Edge- and face-width of projective quadrangulations

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G-SCOP

Journées Graphes & Surfaces
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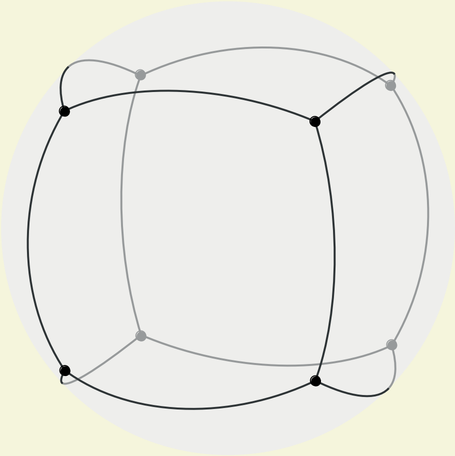
Quadrangulations of surfaces

Definition

A quadrangulation of a surface Σ is an embedding of a simple graph G in Σ such that every face is bounded by 4 edges.



Projective quadrangulations

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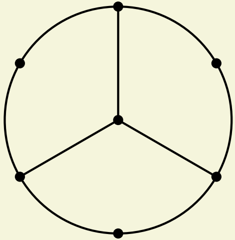
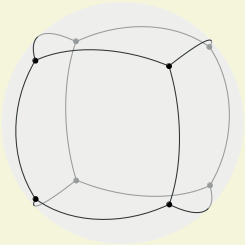
- Equivalently, by identifying all pairs of antipodal points in the sphere.
- A projective quadrangulation is a graph embedded in the projective plane so that the boundary of every face is a 4-cycle.

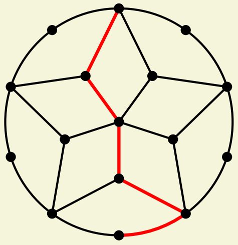




Edge- and face-width of embedded graphs

Edge-width

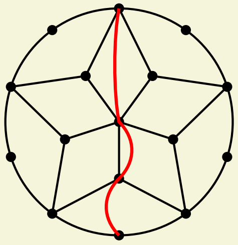
Minimum length of a non-contractible cycle in G



$\mathrm{ew}(G) = 5$

Face-width

Minimum number of points that γ and G have in common, over all non-contractible closed curves γ



$\mathrm{fw}(G) = 3$

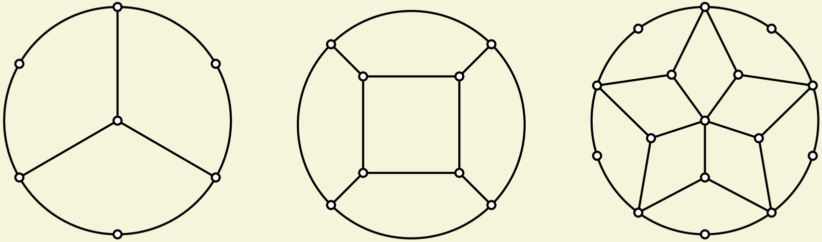
Parity of cycles in quadrangulations

Observation

In every quadrangulation of a surface, homologous cycles have the same parity.

Lemma

In every quadrangulation of the projective plane, the non-bounding cycles are odd if and only if G is non-bipartite.



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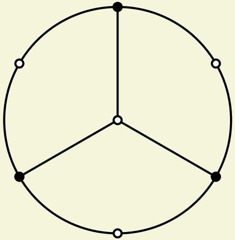
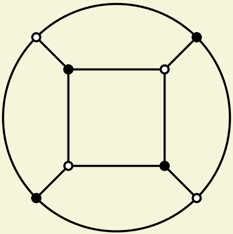
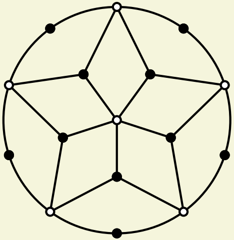
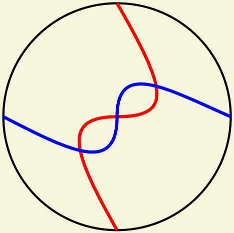
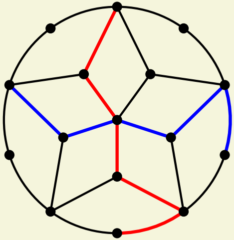




Odd cycles in projective quadrangulations

Key property of the projective plane

Two non-contractible simple closed curves in the projective plane intersect an odd number of times.

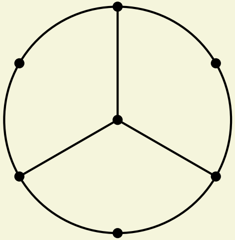




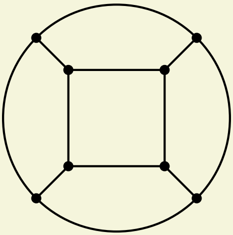
Corollary

If G is a non-bipartite projective quadrangulation, any two odd cycles must intersect.

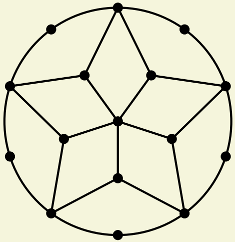
Colouring projective quadrangulations



K_4

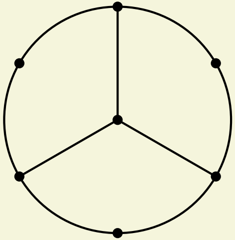


$K_{3,3}$



Grötzsch graph

Colouring projective quadrangulations



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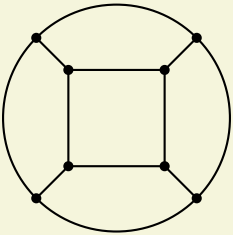

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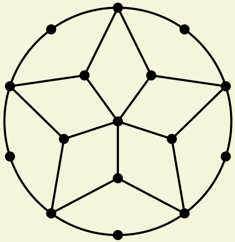


Grötzsch graph

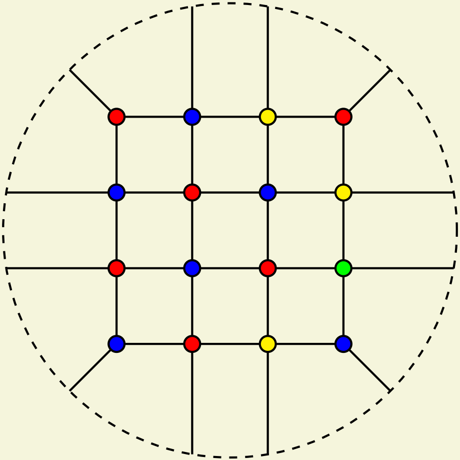
Theorem (Youngs 1996)

If G is a quadrangulation of the real projective plane P^2, then $\chi(G) = 2$ or $\chi(G) = 4$. Moreover, G is 4-edge-critical if and only if G has no separating 4-cycle.

A question of Nakamoto and Ozeki

If G is a non-bipartite projective quadrangulation, is there a 4-colouring of G such that one colour class has size 1 and another has size $o(n)$?

Example



A first approach

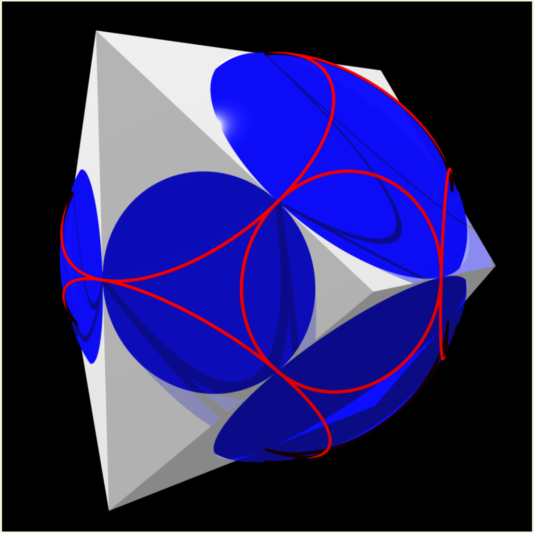
Weaker question

If G is a non-bipartite projective quadrangulation, is there an odd cycle transversal of length $o(n)$?

The Koebe–Andreev–Thurston Theorem

- For every planar 3-connected graph, there is a representation as the graph of a 3-polytope whose edges are all tangent to the unit sphere $S^2 \subset \mathbb{R}^3$, and such that 0 is the barycentre of the contact points.
- This representation is unique up to rotations and reflections of the polytope in $\mathbb{R}^3$.
- In particular, in this representation every combinatorial symmetry of the graph is realised by a symmetry of the polytope.

Example



(picture by David Eppstein)

Upper bound on face-width via circle packing (1/2)

- Represent G as an antipodally symmetric quadrangulation $\tilde{G}$ of the sphere with $2n$ vertices.
- By the Koebe–Andreev–Thurston Theorem, there is a 3-polytope P whose edges are all tangent to the sphere, such that the 1-skeleton of P is isomorphic to $\tilde{G}$, and the polytope P is antipodally symmetric.
- This gives rise to an antipodally symmetric packing of spherical caps $\mathcal{C} = (C_v : v \in \tilde{V})$, where each C_v is the spherical cap consisting of all the points on the unit sphere that are "visible" from v.
- Let ρ_v be the spherical radius of C_v.
- A random plane through the centre of the sphere intersects $\sum_{v \in V} \sin \rho_v$ caps.

Upper bound on face-width via circle packing (2/2)

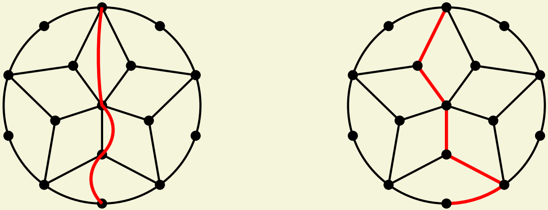
- The cap C_v has area $4\pi \sin^2 \frac{\rho_v}{2}$.
- The caps are disjoint, so $\sum_{v \in V} 4\pi \sin^2 \frac{\rho_v}{2} < 4\pi$
- It follows that $\sum_{v \in V} \sin \rho_v < 2\sqrt{2n}$.
- Hence there is a plane thought the centre of the sphere intersecting less than $2\sqrt{2n}$ caps.
- These caps correspond to an antipodally symmetric separator $\tilde{S} \subseteq \tilde{V}$ of $\tilde{G}$ of size less than $2\sqrt{2n}$.
- Identifying the antipodal points in $\tilde{S}$ gives a subset $S \subseteq V$ of size less than $\sqrt{2n}$ such that $G - S$ is planar.
- Since every cycle in $G - S$ is even, S is an odd cycle transversal of size at most $\sqrt{2n}$.
- Hence $\mathrm{fw}(G) < \sqrt{2n}$.

What about edge-width?

- More detailed analysis gives:

Theorem

Let G be a non-bipartite projective quadrangulation on n vertices. Then $\mathrm{fw}(G) \le (1 + o(1))\sqrt{\frac{\pi n}{\sqrt{3}}} \approx 1.347\sqrt{n}$.



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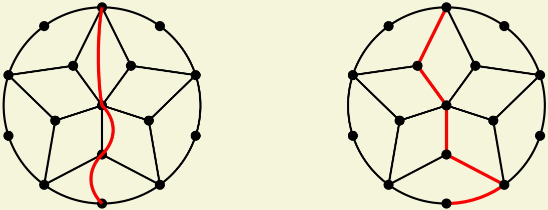


- Using the fact that $\mathrm{ew}(G) \leq 2\,\mathrm{fw}(G) - 1$:

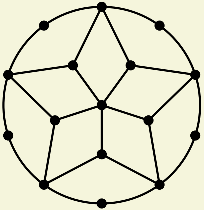
Corollary

Let G be a non-bipartite projective quadrangulation on n vertices. Then $\mathrm{ew}(G) \leq (2 + o(1))\sqrt{\frac{2\pi n}{\sqrt{3}}} \approx 2.694\sqrt{n}$.

A min–max theorem

Theorem (Lins 1981)

Let $G = (V, E)$ be a graph embedded in the projective plane with all faces bounded by an even number of edges. Then the length of a shortest non-contractible cycle is equal to the maximum size of a packing of non-contractible co-cycles.



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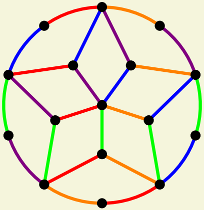

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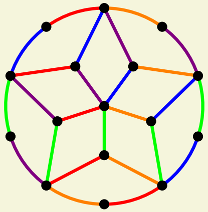


Corollary

Let $G = (V, E)$ be a graph embedded in the projective plane with all faces bounded by an even number of edges, with a shortest non-contractible cycle of length ℓ and a shortest non-contractible cycle of length ℓ^* in its dual graph G^*. Then $|E| \geq \ell \cdot \ell^*$.

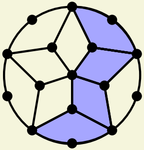
The edge-width of projective quadrangulations

Theorem (Esperet, S. 2015+)

Let G be a non-bipartite projective quadrangulation on n vertices. Then $\mathrm{ew}(G) \leq \frac{1}{2}(1 + \sqrt{8n-7})$.

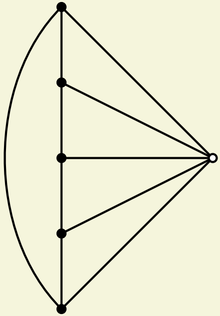
Proof.

- Let C^* be shortest non-contractible cycle in G^*, of length ℓ^*.
- This gives rise to a cycle in the planar representation of G of length $2\ell^* + 2$.



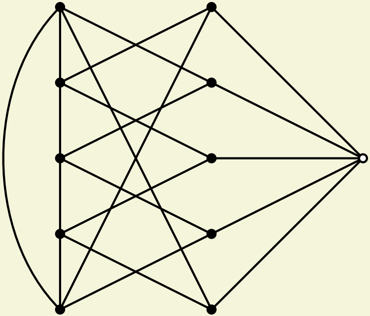
- Hence, $\ell \leq \ell^* + 1$.
- By Lins and Euler, $\ell(\ell - 1) \leq \ell \cdot \ell^* \leq m = 2n - 2$.
- It follows that $\ell \leq \frac{1}{2}(1 + \sqrt{8n-7})$. $\square$

The (generalised) Mycielski construction $M_r(G)$



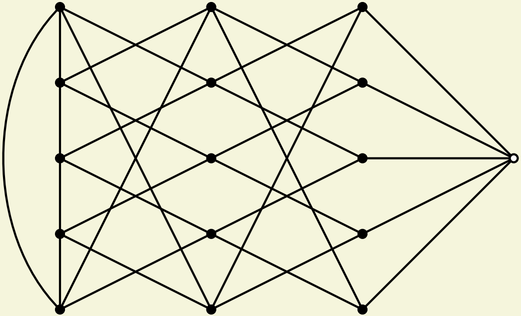
$M_1(C_5)$

The (generalised) Mycielski construction $M_r(G)$



$M_2(C_5)$

The (generalised) Mycielski construction $M_r(G)$



$M_3(C_5)$

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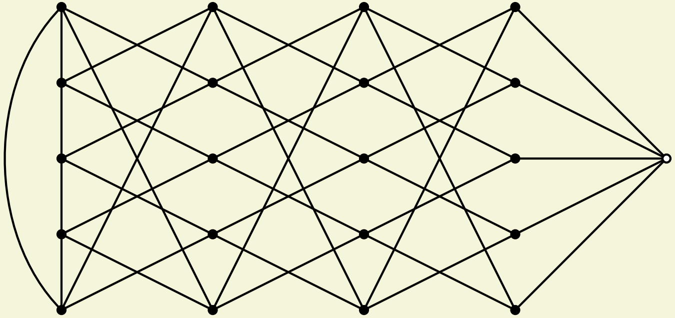
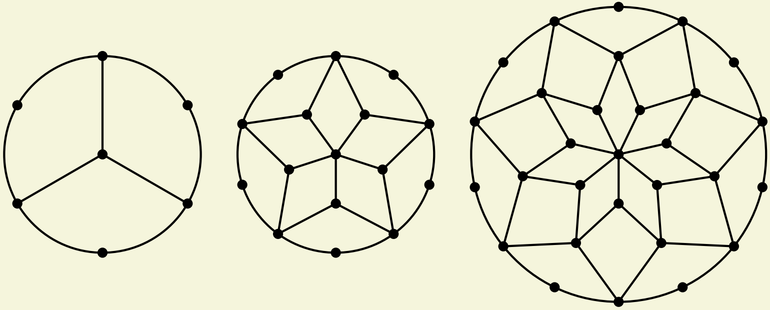


$M_4(C_5)$

The bound is tight!



$M_1(C_3)$ $M_2(C_5)$ $M_3(C_7)$

- The graph $M_k(C_{2k+1})$ is a projective quadrangulation with $n = k(2k+1)+1$ vertices and odd girth $\ell = 2k+1$.
- Hence $\ell(\ell-1) = 2n-2$, so $\ell = \frac{1}{2}(1+\sqrt{8n-7})$.

Graphs without two vertex-disjoint odd cycles

Theorem (Lovász ~1990; Kawarabayashi and Ozeki 2013)

Let G be an internally 4-connected graph. Then G has no two vertex-disjoint odd cycles if and only if G satisfies one of the following conditions:

1. $G - v$ is bipartite, for some $v \in V(G)$;
2. $G - \{e_1, e_2, e_3\}$ is bipartite for some edges $e_1, e_2, e_3 \in E(G)$ such that e_1, e_2, e_3 form a triangle;
3. $|V(G)| \leq 5$;
4. G can be embedded into the projective plane so that every face boundary has even length.

Theorem (Esperet, S. 2015+)

Let G be a 4-chromatic graph on n vertices without two vertex-disjoint odd cycles. Then G contains an odd cycle of length at most $\frac{1}{2}(1 + \sqrt{8n-7})$.

Related questions

Question (Erdős 1974)

Is there is a constant c such that every n-vertex 4-chromatic graph has an odd cycle of length at most $c\sqrt{n}$?

- Kierstead, Szemerédi and Trotter 1984: YES, $c \leq 8$.
- Nilli 1999: $c \leq \sqrt{8}$
- Jiang 2001: $c \leq 2$
- Gallai 1963: $c > 1$
- Ngoc and Tuza 1995, Youngs 1996: $c > \sqrt{2}$ (using generalised Mycielski graphs).

Open question (Esperet, S. 2015+)

Does every n-vertex 4-chromatic graph contain an odd cycle of length at most $\frac{1}{2}(1 + \sqrt{8n - 7})$?

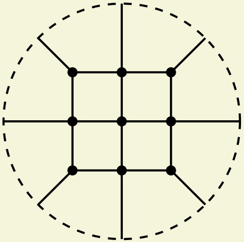
Back to face-width...

- Recall that every odd cycle in a non-bipartite projective quadrangulation is an odd cycle transversal.
- So $\mathrm{fw}(G) \leq \mathrm{ew}(G)$.
- Our bound on face-width using circle packing is better (though not by much...)

Minimal graphs of given face-width

Definition

A (multi)graph G embedded in a surface is *minimal* with respect to the face-width if contracting or deleting any edge decreases the face-width.



Theorem (Randby 1997)

For any integer k, if a multigraph embedded in the projective plane is minimal of face-width k, then it contains exactly $2k^2 - k$ edges.

Face-width of projective quadrangulations

Theorem (Esperet, S. 2015+)

Let G be a non-bipartite projective quadrangulation on n vertices. Then $\mathrm{fw}(G) \leq \frac{1}{4} + \sqrt{n - \frac{15}{16}}$.

Proof.

- Let G be a non-bipartite projective quadrangulation on n vertices.
- By Euler's formula, G has $m = 2n - 2$ edges.
- Let k be the face-width of G.
- Delete or contract edges of G until we obtain a (multi)graph H that is minimal with face-width k.
- H has at most $m = 2n - 2$ edges by construction, and exactly $2k^2 - k$ edges by Randby's theorem.
- Hence $2k^2 - k \leq 2n - 2$ and so $k \leq \frac{1}{4} + \sqrt{n - \frac{15}{16}}$. $\square$

Extension to graphs without two vertex-disjoint odd cycles

- Using the characterisation of graphs without two vertex-disjoint odd cycles:

Theorem (Esperet, S. 2015+)

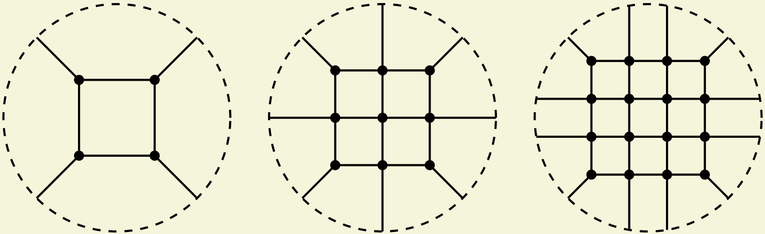
Let G be a 4-vertex-critical graph on n vertices without two vertex-disjoint odd cycles. Then G has an odd cycle transversal of size at most $\frac{1}{4} + \sqrt{n - \frac{15}{16}}$.

- We were unable to extend it to all 4-chromatic graphs without two vertex-disjoint odd cycles

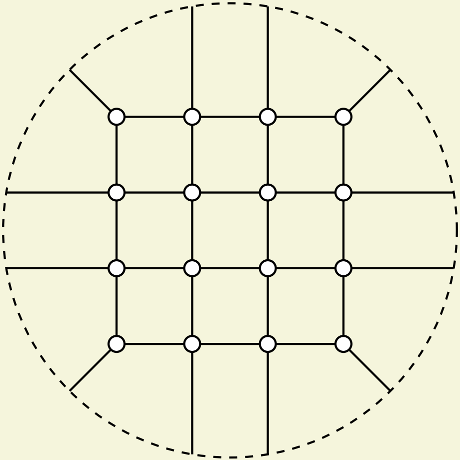
The bound is almost sharp...

Theorem (Esperet, S. 2015+)

There are infinitely many values of n for which there are non-bipartite projective quadrangulations on n vertices containing no odd cycle transversal of size less than $\sqrt{n}$.



Proof by picture



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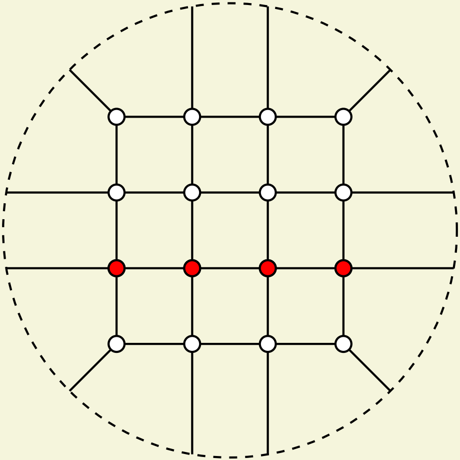

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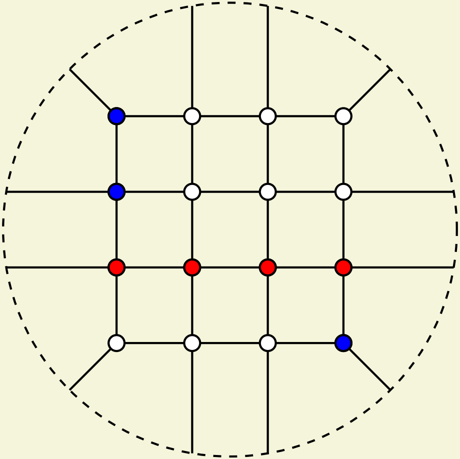

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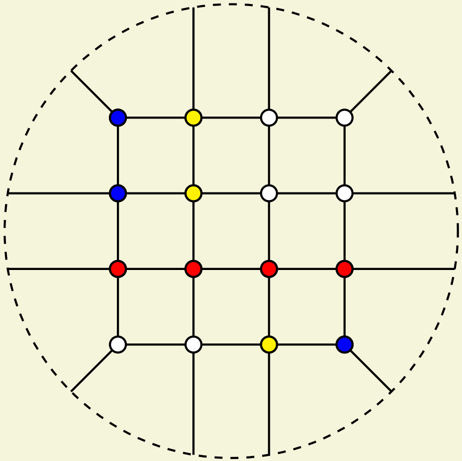

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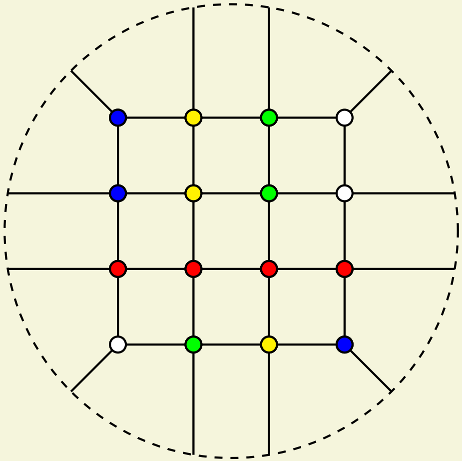

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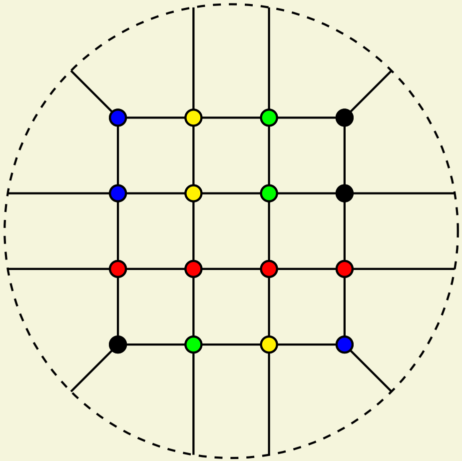

A partial answer to Nakamoto and Ozeki's question

Theorem (Esperet, S. 2015+)

Let G be a non-bipartite projective quadrangulation on n vertices, with maximum degree Δ. Then G has an odd cycle transversal of size at most $\sqrt{2\Delta n}$ inducing a single edge.