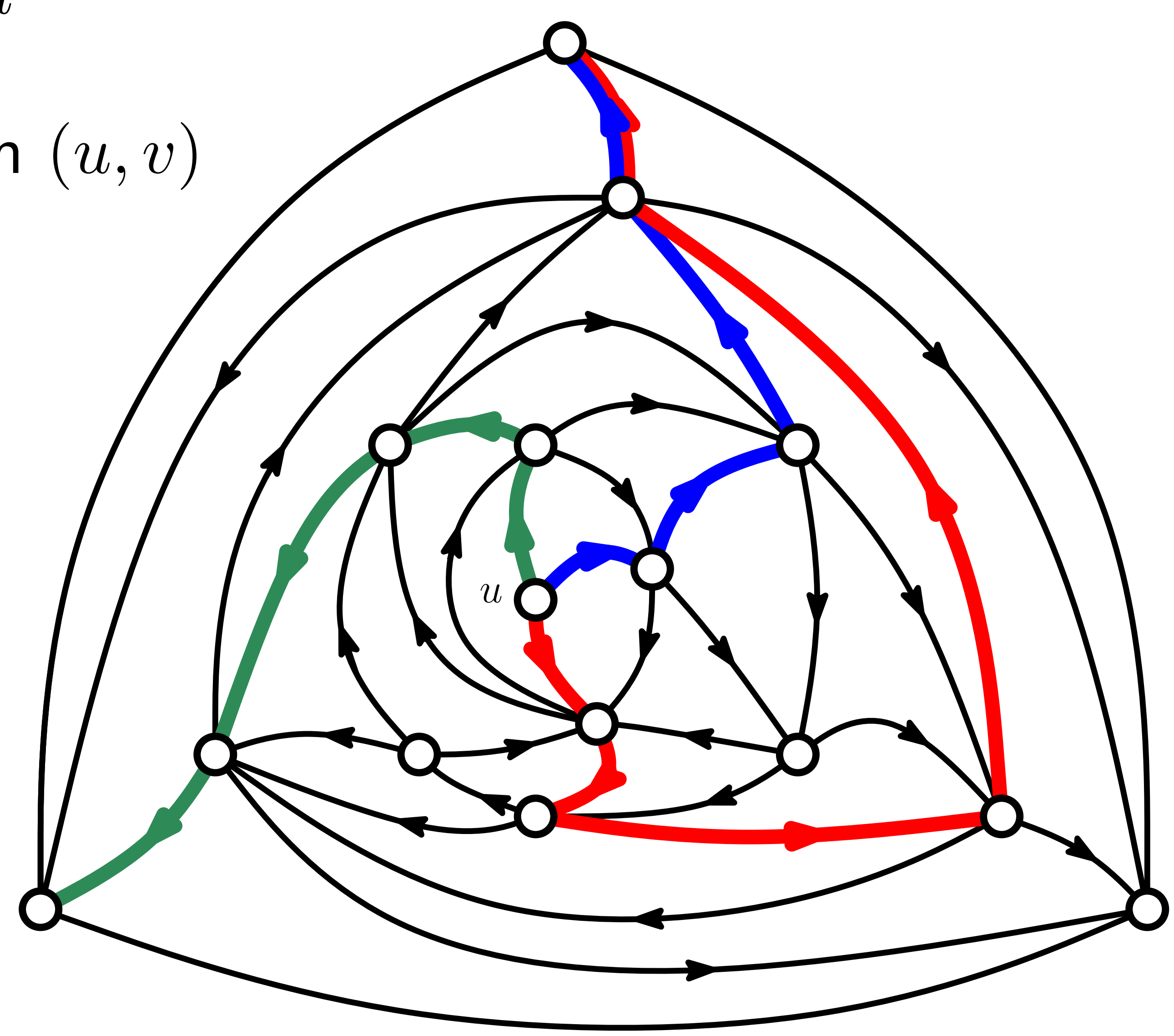

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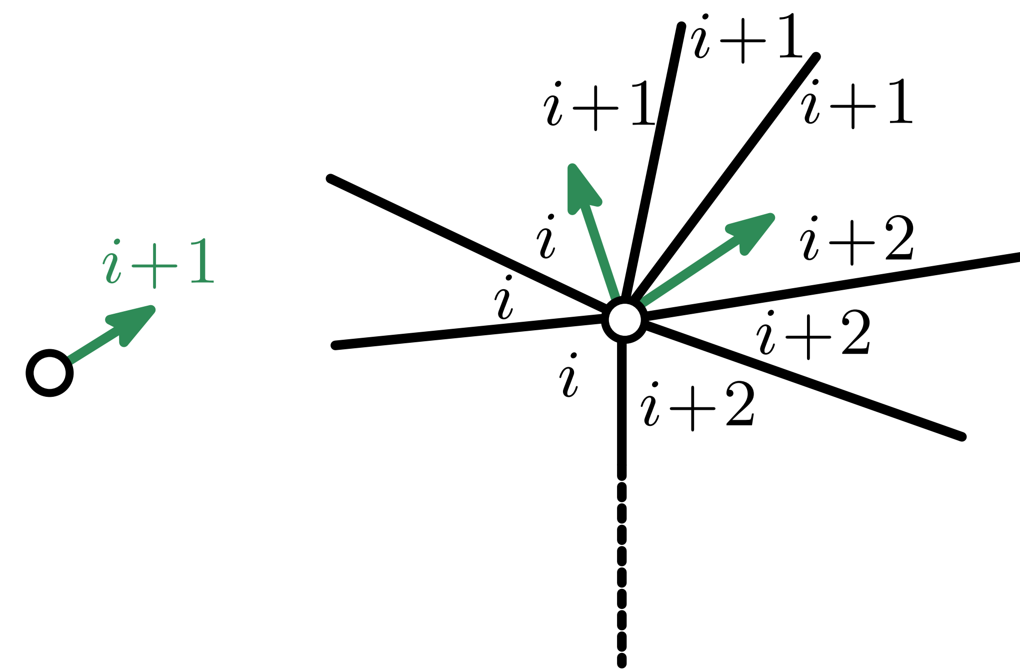

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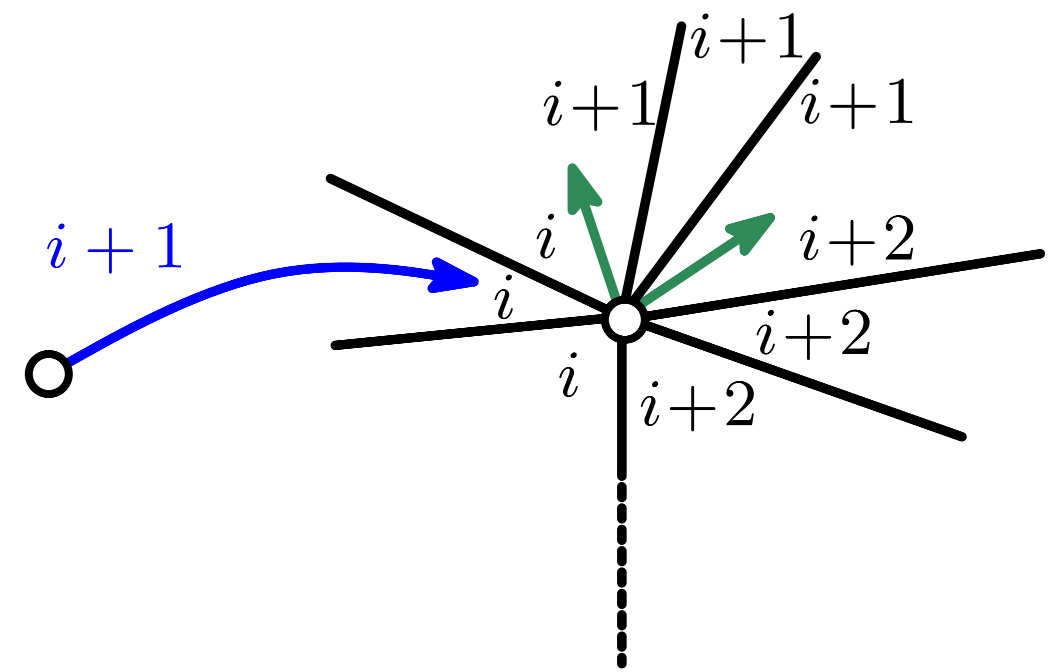

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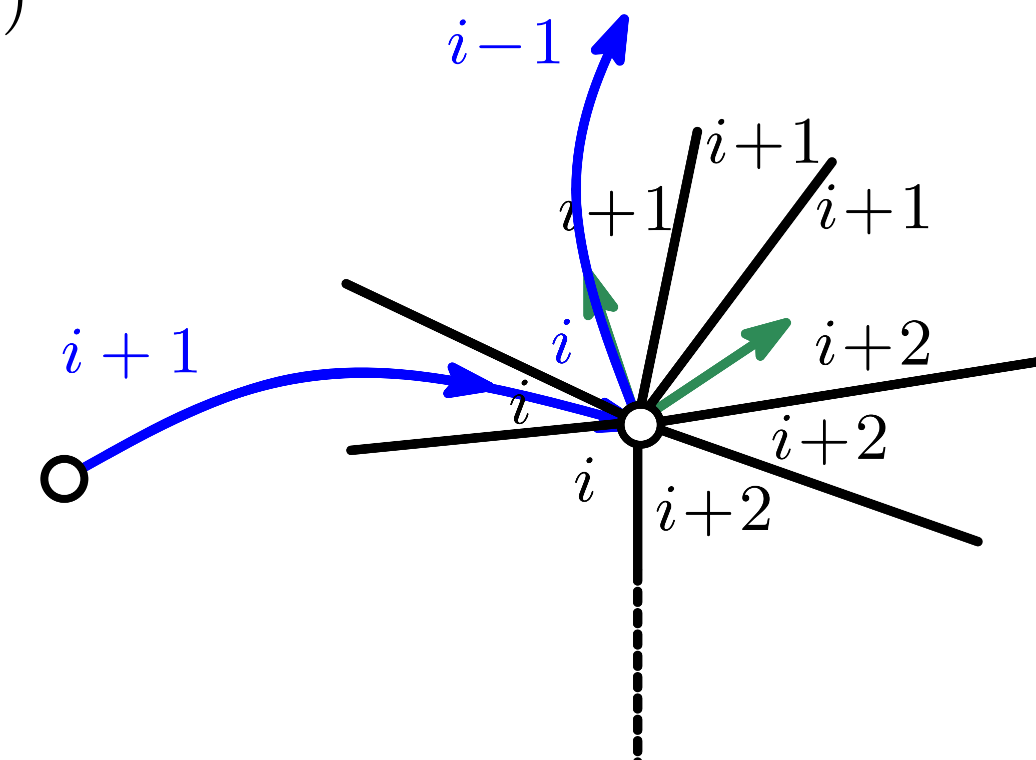
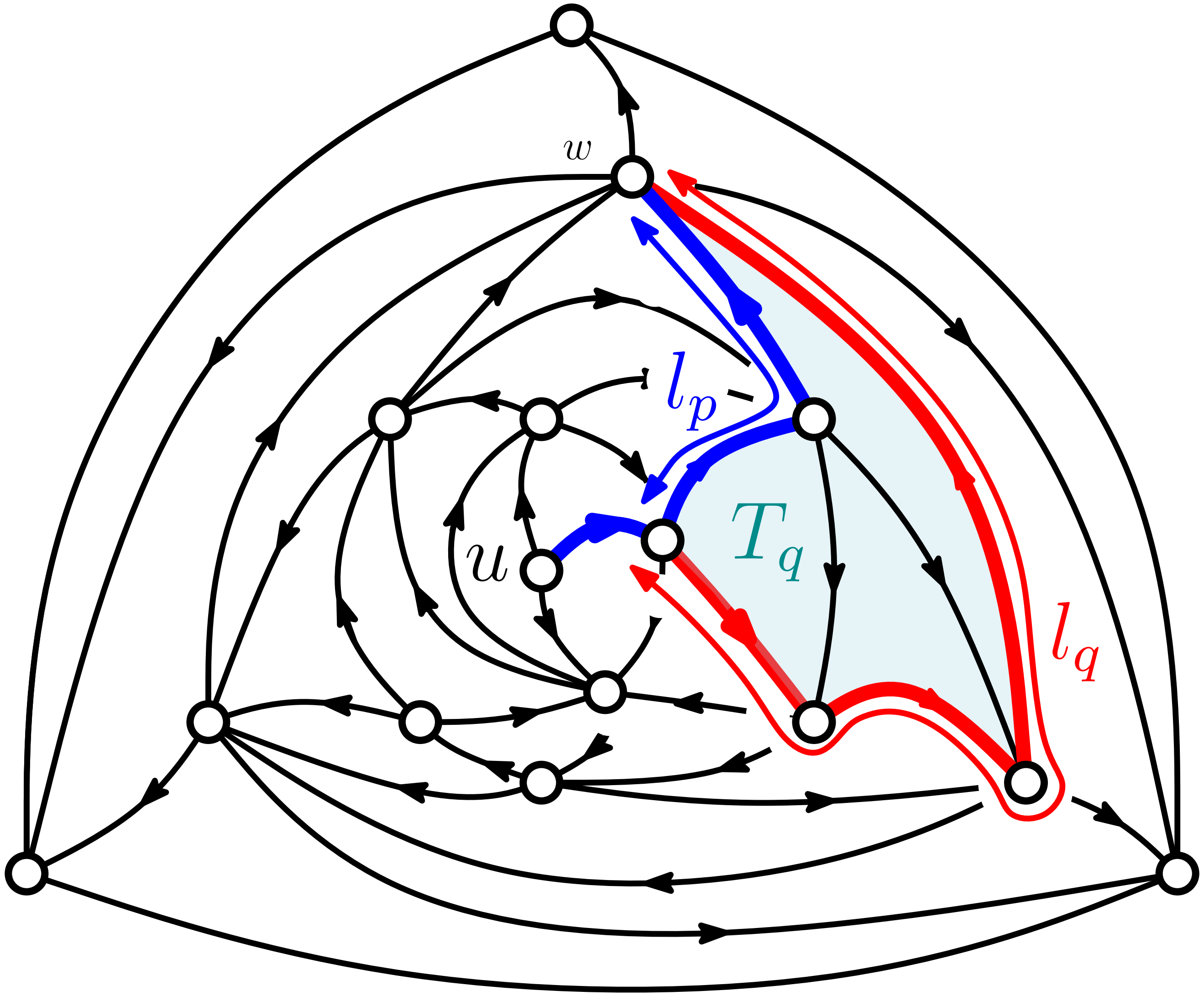


LMP are almost geodesic



Leftmost path

Another path: can it be shorter ?

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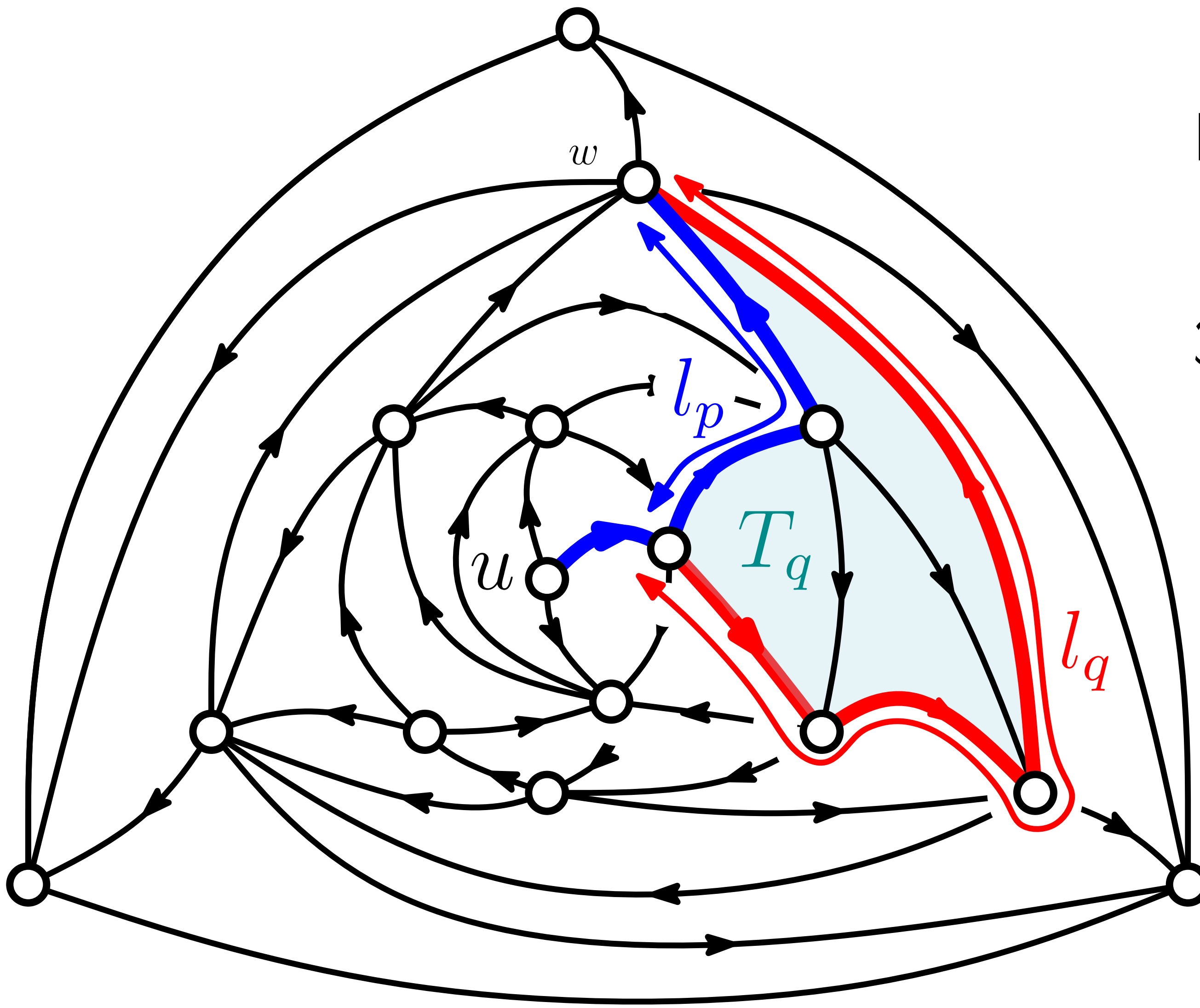

Euler Formula :

$$|E(T_q)| = 3|V(T_q)| - 3 - (\ell_p + \ell_q)$$

3-orientation + LMP :

$$|E(T_q)| \geq 3|V(T_q)| - 2\ell_q - 2$$

$$\implies \ell_q \geq \ell_p + 1$$

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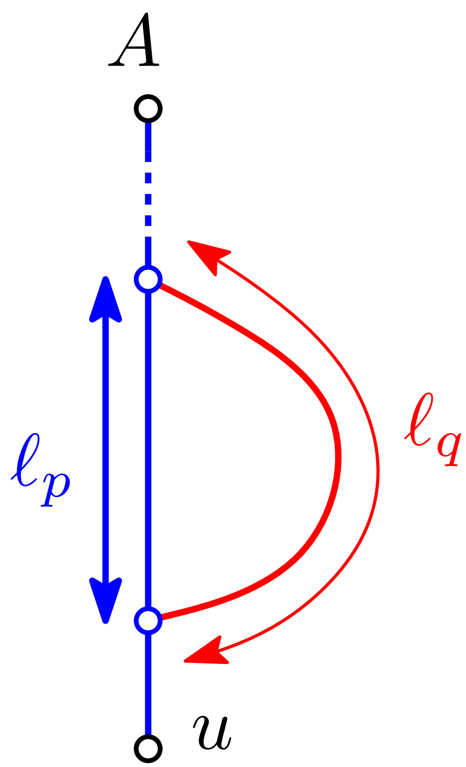

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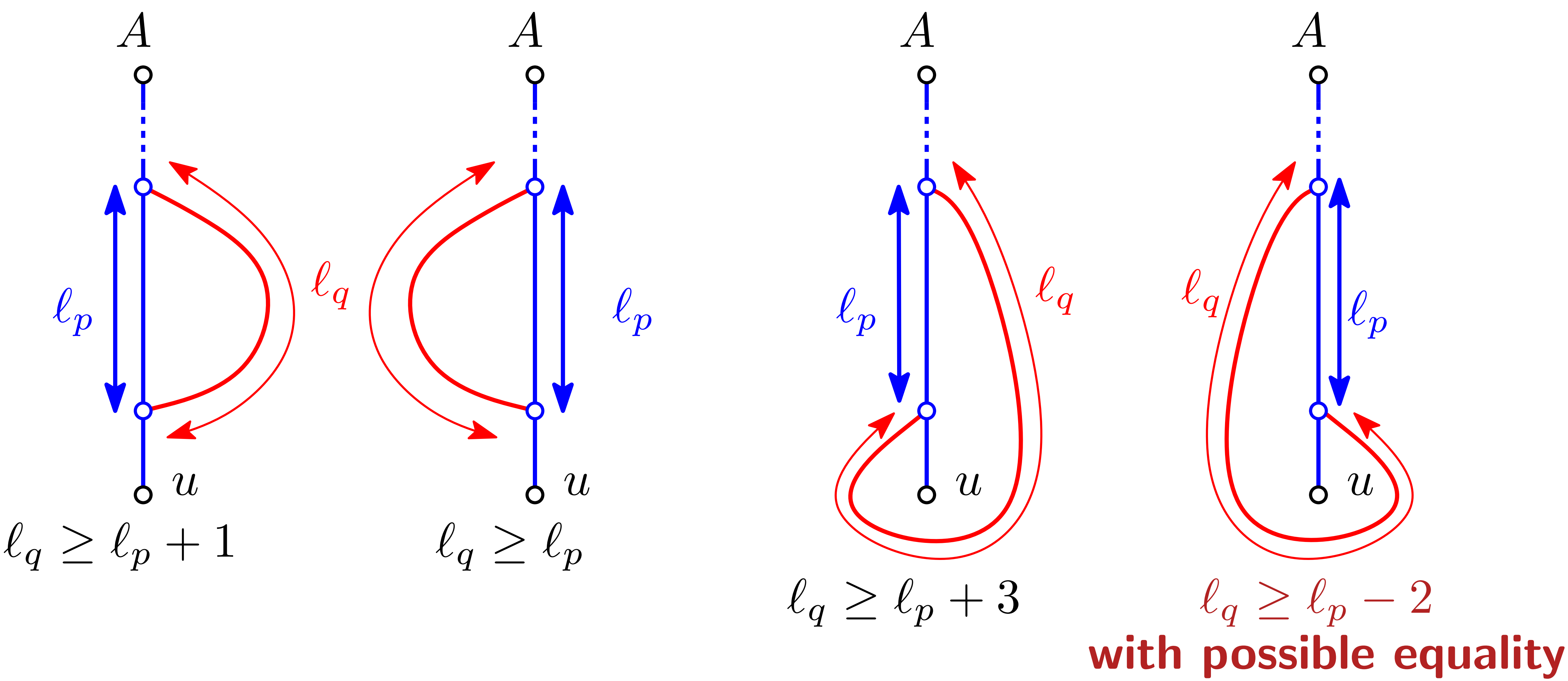

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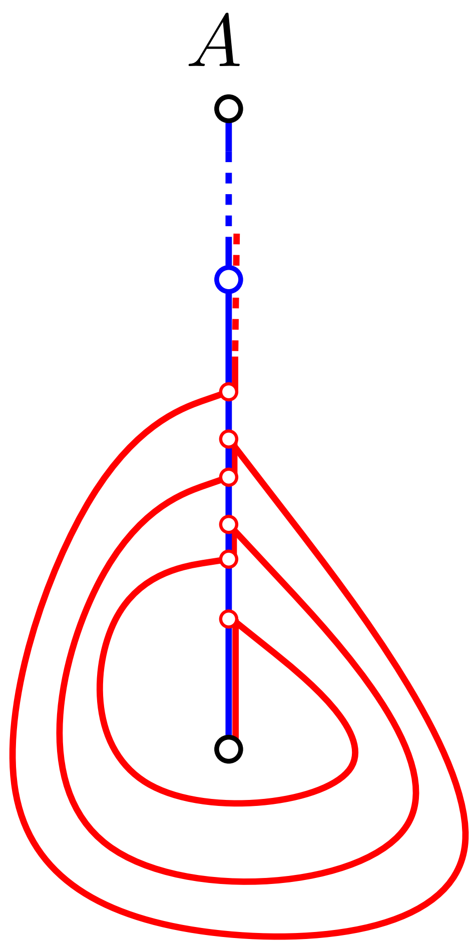

Bad configuration =
too many **windings** around the LMP

But w.h.p a winding cannot be too short.

$\Longrightarrow$ w.h.p the number of windings is $o(n^{1/4})$.

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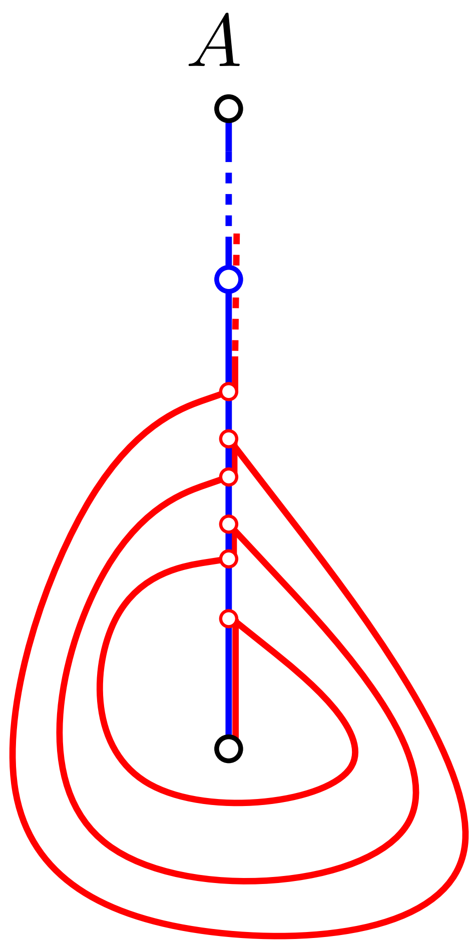

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Proposition:
For $\varepsilon > 0$, let $A_{n,\varepsilon}$ be the event that there exists $u \in M_n$ such that
Label of $u \geq d_{M_n}(u, root) + \varepsilon n^{1/4}$.
Then under the uniform law on $\mathcal{M}_n$, for all $\varepsilon > 0$:

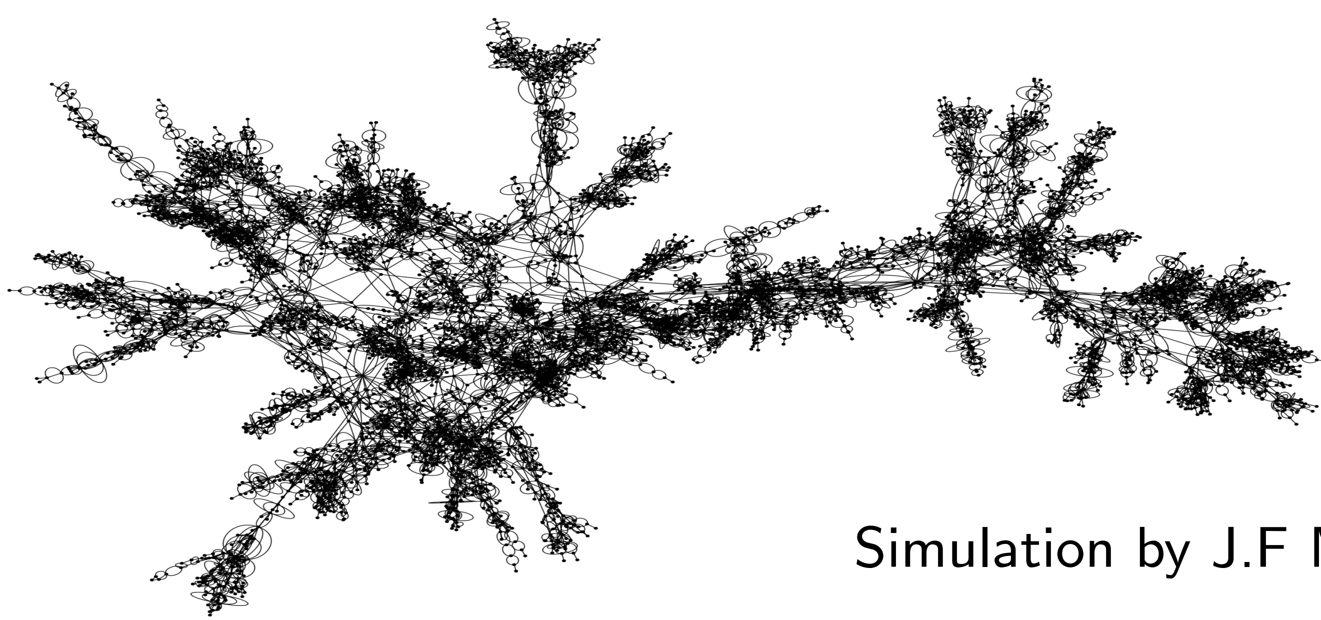
$$\mathbb{P}(A_{n,\varepsilon}) \to 0.$$

The result

Theorem : [Addario-Berry, A.]
(M_n) = sequence of random **simple** triangulations, then:

$$\left(M_n, \left(\frac{3}{4n}\right)^{1/4} d_{M_n}\right) \xrightarrow{(d)} \text{Brownian map}$$

for the distance of Gromov-Hausdorff on the isometry classes of compact metric spaces.



Simulation by J.F Marckert

Beyond the universality

Simple triangulations converge to the Brownian map
$\Rightarrow$ properties of the Brownian map from the simple triangulations ?

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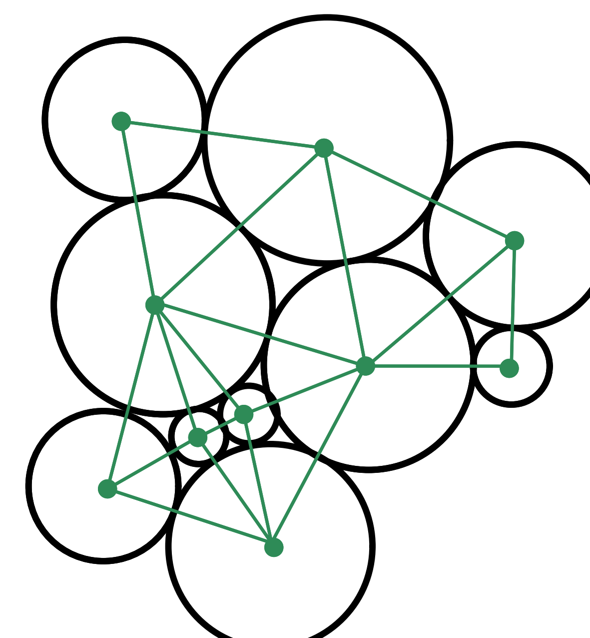
$\Rightarrow$ properties of the Brownian map from the simple triangulations ?

One motivation : Circle-packing theorem

Each simple triangulation M has a unique (up to Möbius transformations and reflections) circle packing whose tangency graph is M.
[Koebe-Andreev-Thurston]

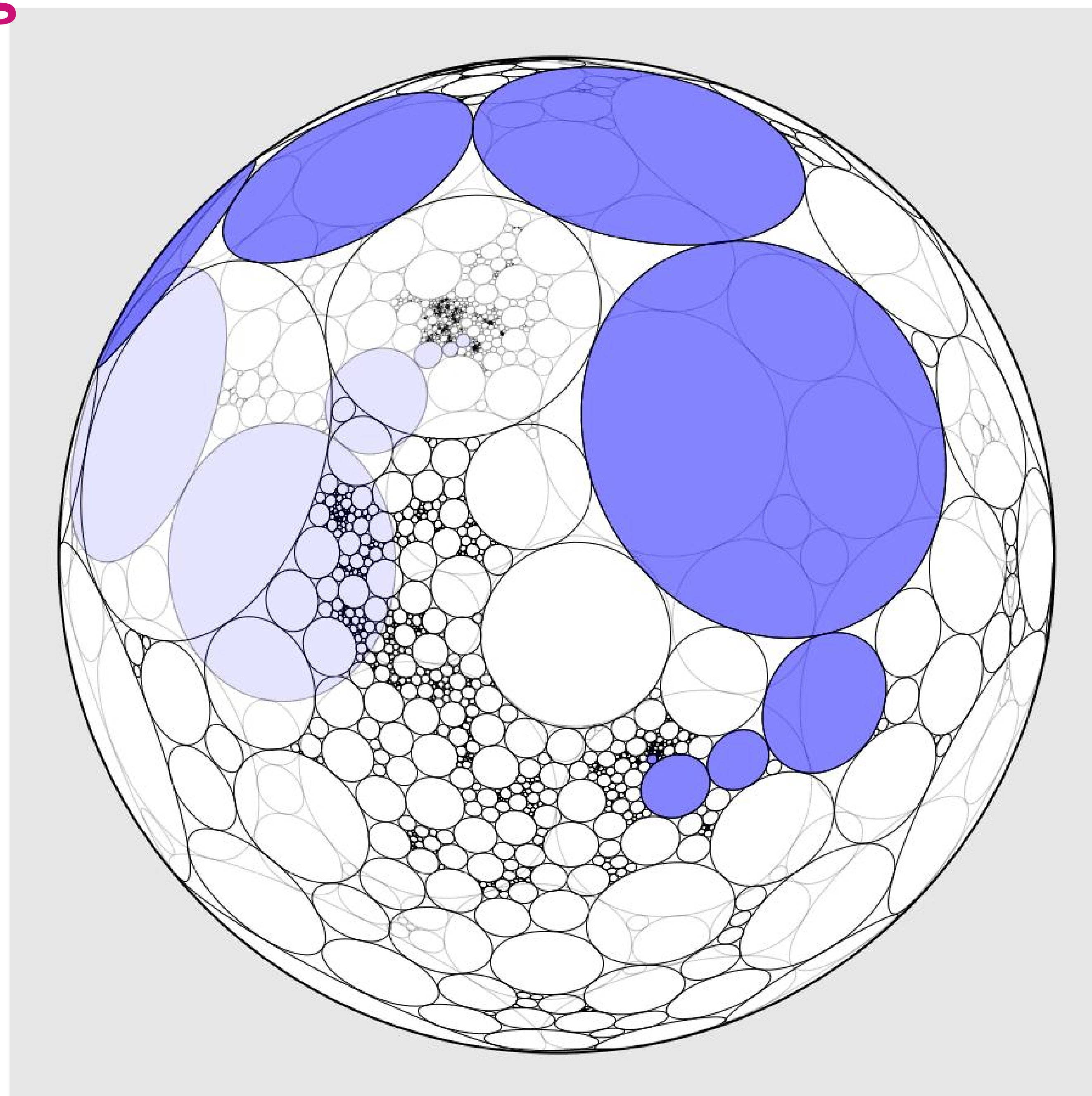


Gives a canonical embedding of simple triangulations in the sphere and possibly of their limit.

Random circle packing

Random circle packing = canonical embedding of random simple triangulation in the sphere.

Gives a way to define a canonical embedding of their limit ?



Team effort : code by Kenneth Stephenson, Eric Fusy and our own.

Perspectives

Same approach works also for simple quadrangulations, and also for simple maps (ongoing work with Bernardi, Collet, Fusy).

Can we make this approach work for the general setting of bijections developped in [A.,Poulalhon] and in [Bernardi, Fusy] or for triangulations on the torus (cf next talk !)?

Can we say something about a random circle packing ?

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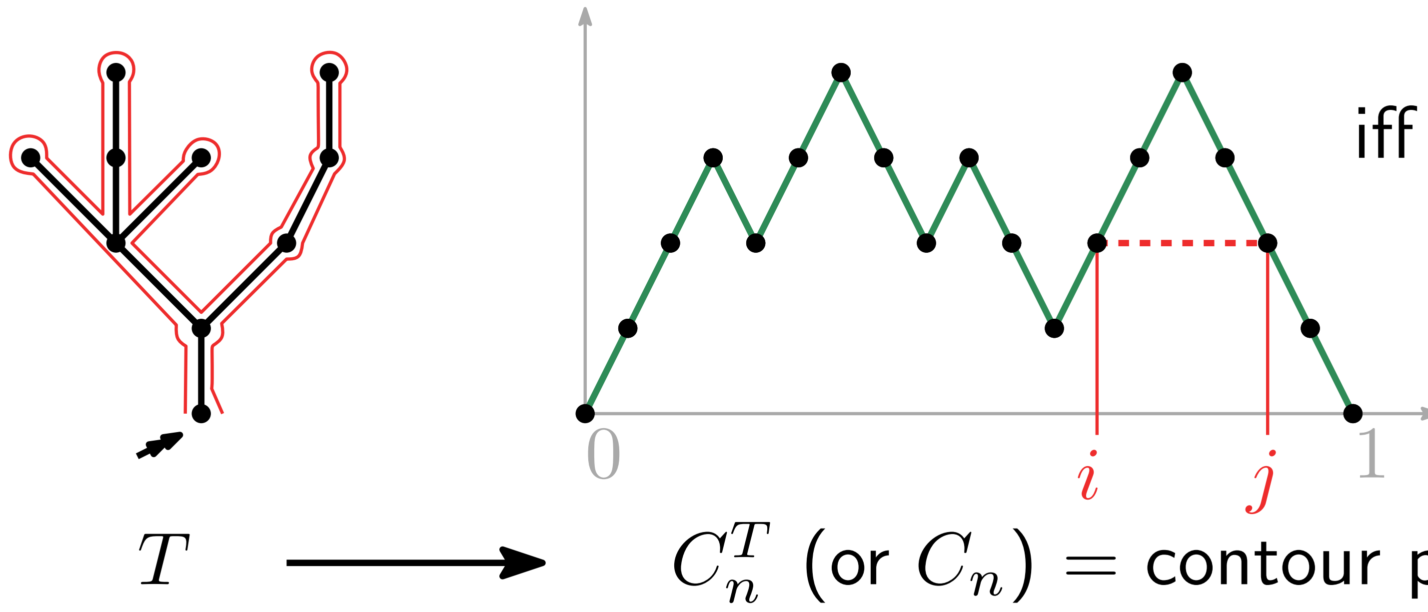
Thank you !

Brownian snake $(e_t, Z_t)_{0\leq t\leq 1}$

1st step : the Brownian tree [Aldous]



i and j = same vertex of T
iff $C_n(i) = C_n(j) = \min_{i\leq k\leq j} C_n(k)$

C_n^T (or C_n) = contour process

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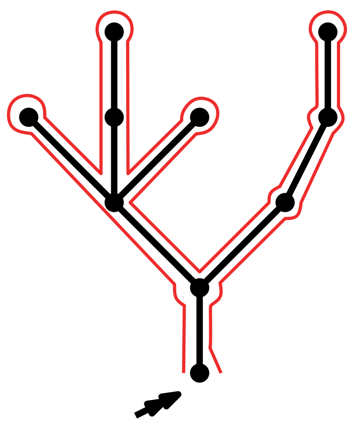

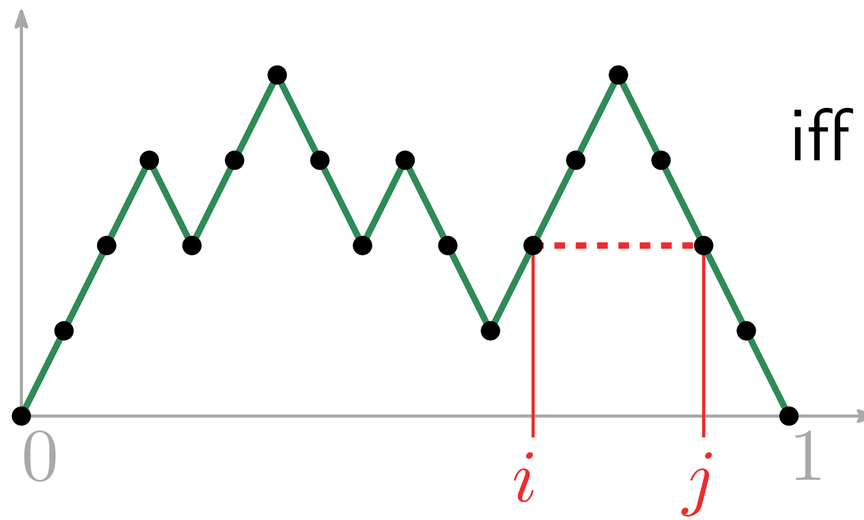

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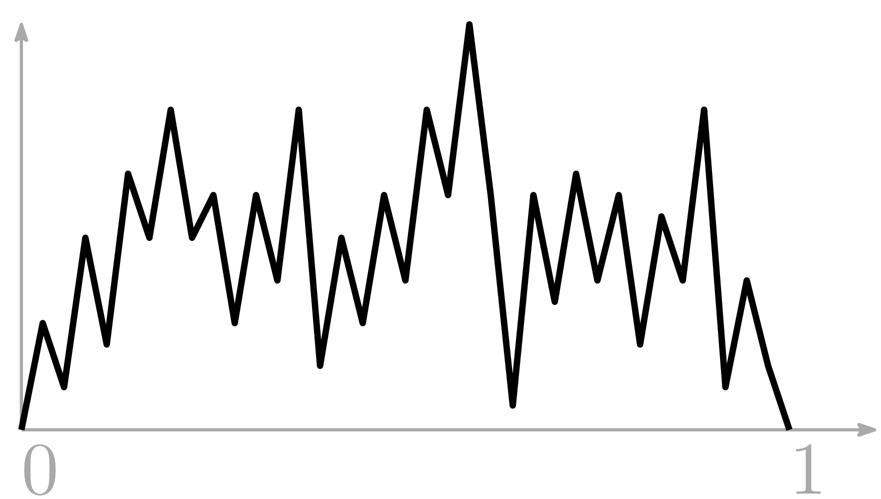

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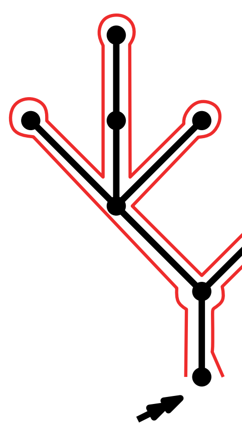

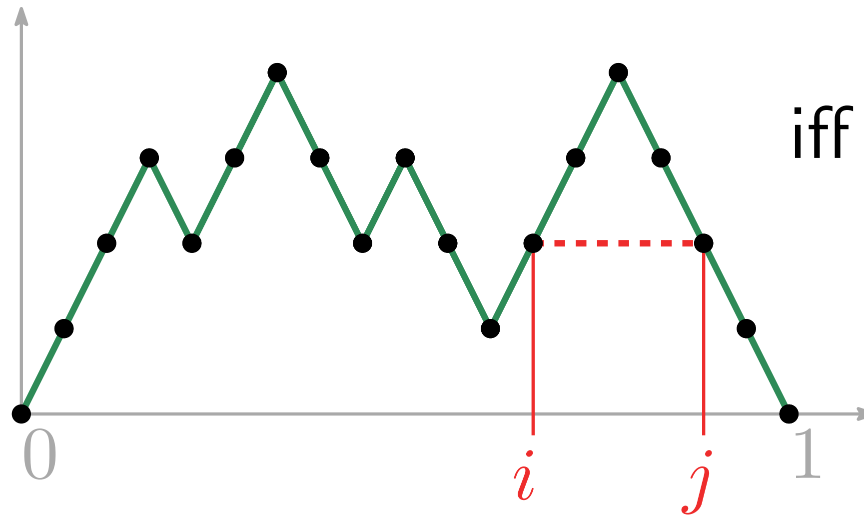

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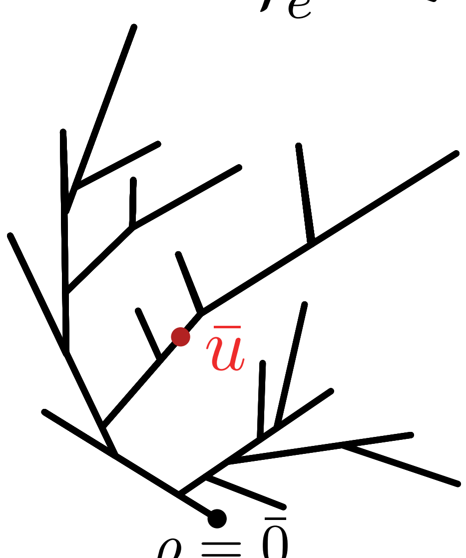

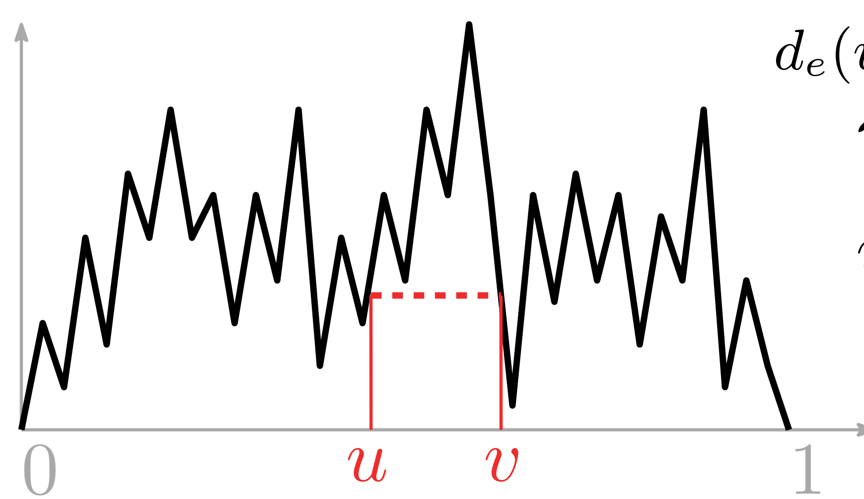

$d_e(u,v) = e_u + e_v - 2\min_{u\le s\le v} e_s$

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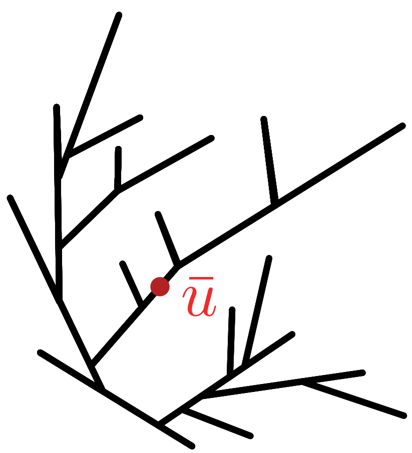

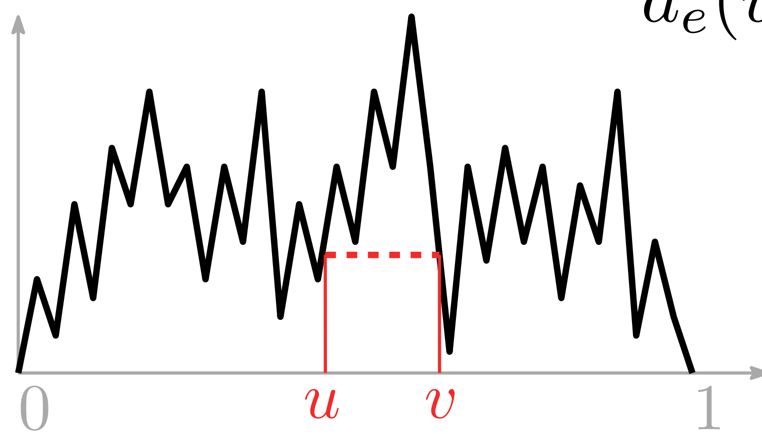

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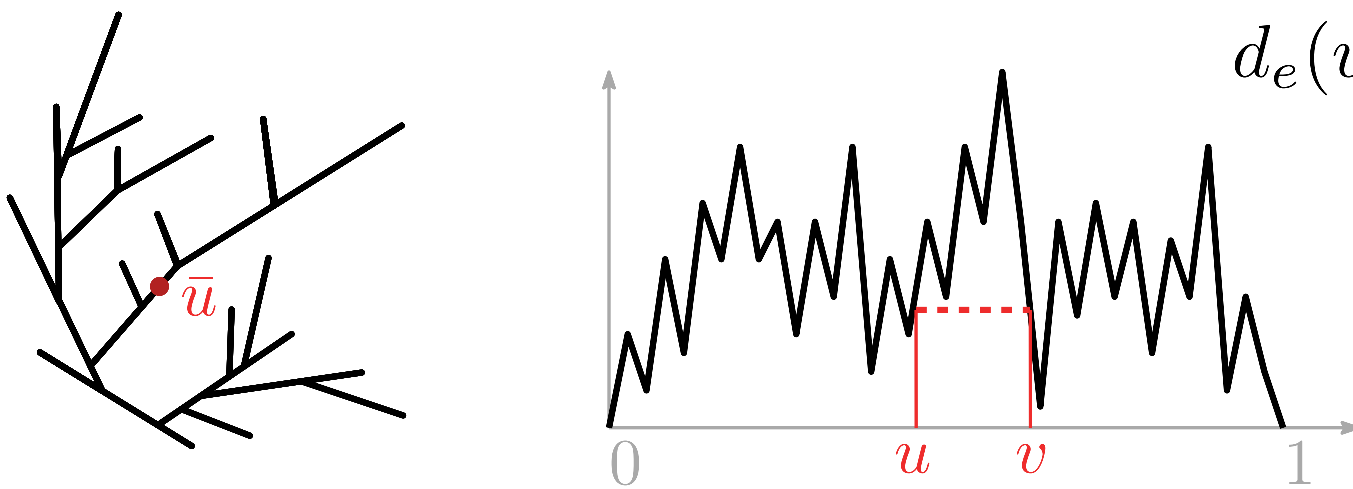

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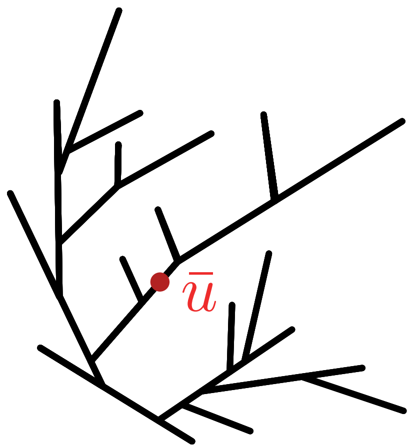
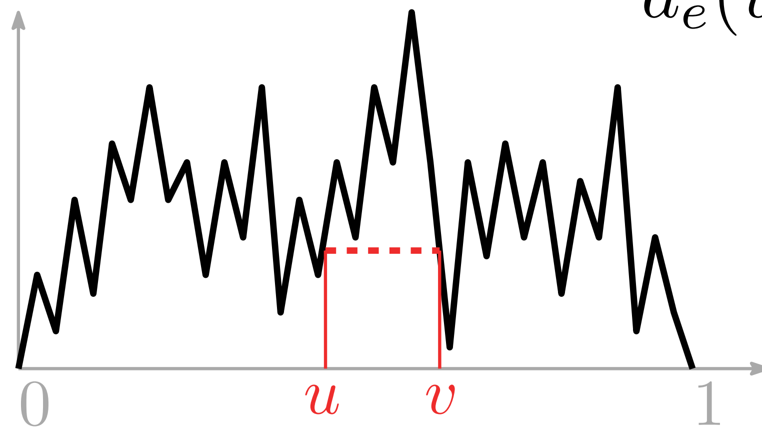
2nd step : Brownian labels

Conditional on $\mathcal{T}_e$, Z a centered Gaussian process with $Z_\rho = 0$ and $E[(Z_s - Z_t)^2] = d_e(s,t)$

$Z \sim$ **Brownian motion on the tree**

The Brownian map





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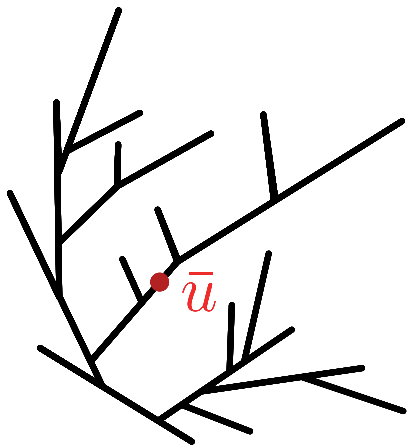

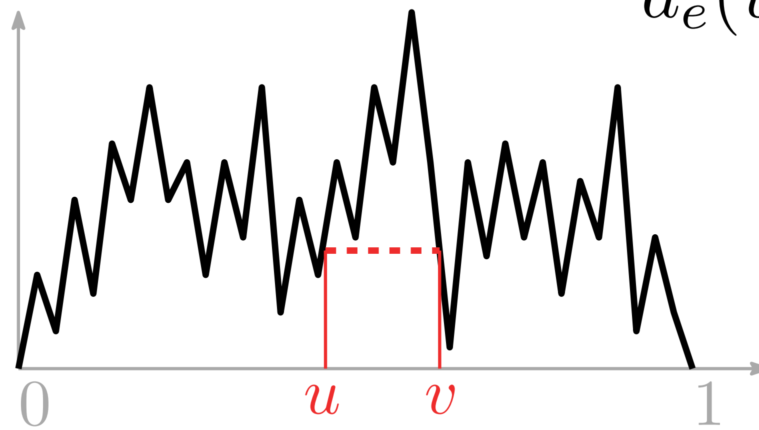

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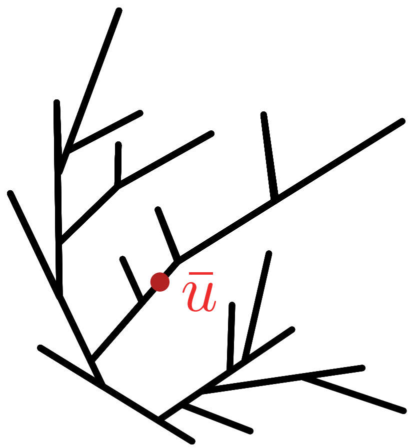

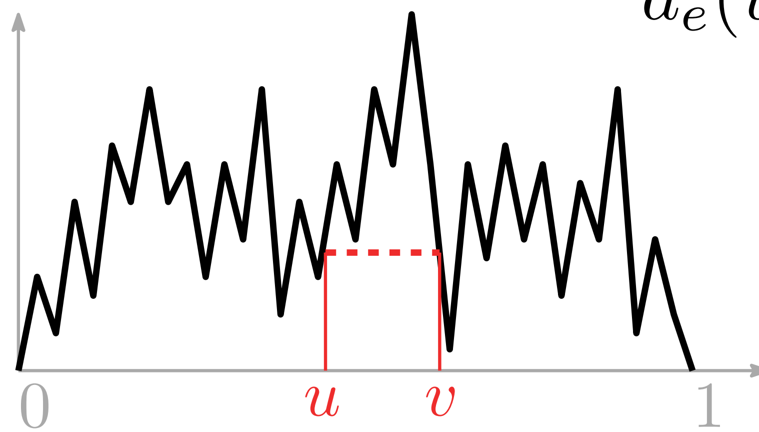

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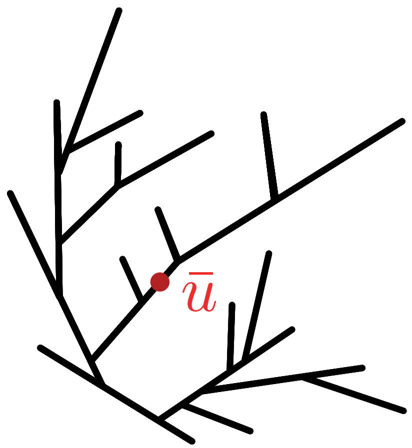

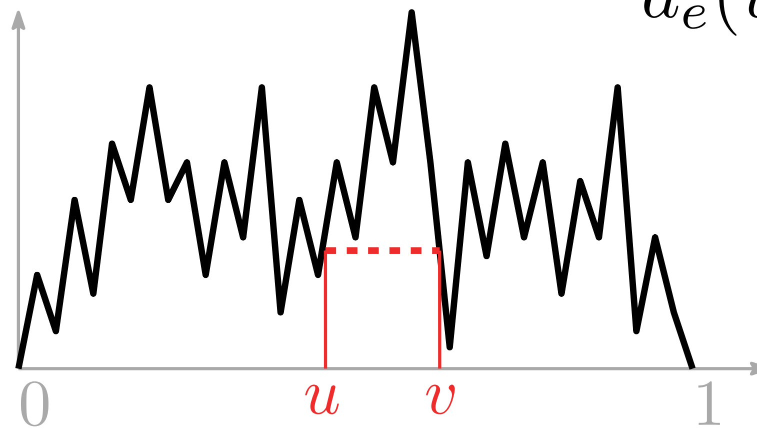

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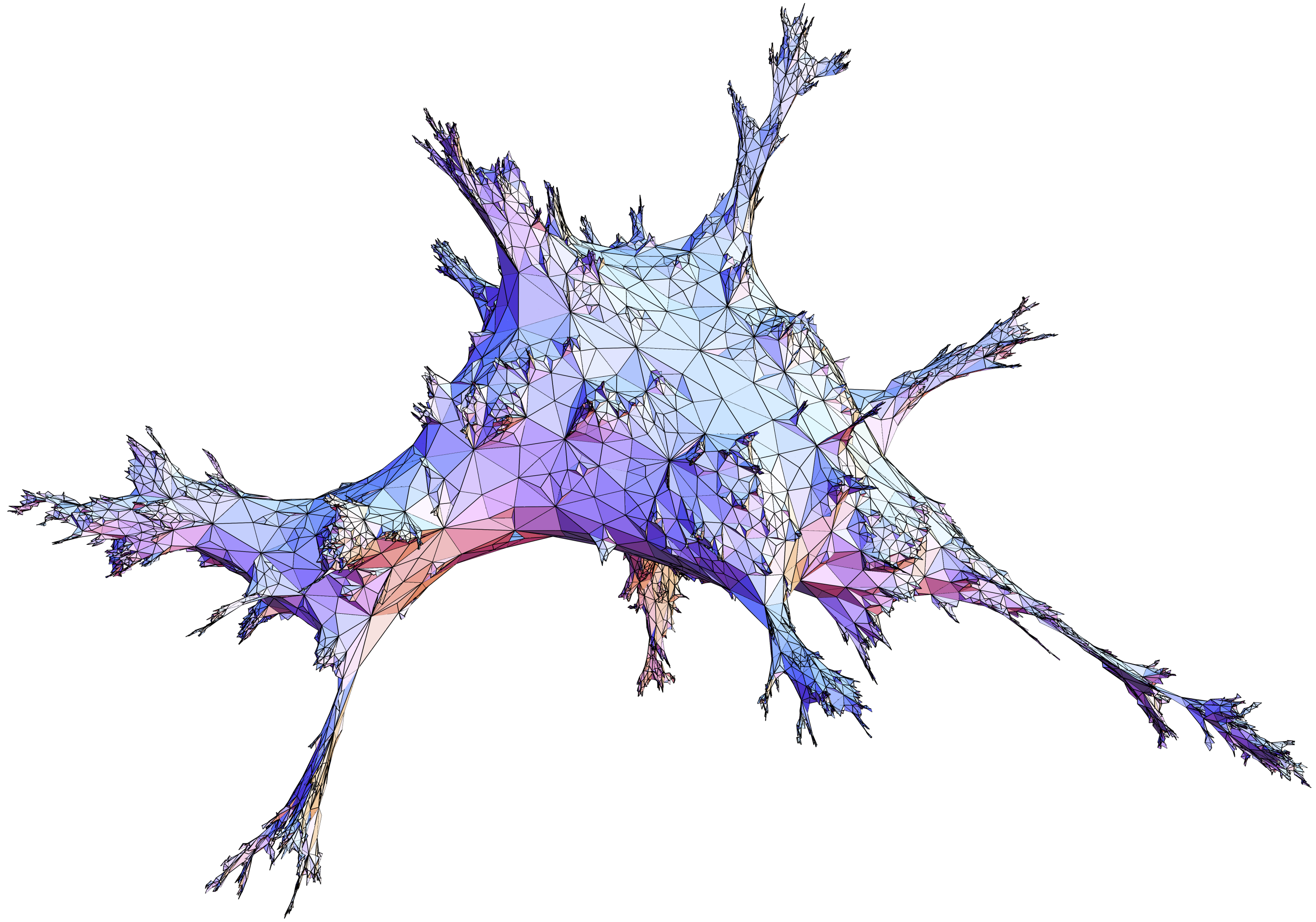

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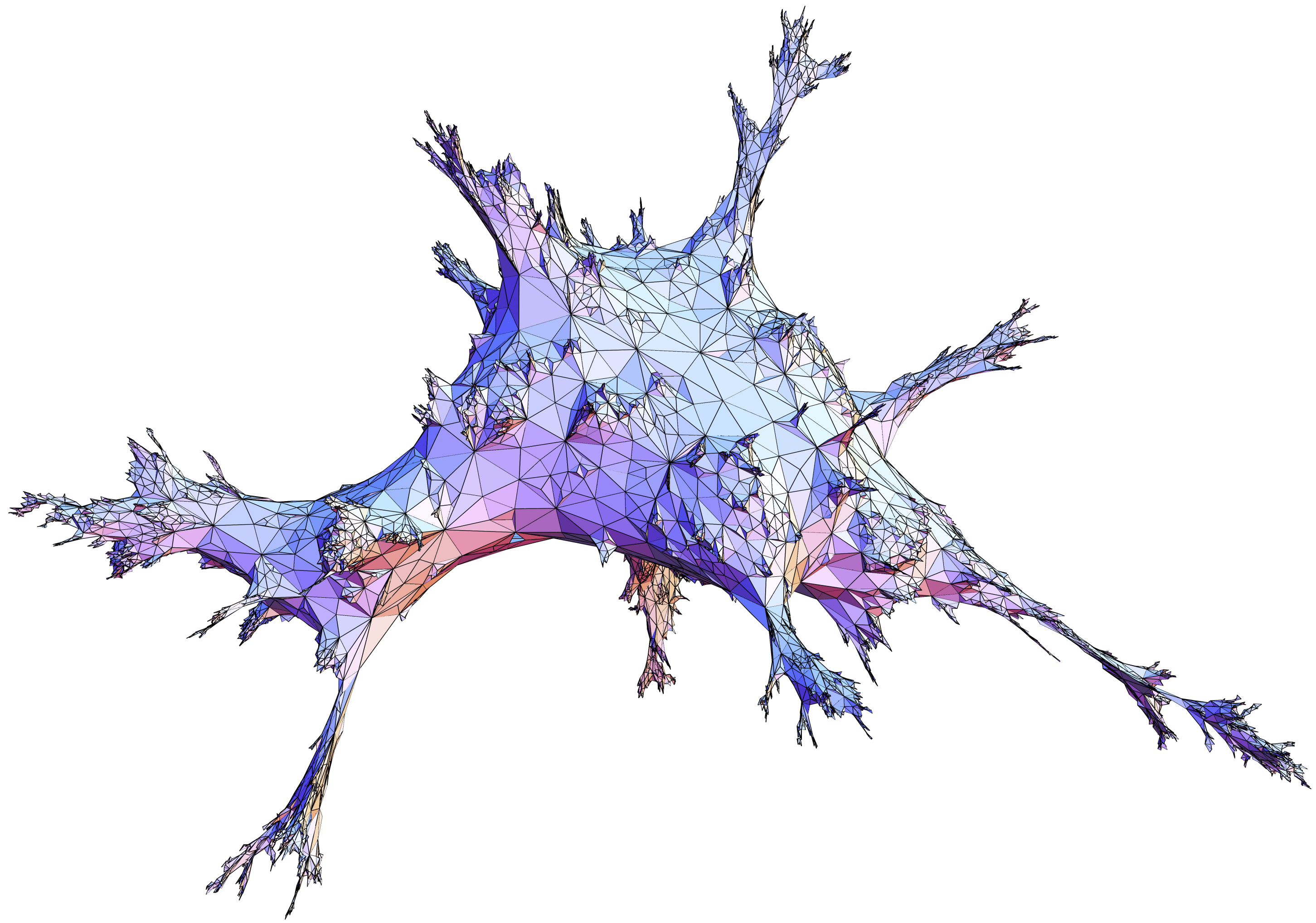

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