Interval Representations of Graphs

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joint work with
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Jonathan Rollin
Peter Stumpf
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Graphes & Surfaces
Grenoble INP

road map

▷ Interval Graphs ◁

Several Intervals per Vertex

- ▷ interval, track and local track number
- ▷ planar graphs

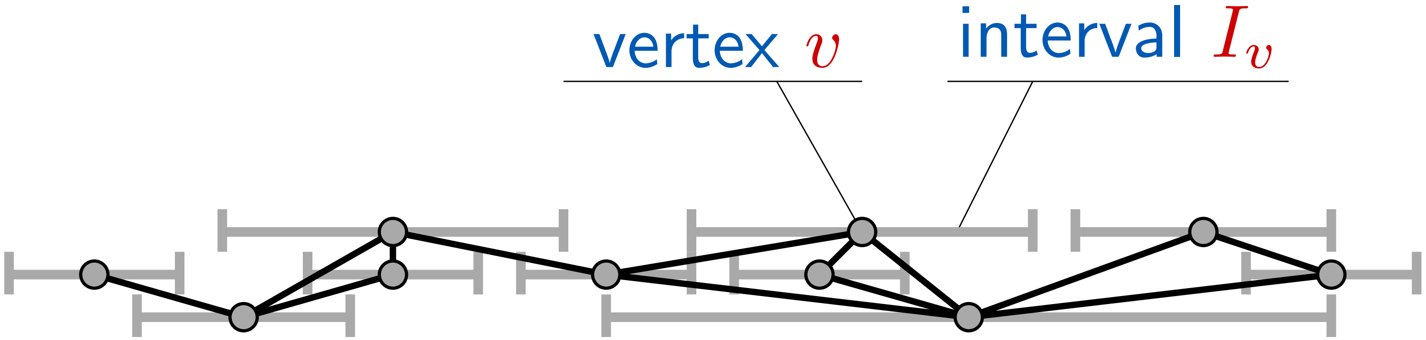
General Covering Parameters

Higher-Dimensional Box per Vertex

- ▷ boxicity, local and union boxicity
- ▷ planar graphs

Combining Approaches

- interval graphs $=$ intersection graphs of intervals



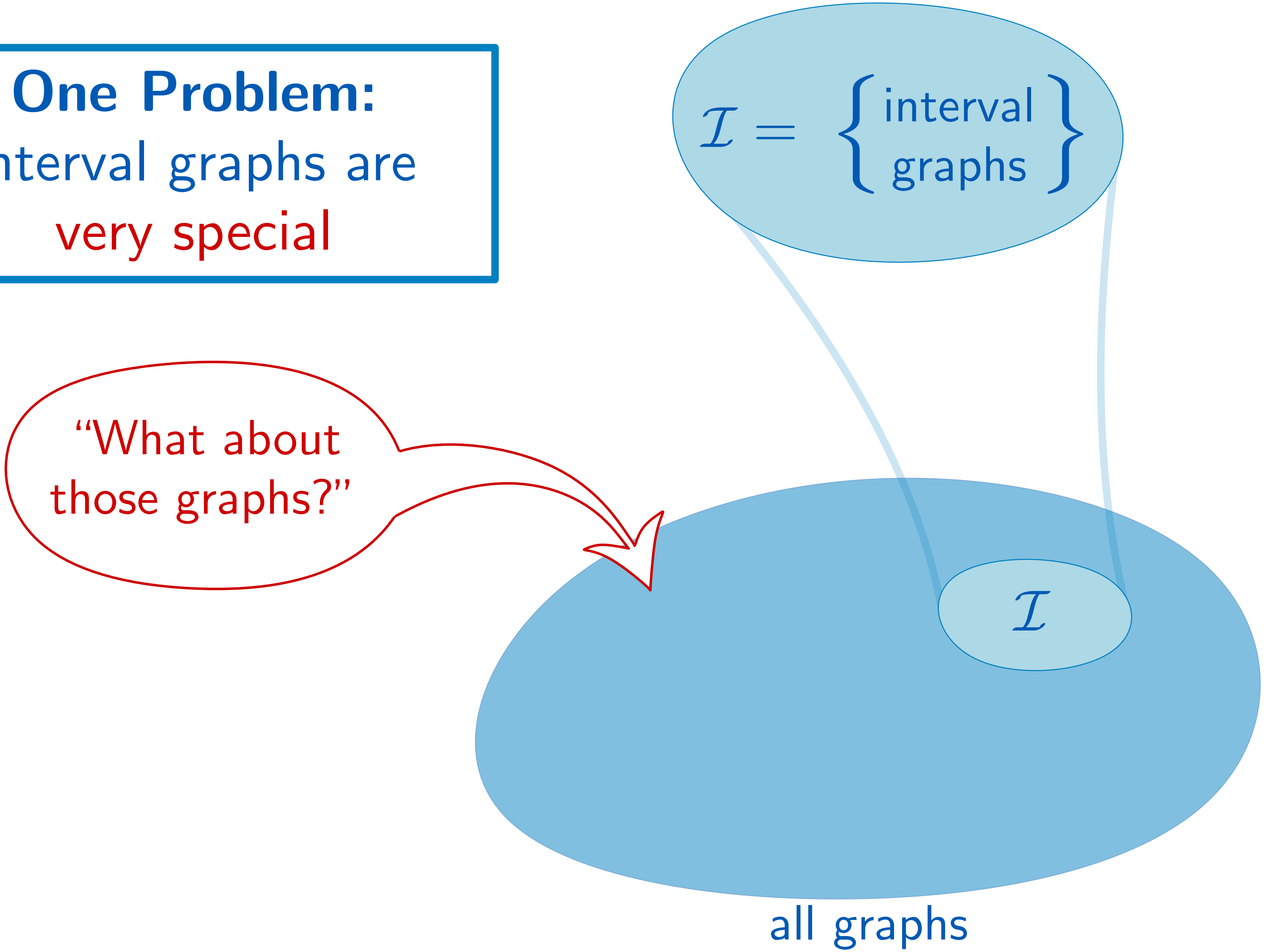
$$v \in V(G) \iff I_v = [a_v, b_v] \subset \mathbb{R}$$

$$uv \in E(G) \iff I_u \cap I_v \neq \emptyset$$

- nice graphs because: applications, easy characterization, recognition, efficient algorithms, foundation of deep theories, . . .

interval graphs

One Problem:
interval graphs are
very special



$\mathcal{I} = \left\{ \begin{array}{c} \text{interval} \\ \text{graphs} \end{array} \right\}$

"What about those graphs?"

$\mathcal{I}$

all graphs

One Problem:
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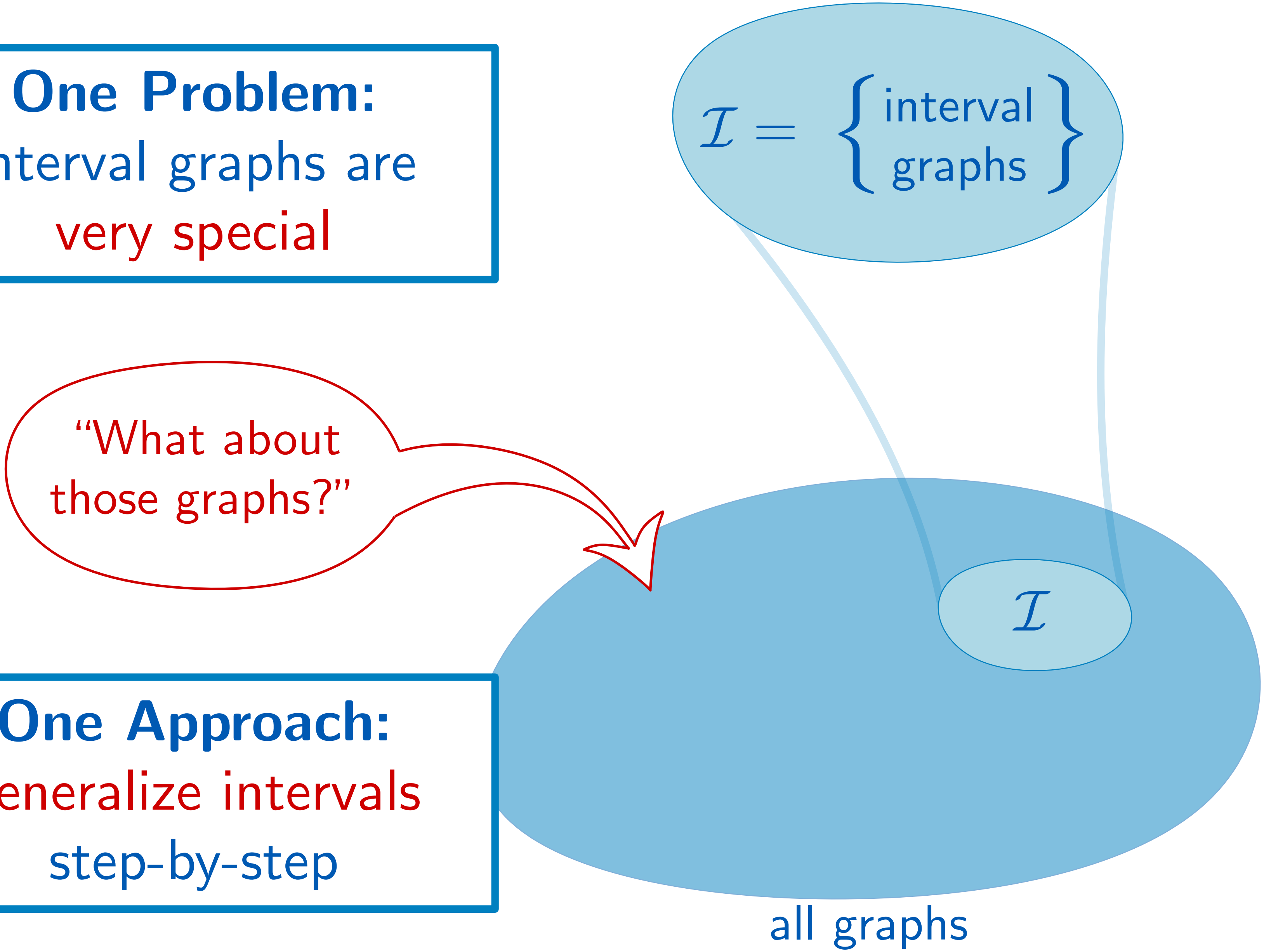

One Approach:
generalize intervals
step-by-step

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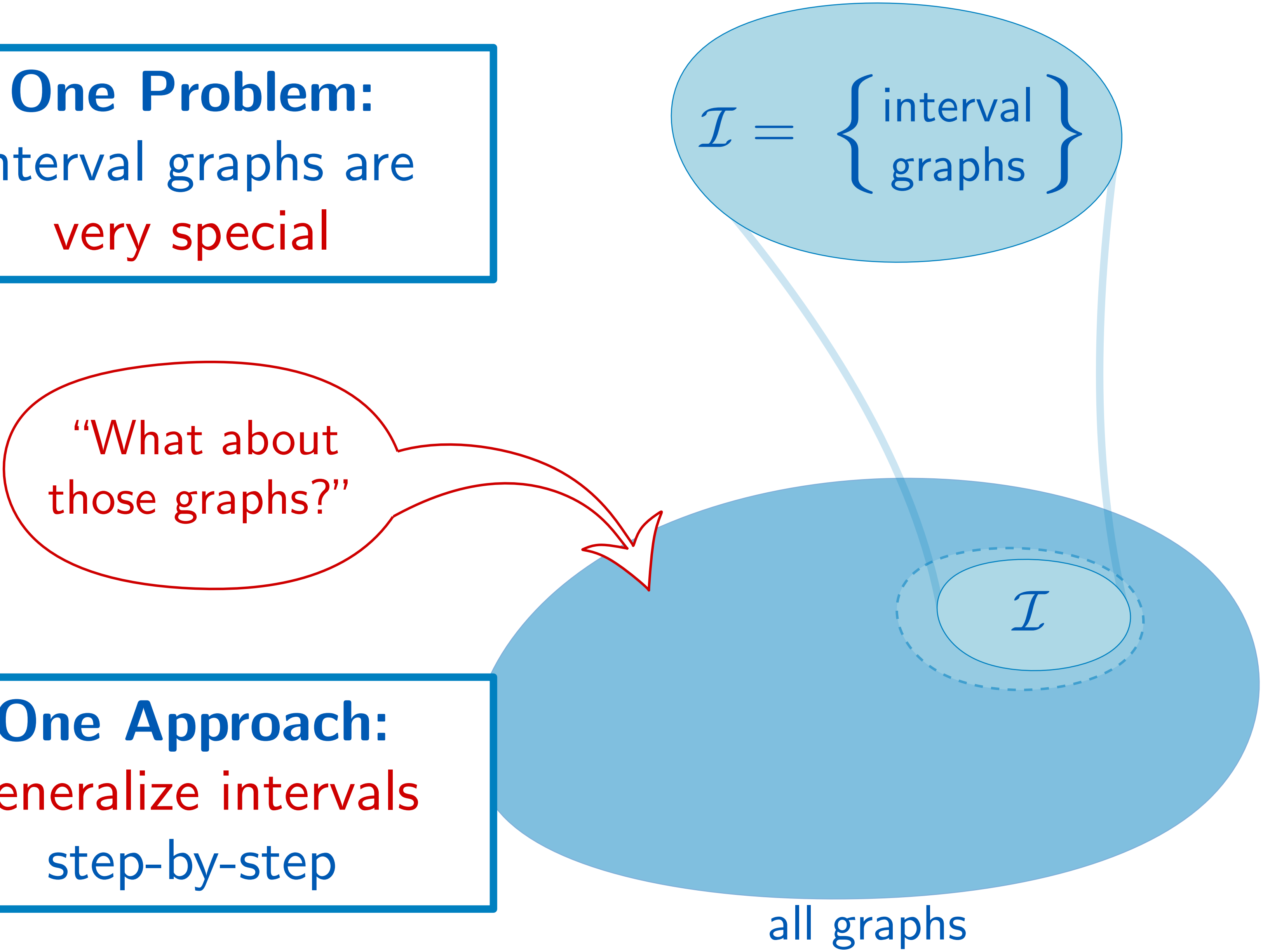
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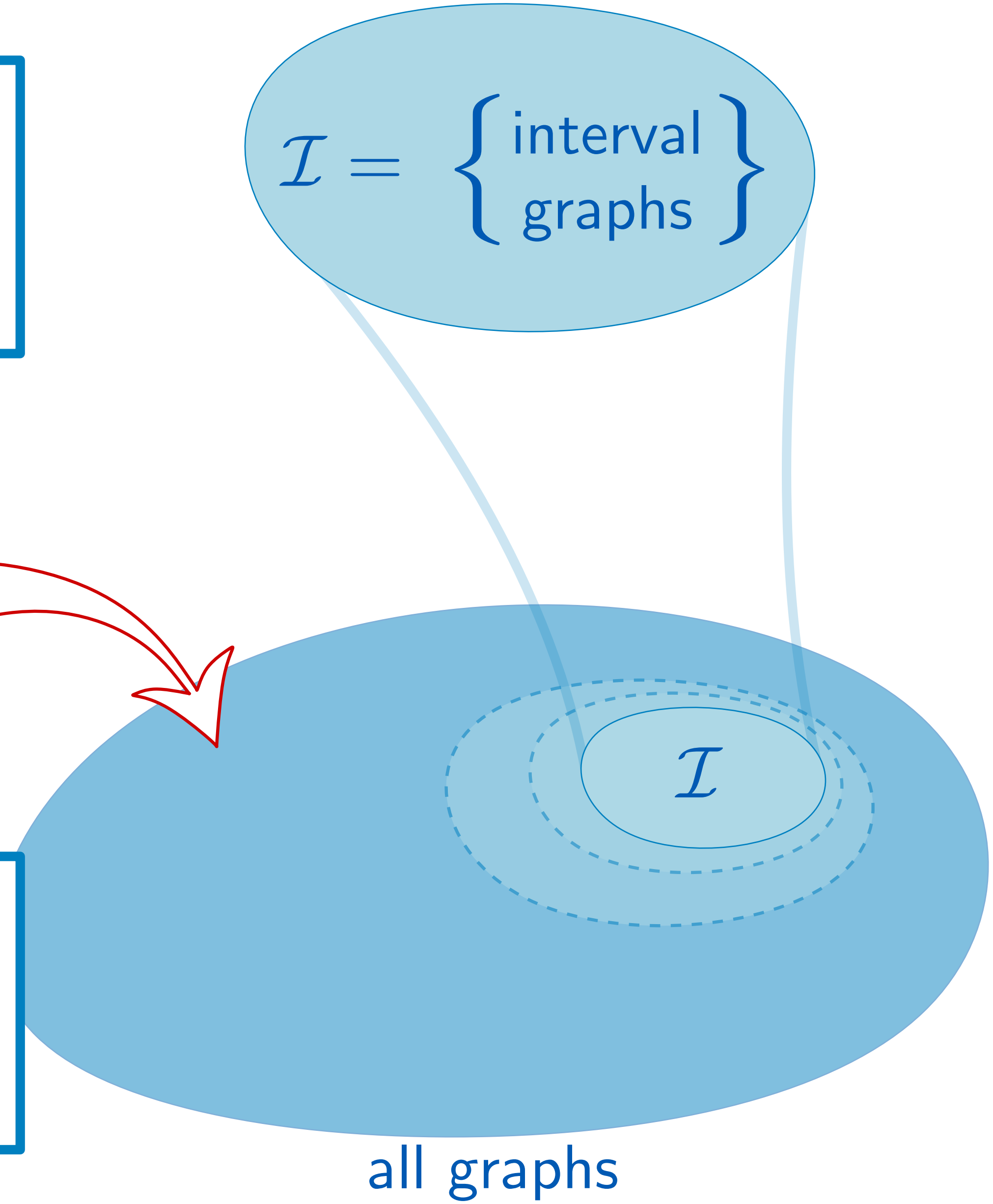

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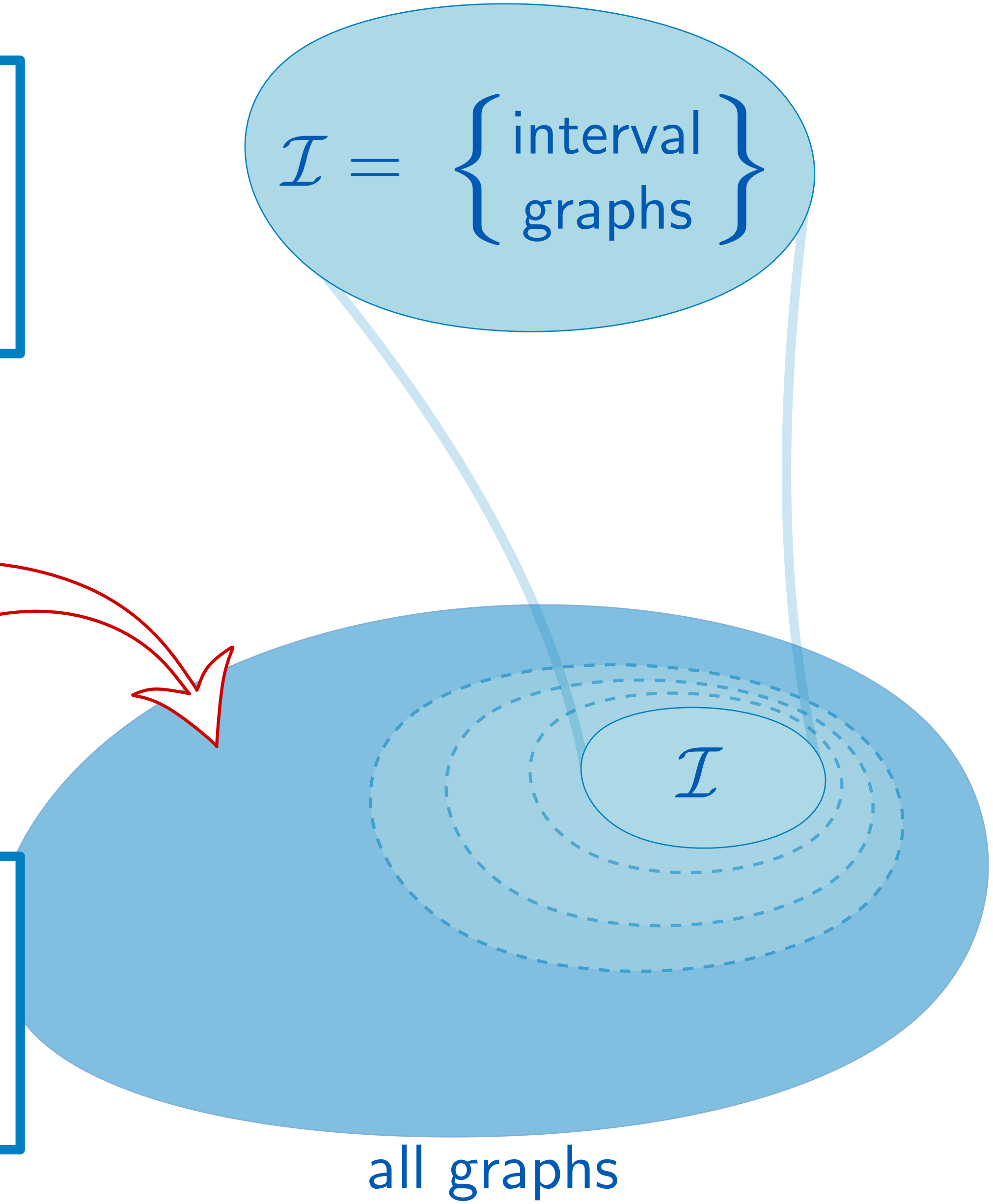

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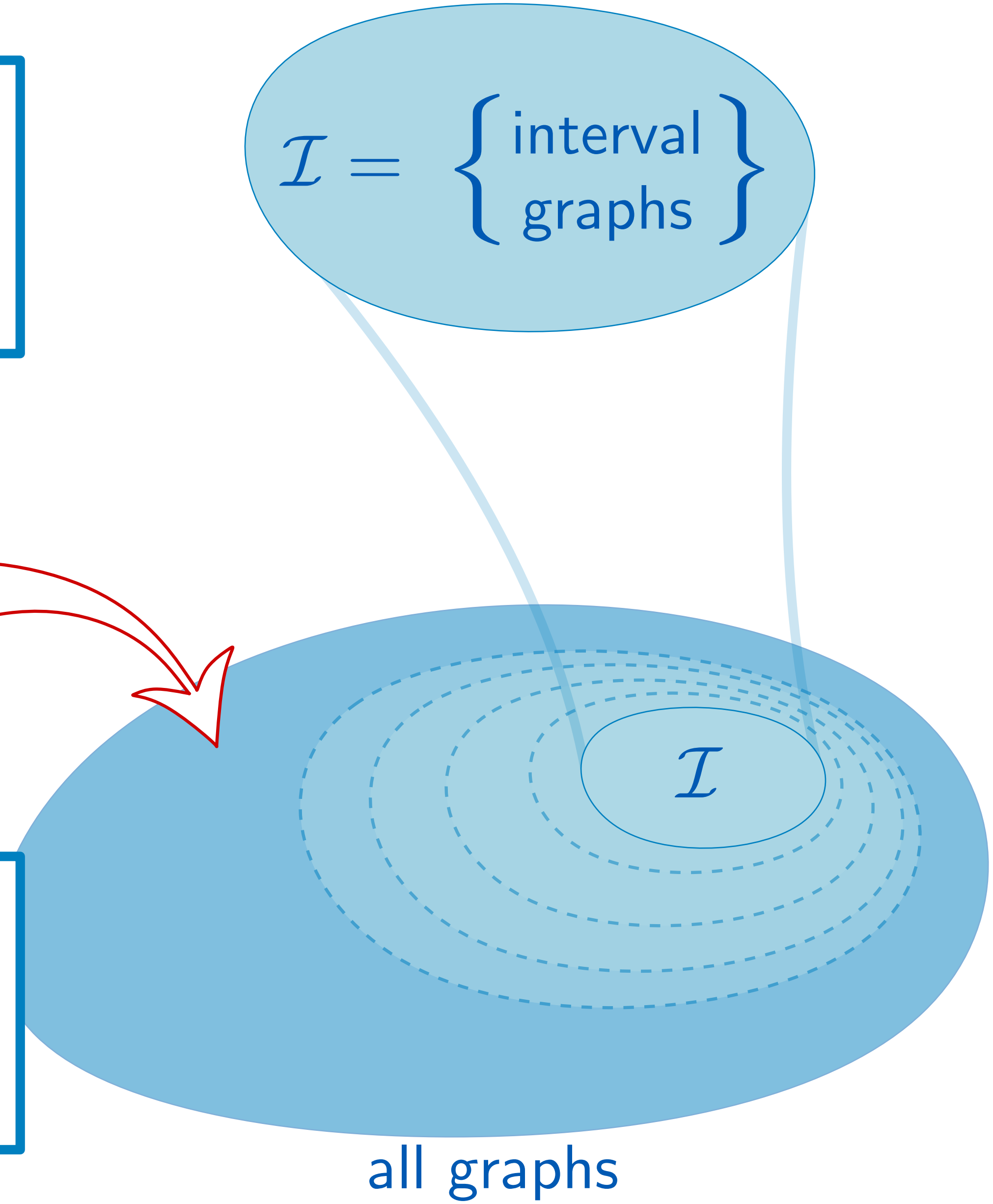

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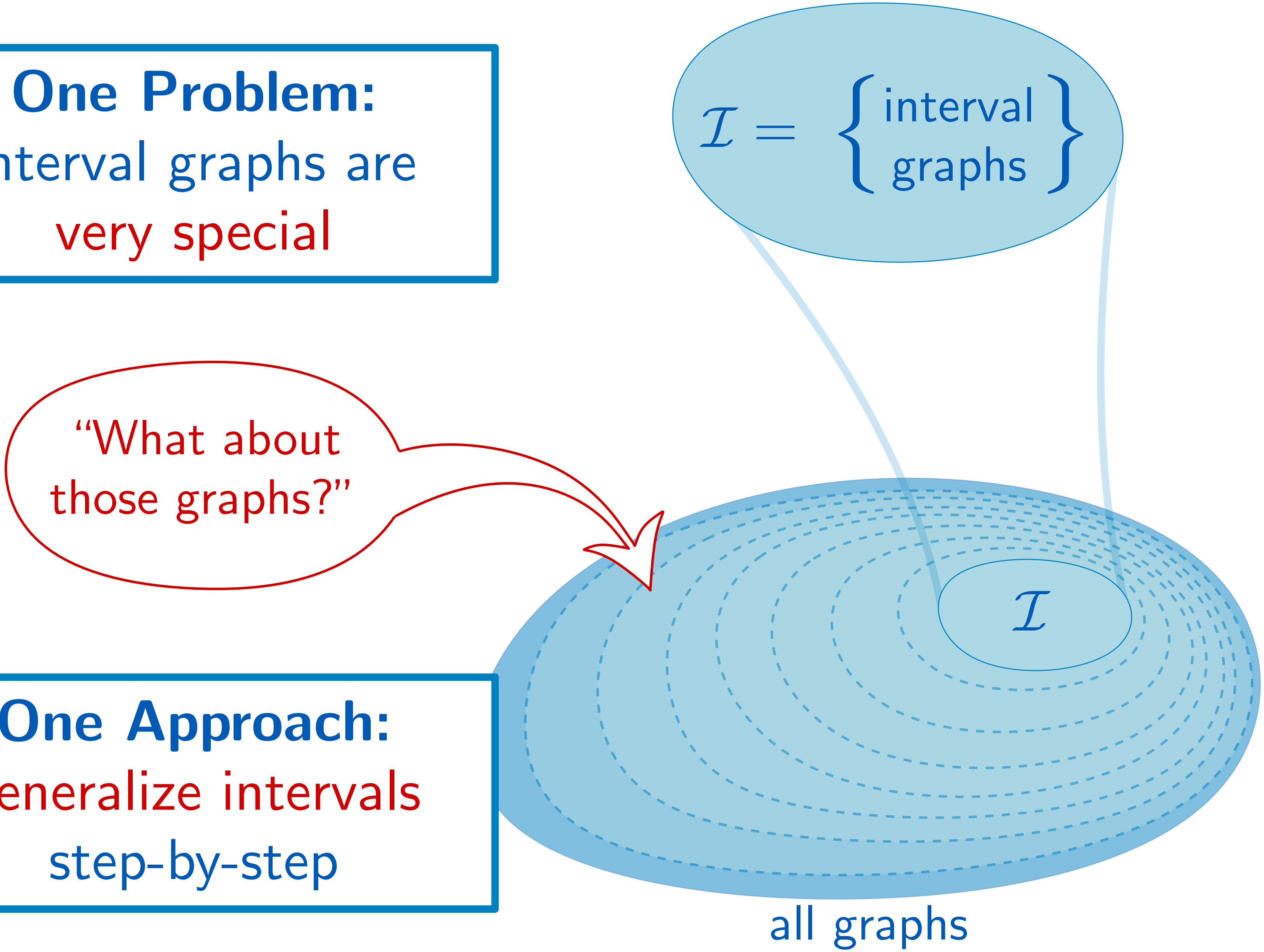

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Interval Graphs

▷ Several Intervals per Vertex ◁

- ▷ interval, track and local track number
- ▷ planar graphs

General Covering Parameters

Higher-Dimensional Box per Vertex

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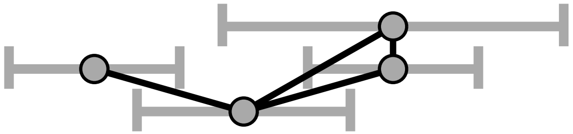
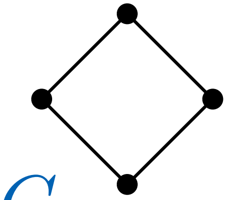
Combining Approaches

generalizing intervals

. . . using several intervals

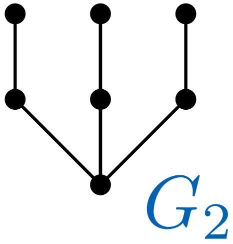


$\mathcal{I} = \{$interval graphs$\}$



graphs

$G_1, G_2 \notin \mathcal{I}$



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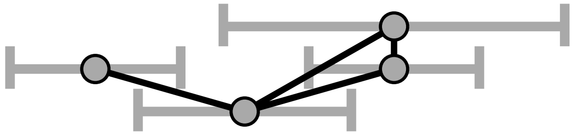
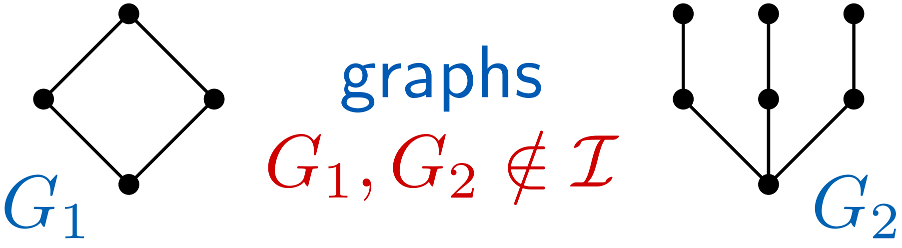
folded local global

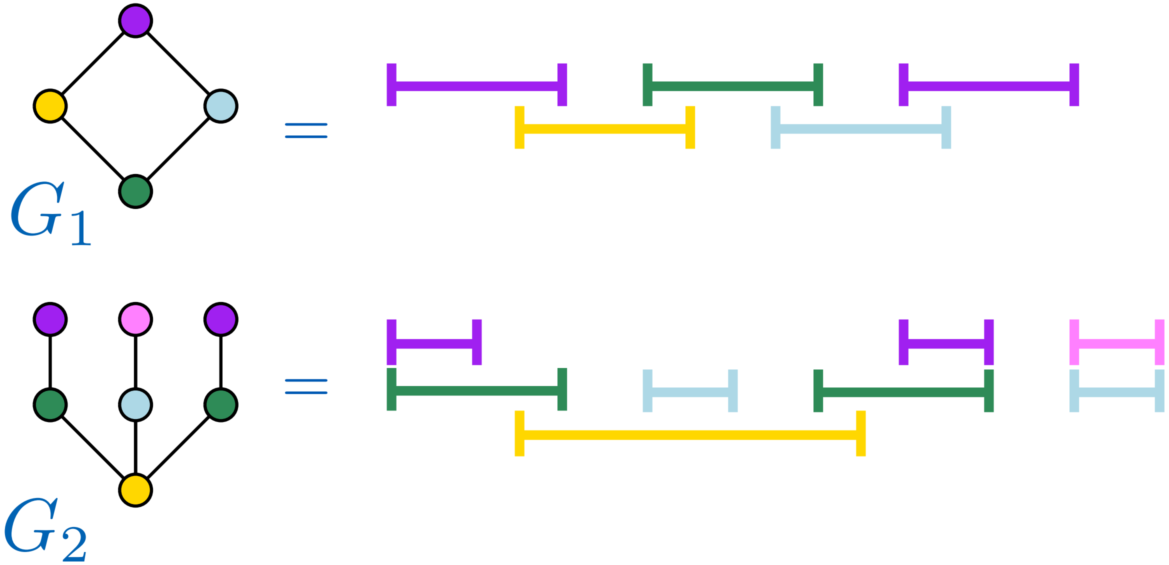
...using several intervals



$\mathcal{I} = \{\text{interval graphs}\}$



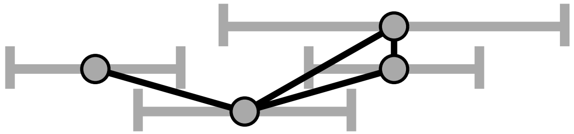
more intervals per vertex



folded

generalizing intervals

…using several intervals



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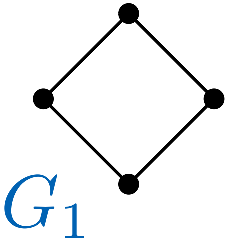
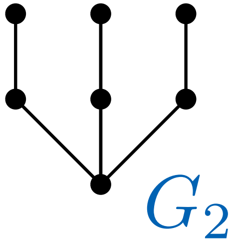


graphs
$G_1, G_2 \notin \mathcal{I}$



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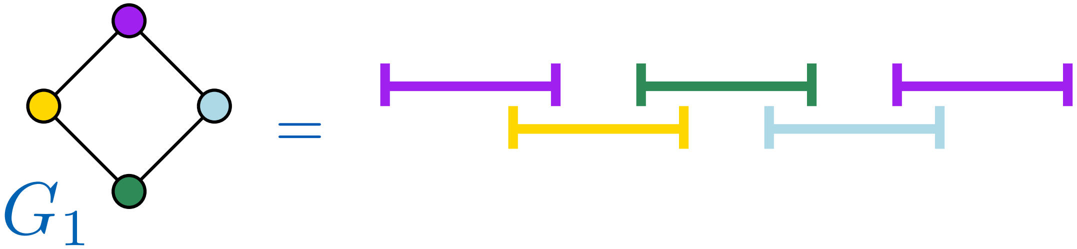
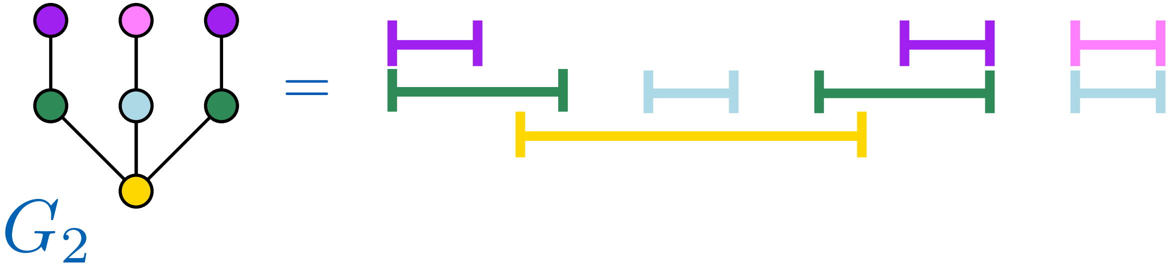
interval number

$$i(G) = \min\{k : k \text{ intervals per vertex suffice}\}$$

more intervals
per vertex

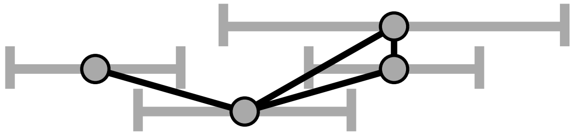




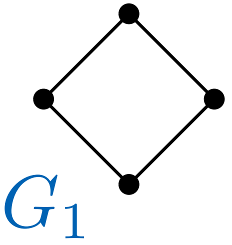
folded

generalizing intervals

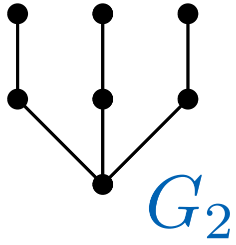
. . . using several intervals

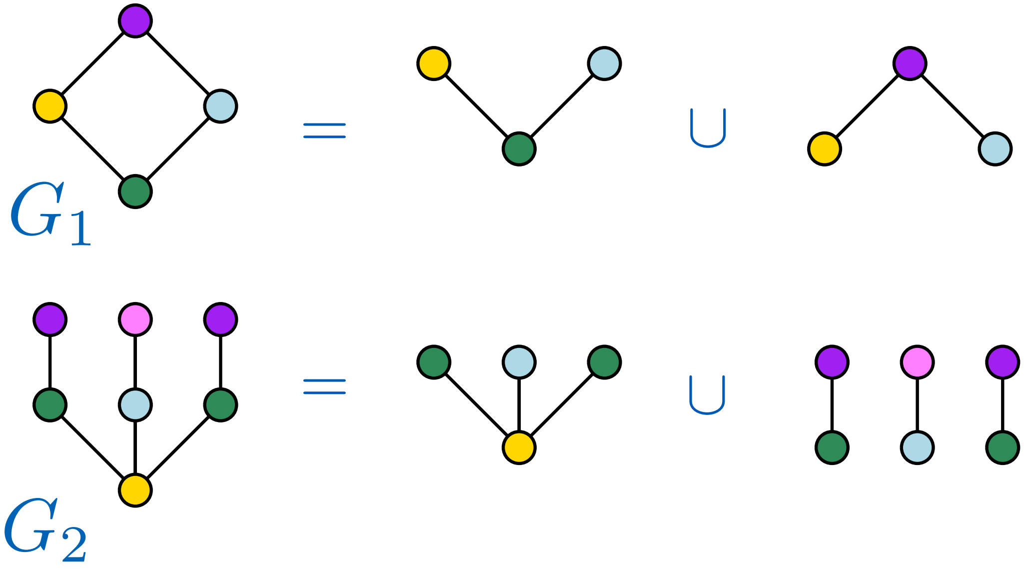


$\mathcal{I} = \{$interval graphs$\}$



graphs $G_1, G_2 \notin \mathcal{I}$



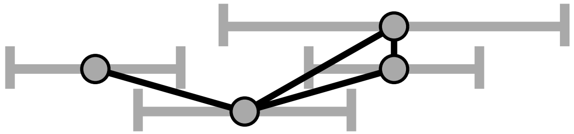


split into interval graphs

global

generalizing intervals

…using several intervals



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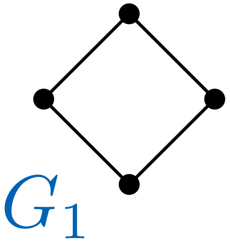
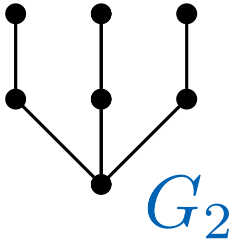


graphs

$G_1, G_2 \notin \mathcal{I}$



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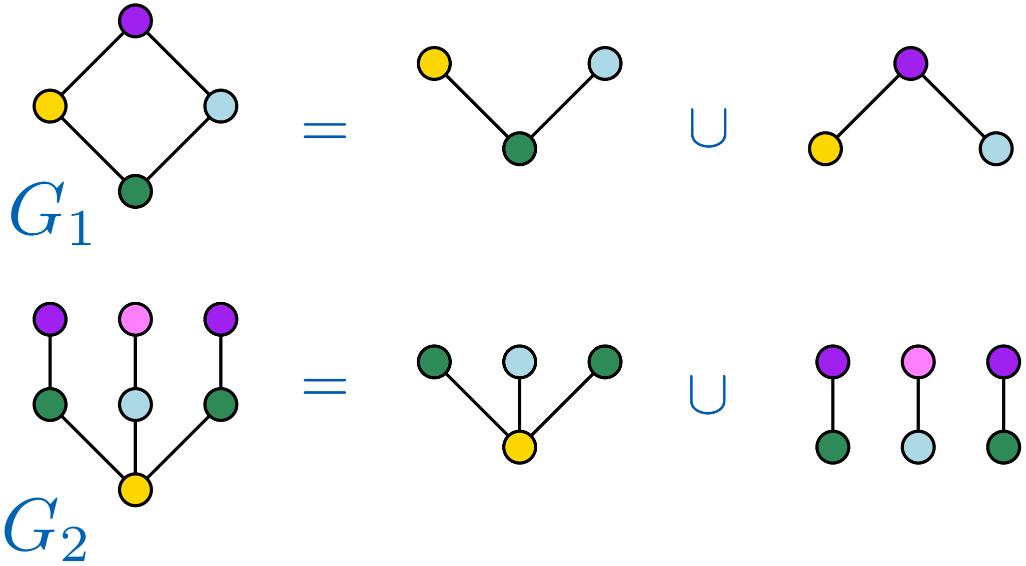
track number

$t(G) = \min\{k : k$ interval graphs suffice$\}$



split into
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global

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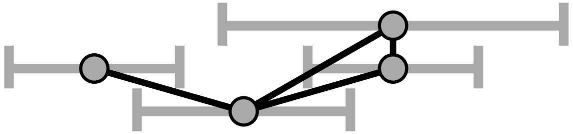

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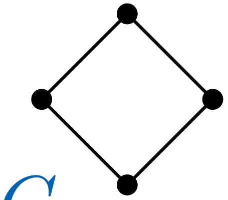
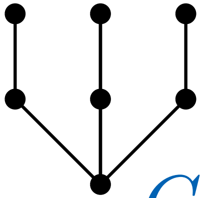


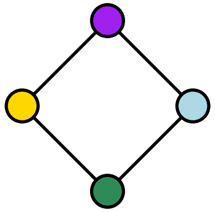
G_1

graphs

$G_1, G_2 \notin \mathcal{I}$



G_2

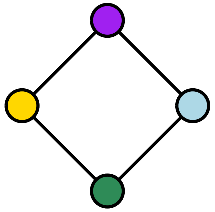


few intervals per vertex

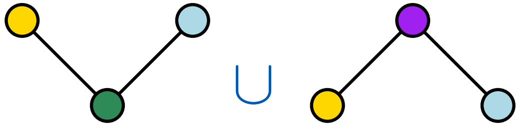
folded

“interpolation”

local



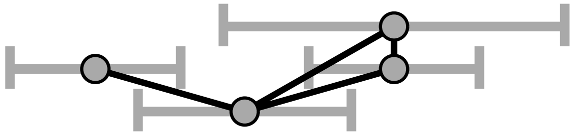
few interval graphs



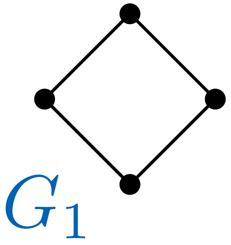
global

generalizing intervals

… using several intervals

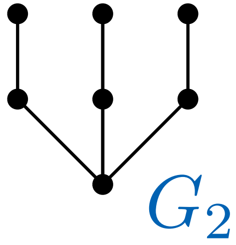


$\mathcal{I} = \{$interval graphs$\}$



graphs

$G_1, G_2 \notin \mathcal{I}$



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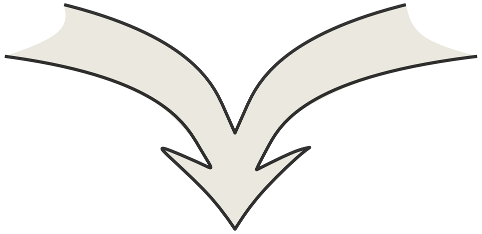
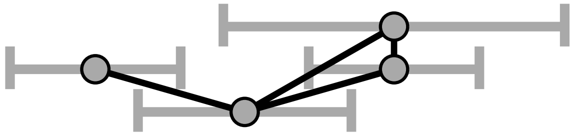
interval number $i(G)$
(few intervals)



track number $t(G)$
(separated intervals)

generalizing intervals

…using several intervals



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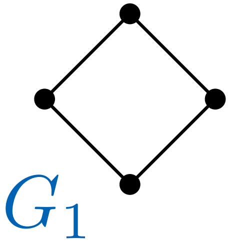
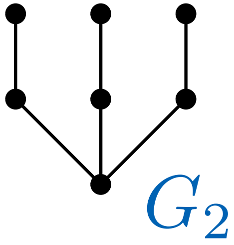


graphs
$G_1, G_2 \notin \mathcal{I}$



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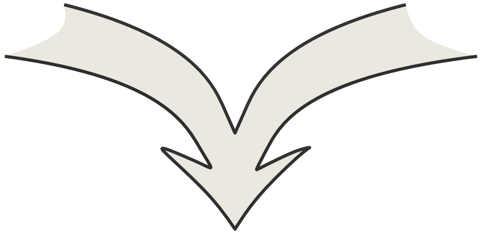
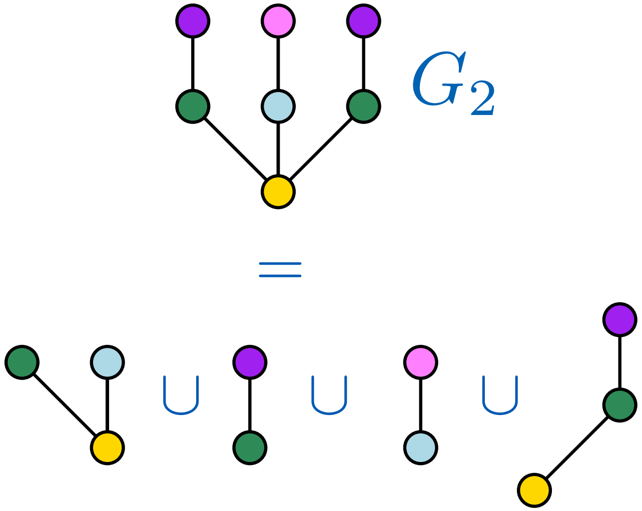
interval number $i(G)$
(few intervals)

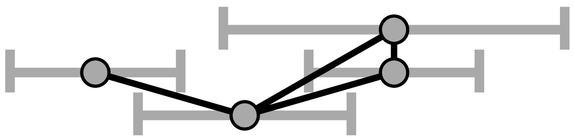
track number $t(G)$
(separated intervals)



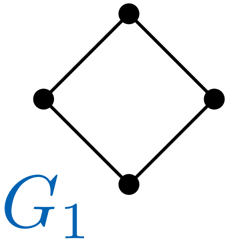


- ▷ arbitrarily many interval graphs
- ▷ each vertex in only few of them

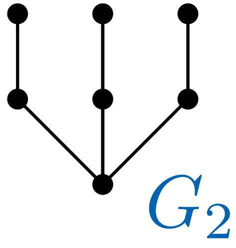
. . . using several intervals



$\mathcal{I} = \{$interval graphs$\}$

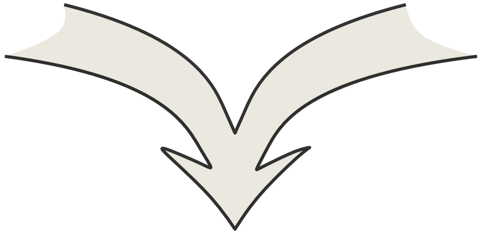


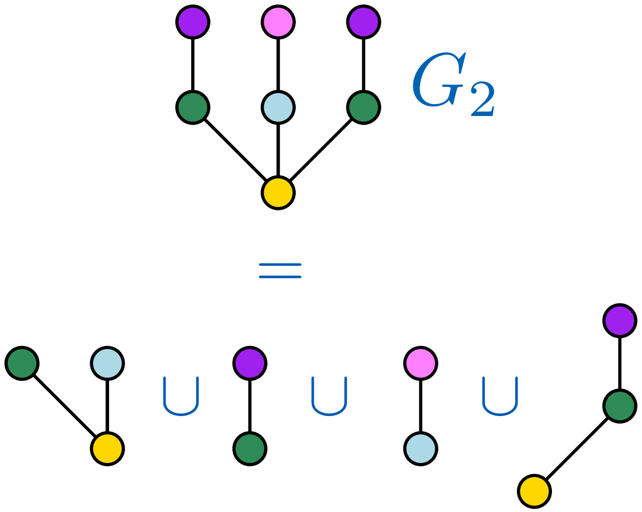
graphs
$G_1, G_2 \notin \mathcal{I}$



interval number $i(G)$
(few intervals)

track number $t(G)$
(separated intervals)



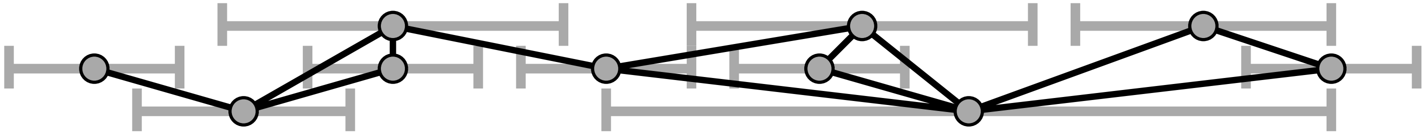


- ▷ arbitrarily many interval graphs
- ▷ each vertex in only few of them

local track number

$t_\ell(G) = \min\{k :$ every vertex in k interval graphs suffices$\}$

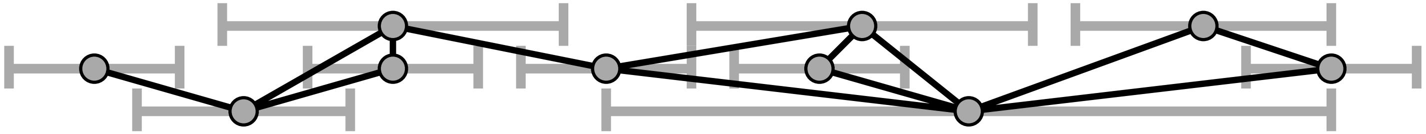
- $\mathcal{I} = \{$interval graphs$\}$ = nice graphs



- $G \notin \mathcal{I} \Rightarrow$ use several intervals per vertex

folded	local	global

- ▷ $\mathcal{I} = \{$interval graphs$\} =$ nice graphs

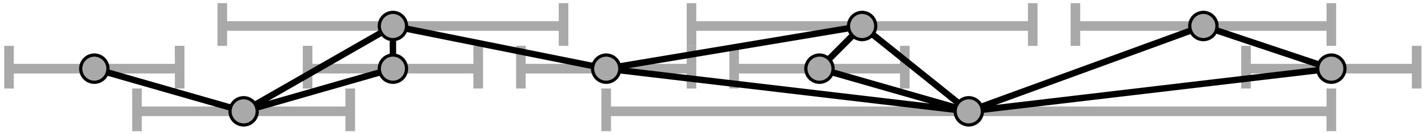


- ▷ $G \notin \mathcal{I} \Rightarrow$ use several intervals per vertex

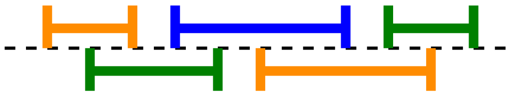
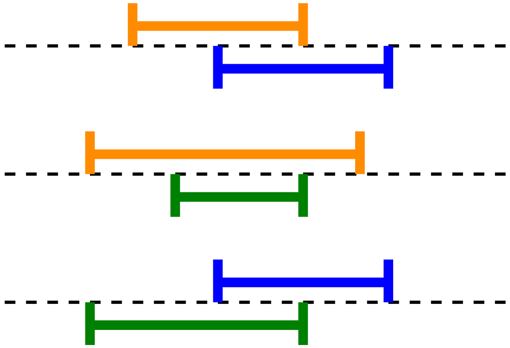
folded	local	global
$i(G)$ (interval number)		

round-up

- ▷ $\mathcal{I} = \{\text{interval graphs}\} = \text{nice graphs}$

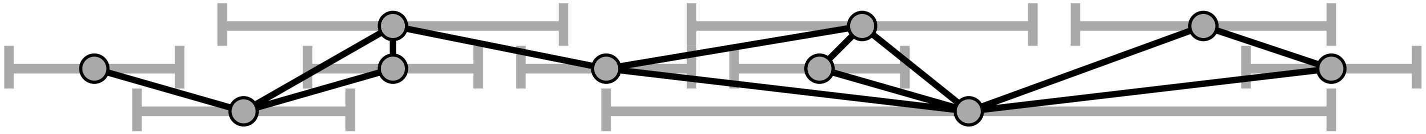


- ▷ $G \notin \mathcal{I} \Rightarrow$ use several intervals per vertex

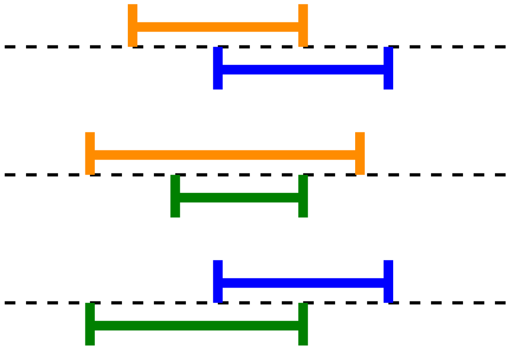
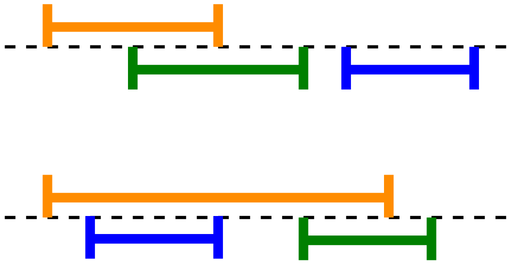
folded	local	global
$i(G)$	$t_\ell(G)$	
(interval number)	(local track number)	
		

round-up

- ▷ $\mathcal{I} = \{$interval graphs$\} =$ nice graphs

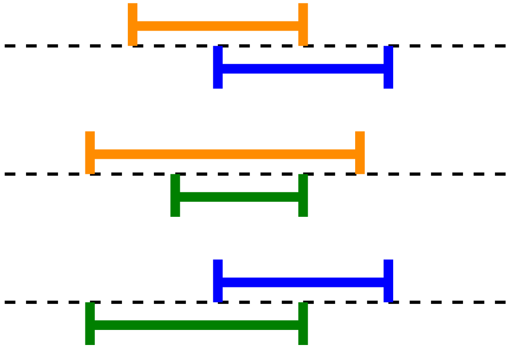
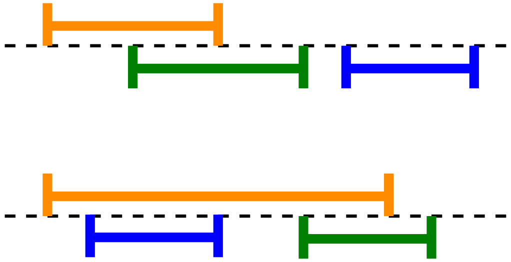


- ▷ $G \notin \mathcal{I} \Rightarrow$ use several intervals per vertex

folded	local	global
$i(G)$	$t_\ell(G)$	$t(G)$
(interval number)	(local track number)	(track number)
		

round-up

$$i(G) \leq t_\ell(G) \leq t(G)$$

folded	local	global
$i(G)$	$t_\ell(G)$	$t(G)$
(interval number)	(local track number)	(track number)
		

Interval Graphs

▷ Several Intervals per Vertex ◁

- interval, track and local track number
- planar graphs

General Covering Parameters

Higher-Dimensional Box per Vertex

- boxicity, local and union boxicity
- planar graphs

Combining Approaches

- ▷ interval graphs

Thm. (Booth-Lueker 1976)
Deciding $G \in \mathcal{I}$ can be done in linear time.

- ▷ interval number

Thm. (Smyos-West 1984)
Deciding $i(G) \leq k$ is NP-complete for every $k \geq 2$.

- ▷ track number

Thm. (Jiang 2013)
Deciding $t(G) \leq k$ is NP-complete for every $k \geq 2$.

- ▷ local track number

Thm. (Bläsius-Stumpf-U. 2016+)
Deciding $t_\ell(G) \leq k$ is NP-complete for every $k \geq 2$.

Que. Maximum $i(G)$ / $t_\ell(G)$ / $t(G)$ if G is planar?

graph class	max $i(G)$	max $t_\ell(G)$	max $t(G)$
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planar bipartite	3 [Scheinerman-West '83]	3 [Knauer-U. '16]	4 [Gonçalves-Ochem '09]
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Open Do we have for planar G that $t_\ell(G) \leq 3$?

some proof ideas

Thm. (Knauer-Rollin-U. 2016+)
For every planar graph G we have $i(G) \leq 3$.

- ▷ w.l.o.g. G is triangulation
- ▷ induction on $\#V(G)$

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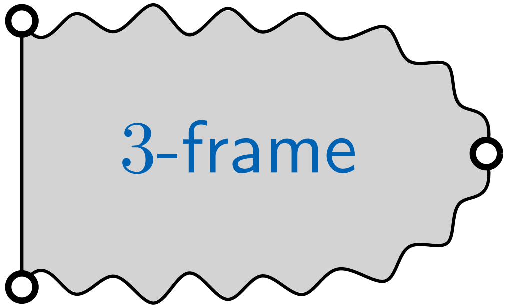
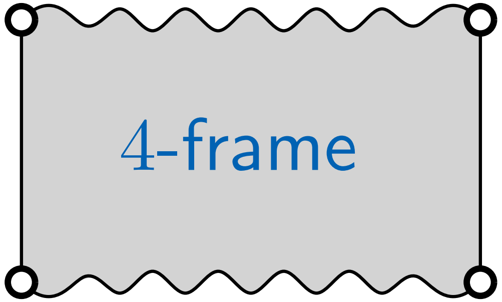
decomposition into **3-frames** and **4-frames**





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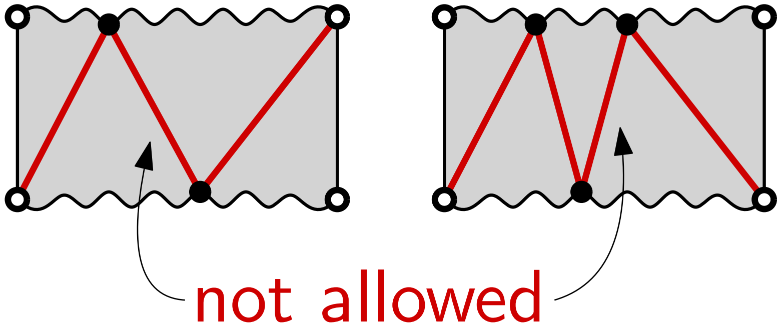
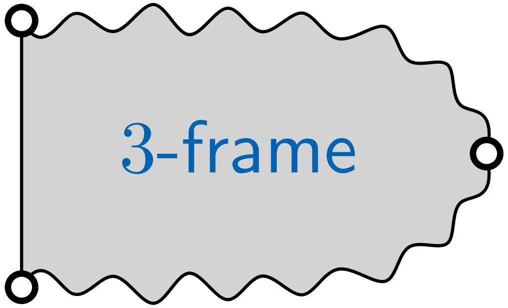
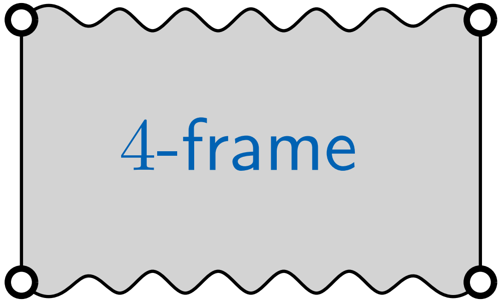
decomposition into 3-frames and 4-frames

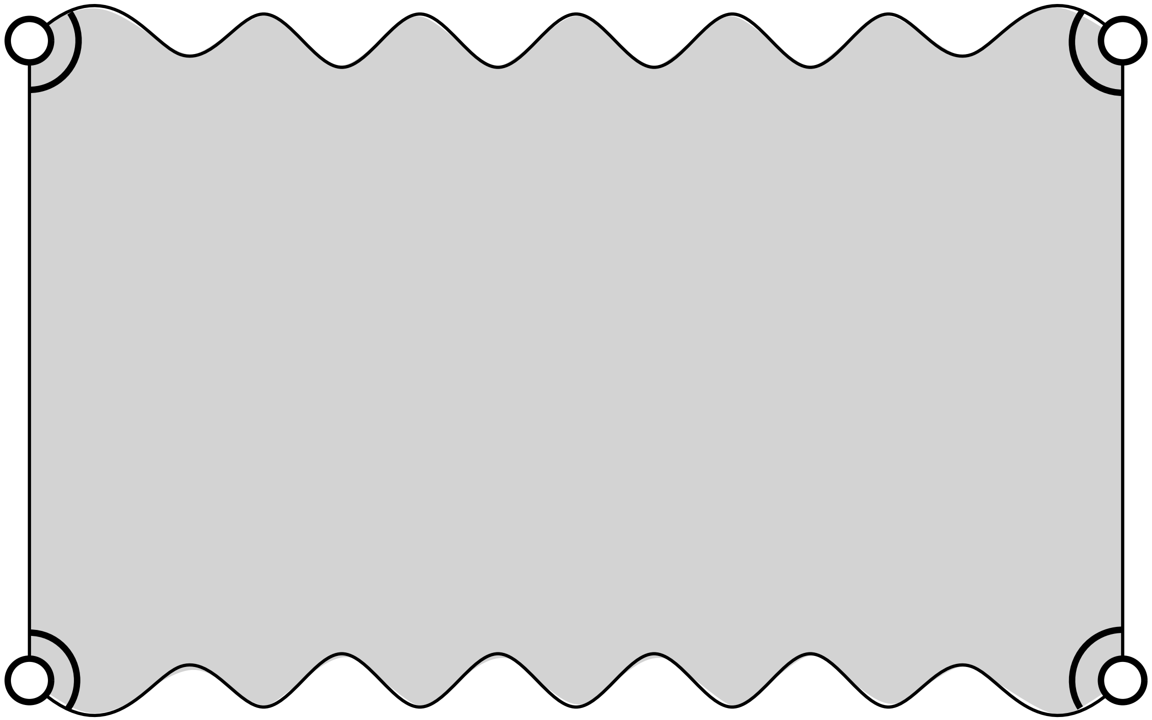
- ▷ inner triangulated
- ▷ ∿ $=$ induced paths
- ▷ no chord-connection







decomposing a **4-frame**

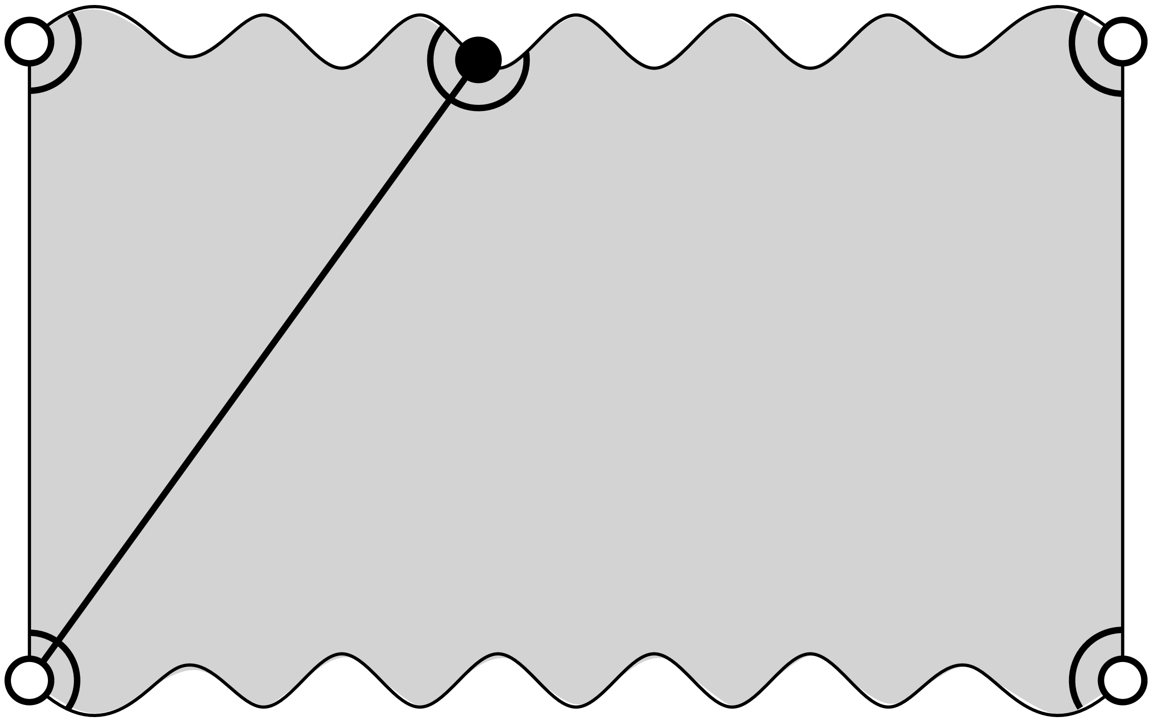


- ▷ Case 1: there is a chord
  - → split into two frames
- ▷ Case 2: chordless boundary
  - → split into several frames

Case 1

Case 2

decomposing a **4-frame**



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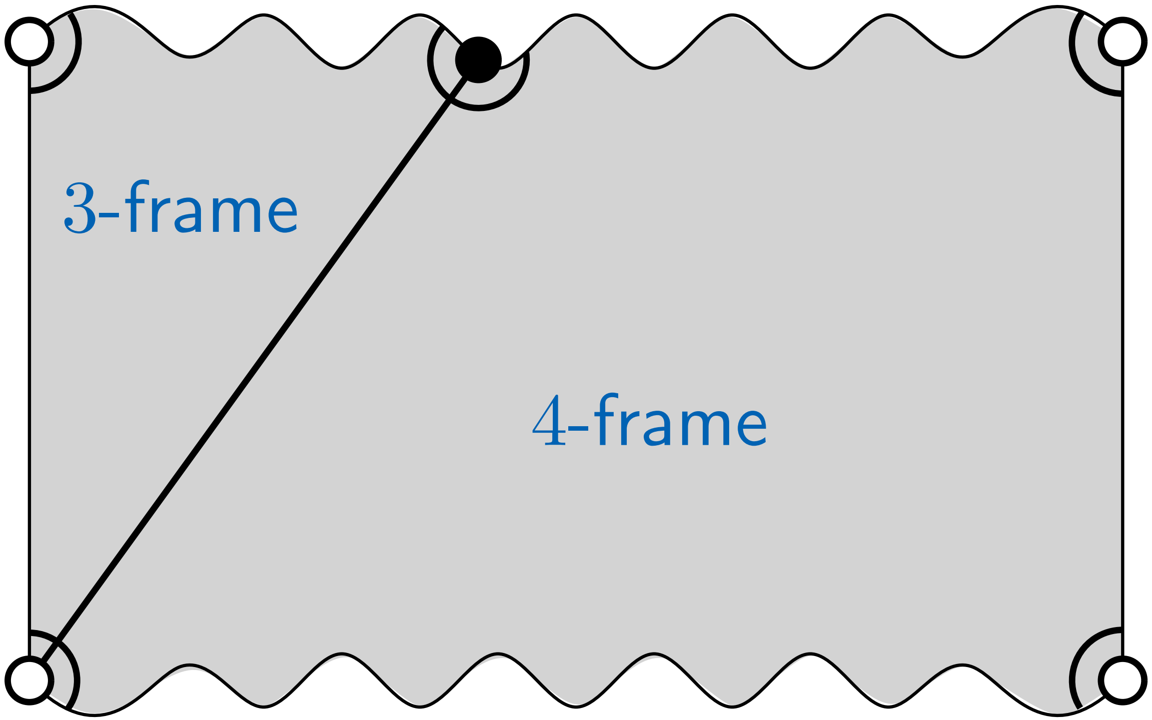

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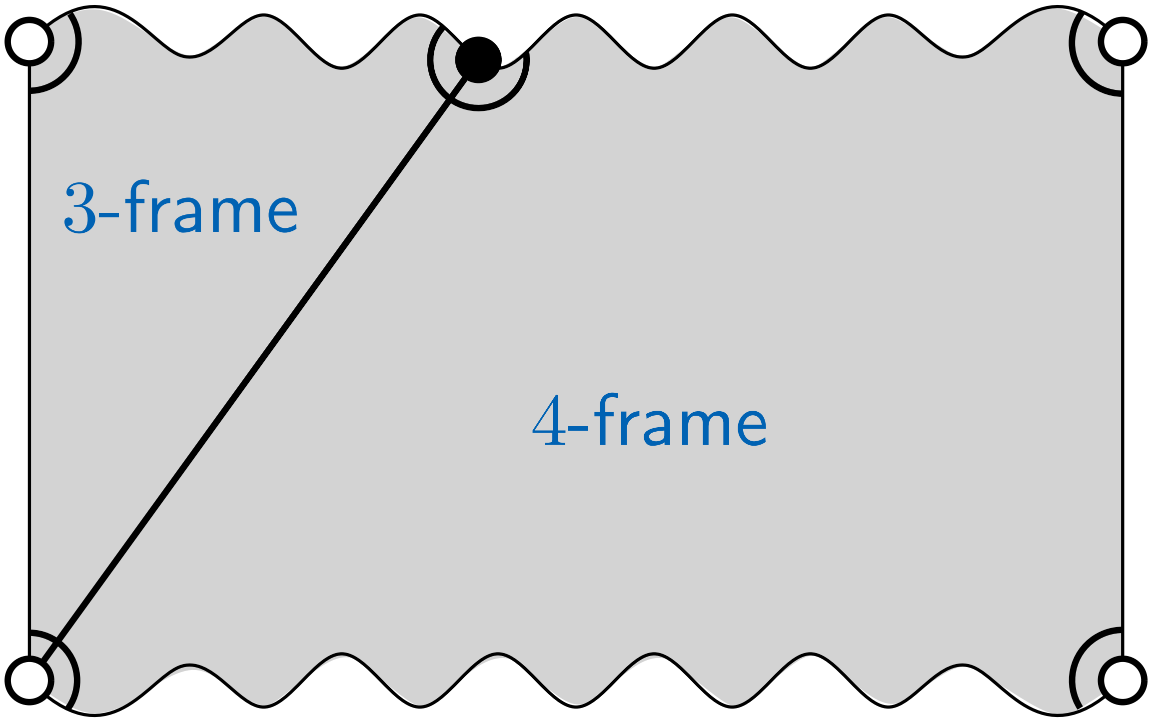

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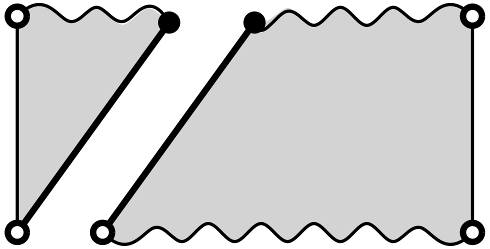
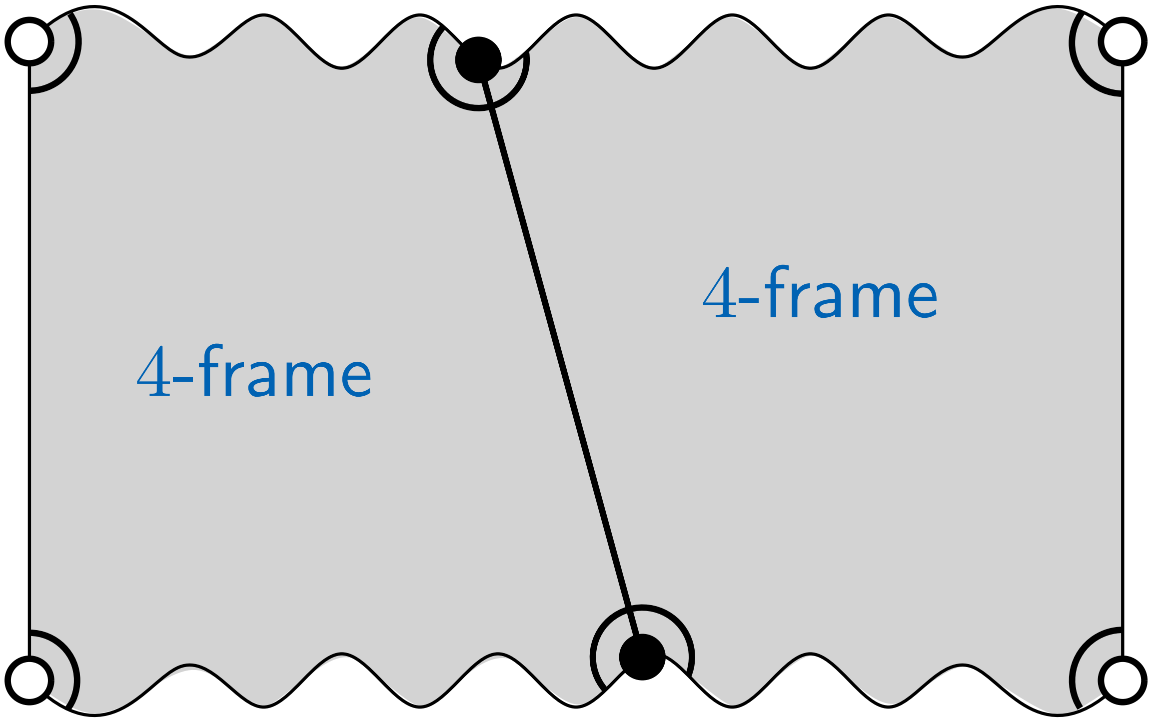


Case 1

Case 2

decomposing a **4-frame**



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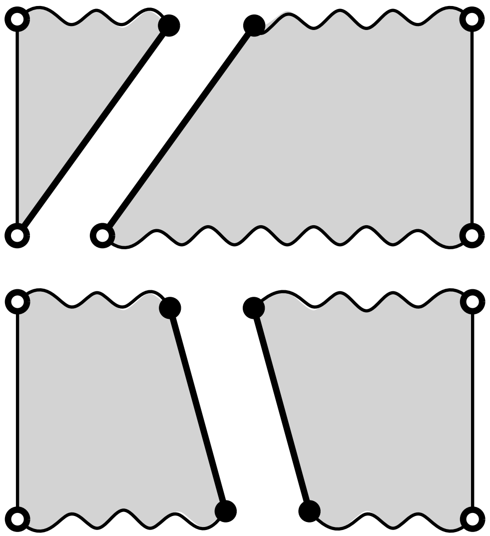
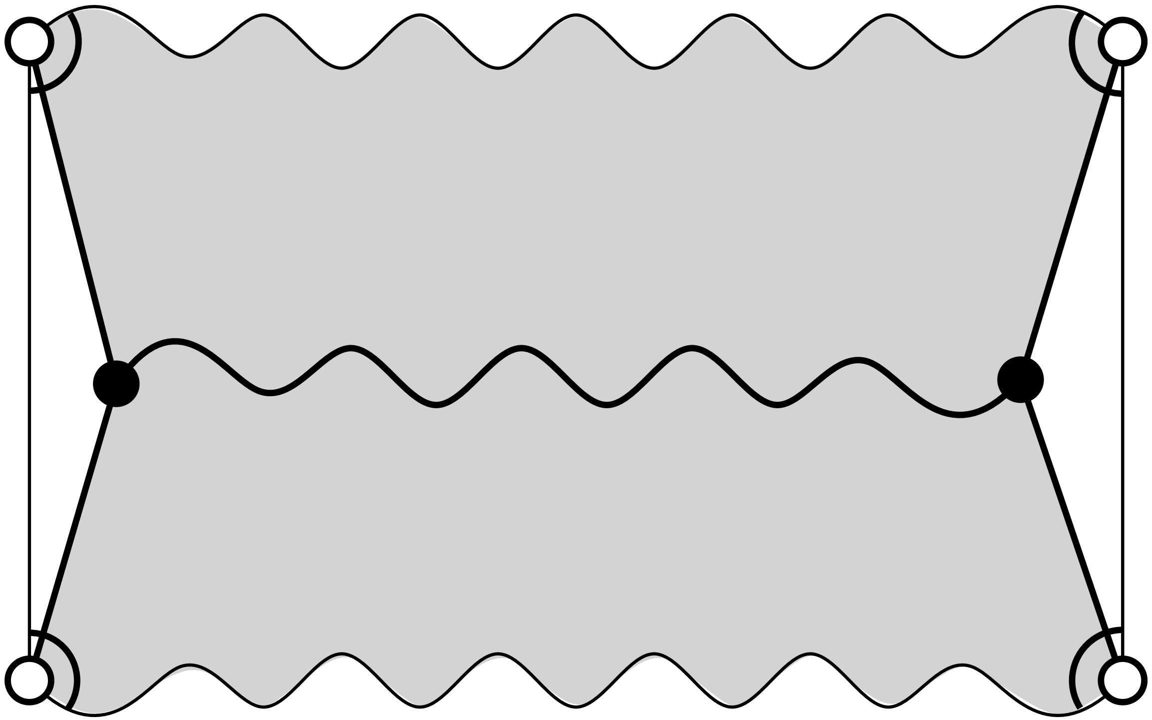


Case 2

decomposing a **4-frame**



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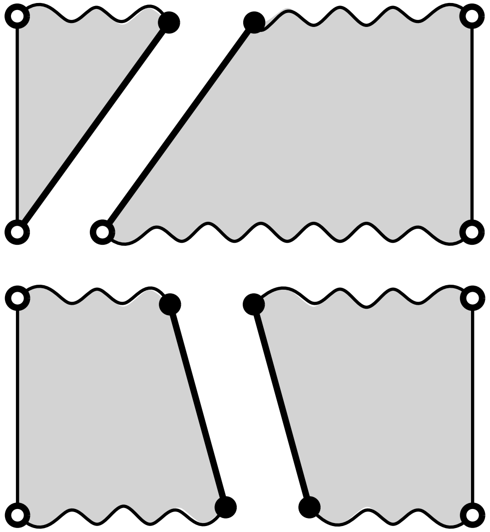
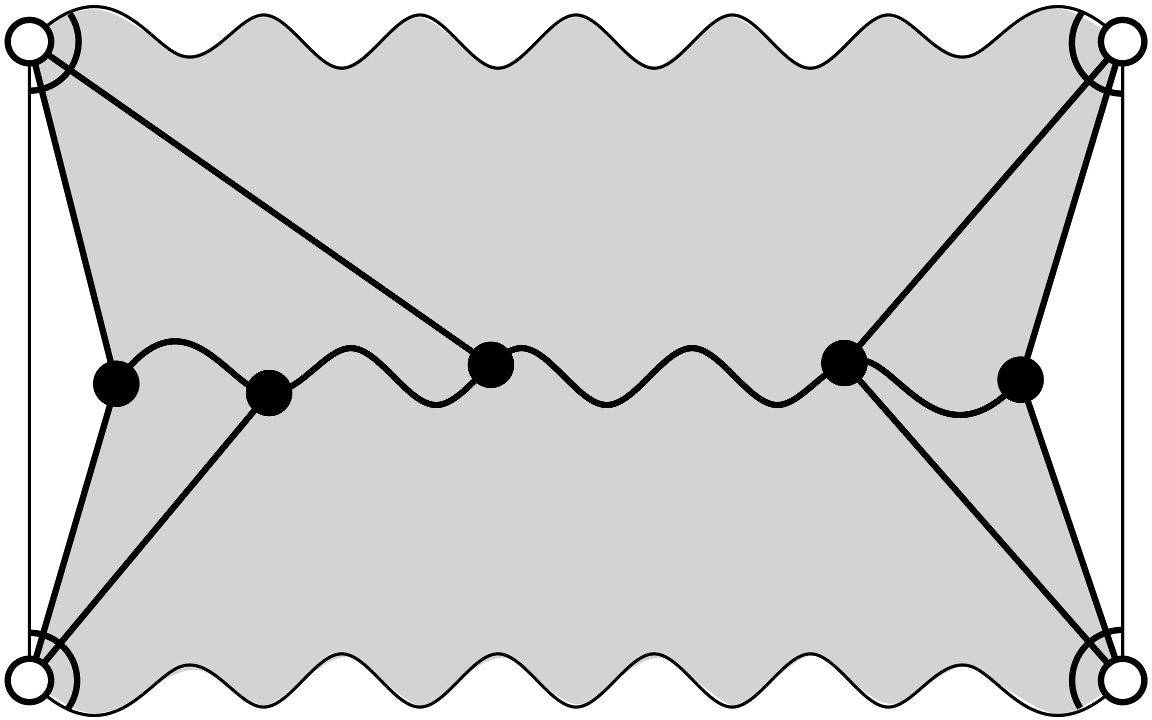


Case 1

Case 2

decomposing a **4-frame**



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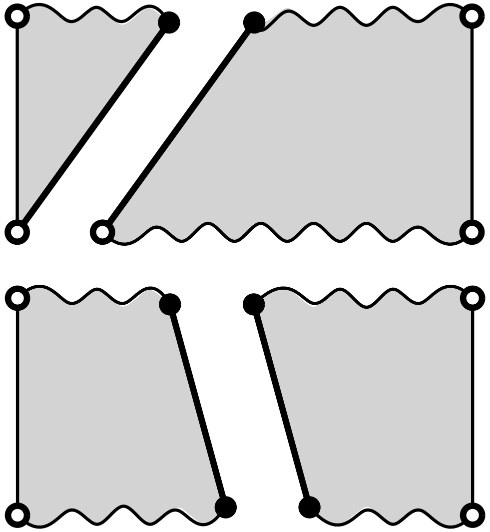

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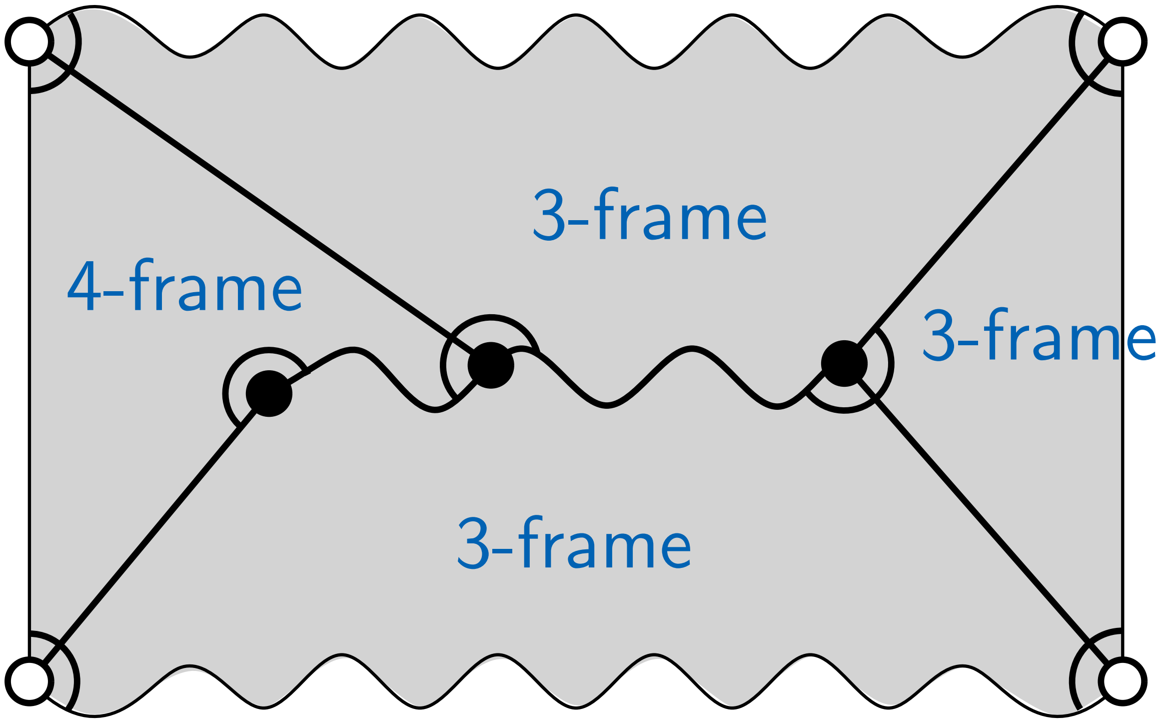

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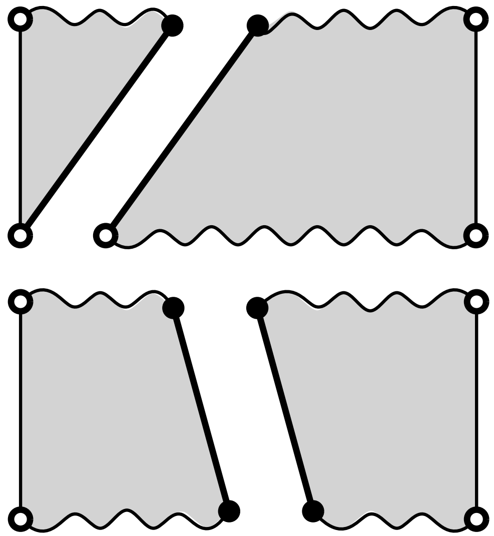

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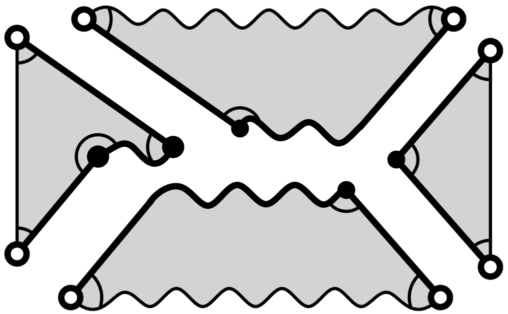
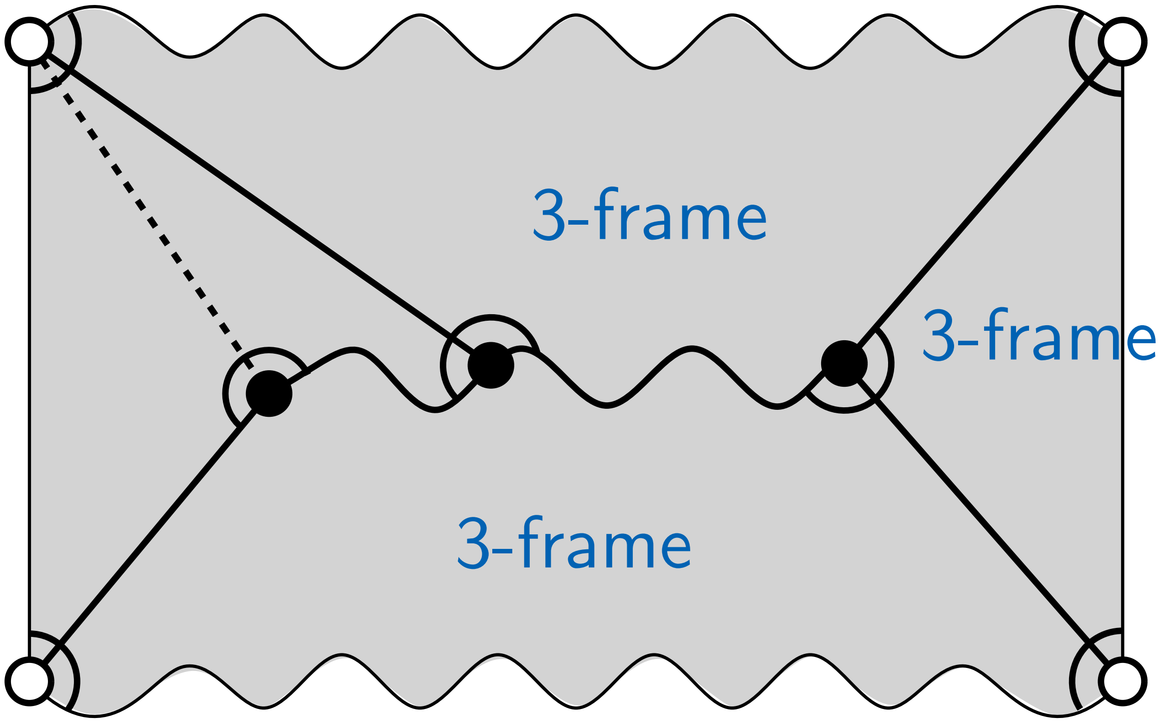


Case 2

decomposing a **4-frame**



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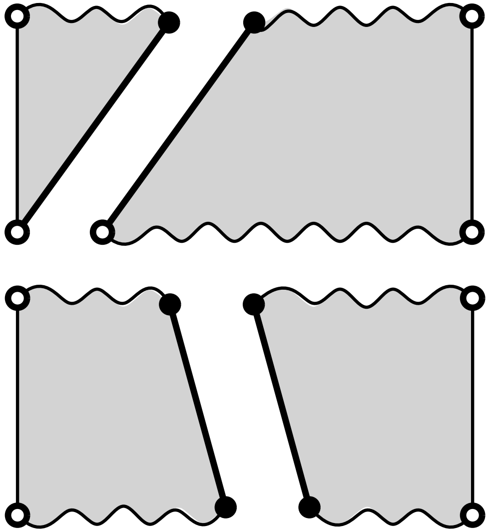

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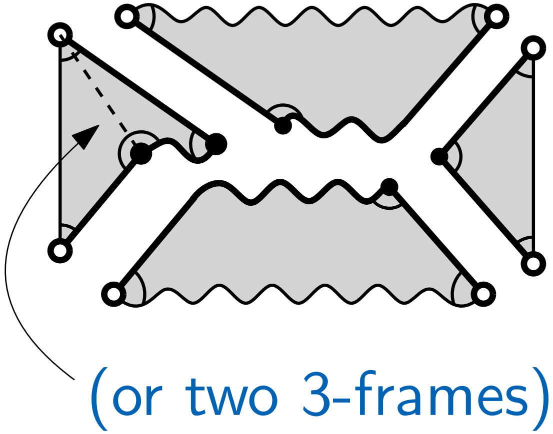


Case 2

defining intervals

Thm. (Knauer-Rollin-?-U. 2016+)
For every planar graph G we have $i(G) \leq 3$.

Proof. We proceed along a frame decomposition (induction).

Invariants:

▷  represented as 

▷ $$\left.\begin{array}{r}\text{corners}\\\text{boundary}\\\text{inner}\end{array}\right\}\text{ can spend }\left\{\begin{array}{l}0\text{ intervals}\\1\text{ interval}\\3\text{ intervals}\end{array}\right.$$

▷ one interval for connection to corners

▷ one interval for induced path

▷ one interval for later frame □

Thm. (Knauer-Rollin-U. 2016+)
For every planar graph G we have $i(G) \leq 3$.

Thm. (Gonçalves 2007)
For every planar graph G we have $t(G) \leq 4$.

Open Do we have for every planar graph G that $t_\ell(G) \leq 3$?

Interval Graphs

Several Intervals per Vertex

- ▷ interval, track and local track number
- ▷ planar graphs

▷ General Covering Parameters ◁

Higher-Dimensional Box per Vertex

- ▷ boxicity, local and union boxicity
- ▷ planar graphs

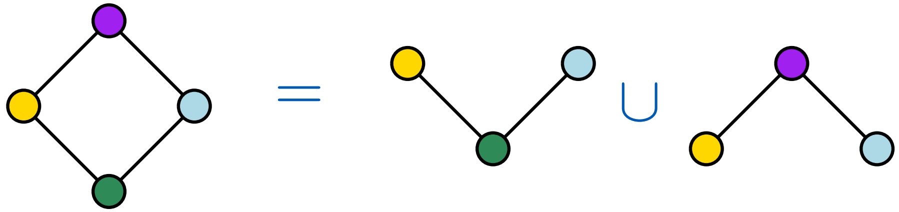
Combining Approaches

a more abstract view

$$\mathcal{H} = \{\text{guest graphs}\}$$

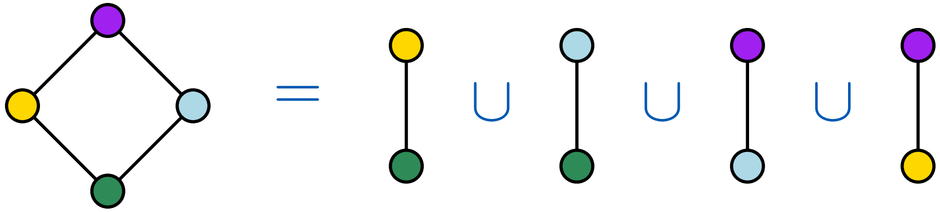
▷ global cover

$$c_g^{\mathcal{H}}(G) = \min\{k : G \text{ union of } k \text{ guest graphs}\}$$



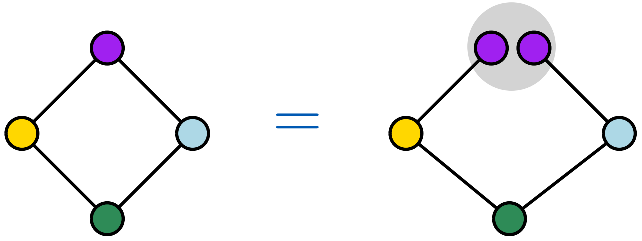
▷ local cover

$$c_\ell^{\mathcal{H}}(G) = \min\{k : G \text{ union of guest graphs, at most } k \text{ at each vertex}\}$$



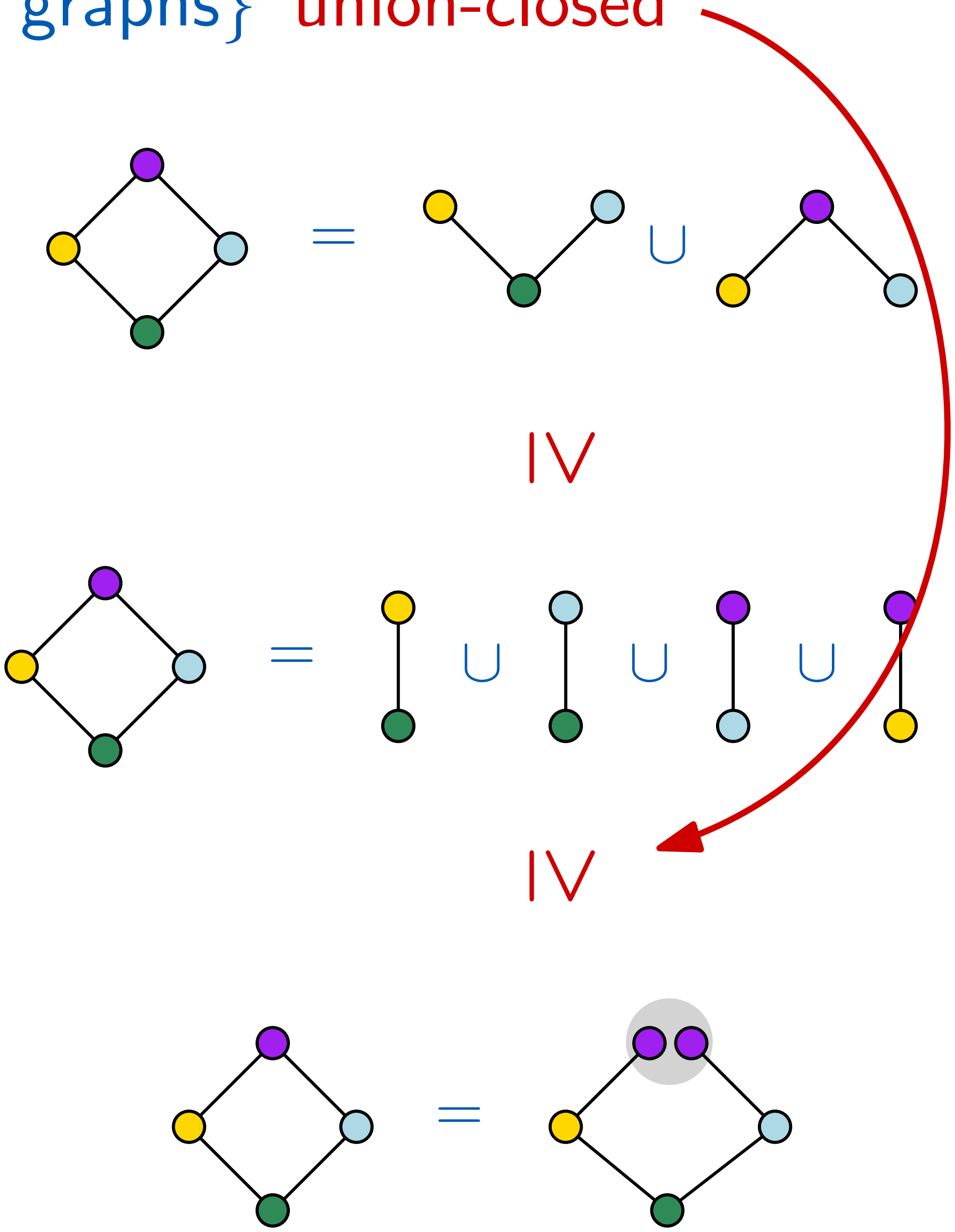
▷ folded cover

$$c_f^{\mathcal{H}}(G) = \min\{k : \text{splitting each vertex into } k \text{ gives a guest graph}\}$$



a more abstract view

$\mathcal{H} = \{$guest graphs$\}$ union-closed



- **global cover**

$c_g^{\mathcal{H}}(G) = \min\{k :$
G union of k guest graphs$\}$

- **local cover**

$c_\ell^{\mathcal{H}}(G) = \min\{k :$
G union of guest graphs,
at most k at each vertex$\}$

- **folded cover**

$c_f^{\mathcal{H}}(G) = \min\{k :$
splitting each vertex into k
gives a guest graph$\}$

road map

Interval Graphs

Several Intervals per Vertex

- ▷ interval, track and local track number
- ▷ planar graphs

General Covering Parameters

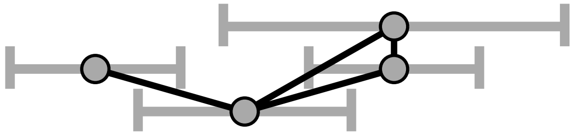
▷ **Higher-Dimensional Box per Vertex** ◁

- ▷ boxicity, local and union boxicity
- ▷ planar graphs

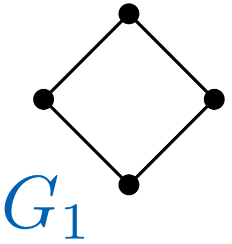
Combining Approaches

generalizing intervals

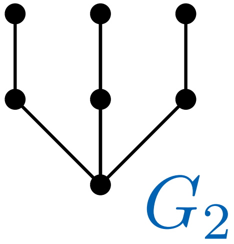
… using higher dimensions



$\mathcal{I} = \{\text{interval graphs}\}$



graphs
$G_1, G_2 \notin \mathcal{I}$



generalizing intervals

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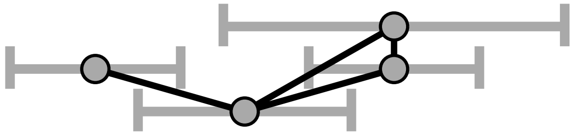

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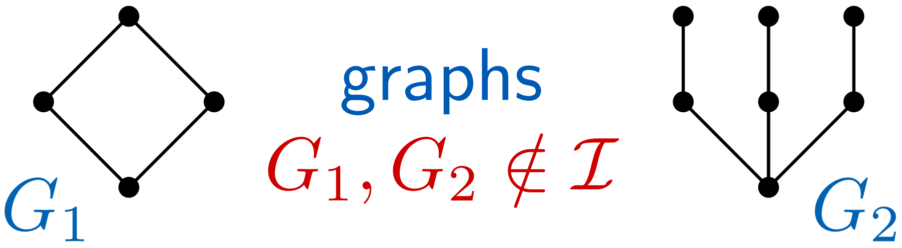

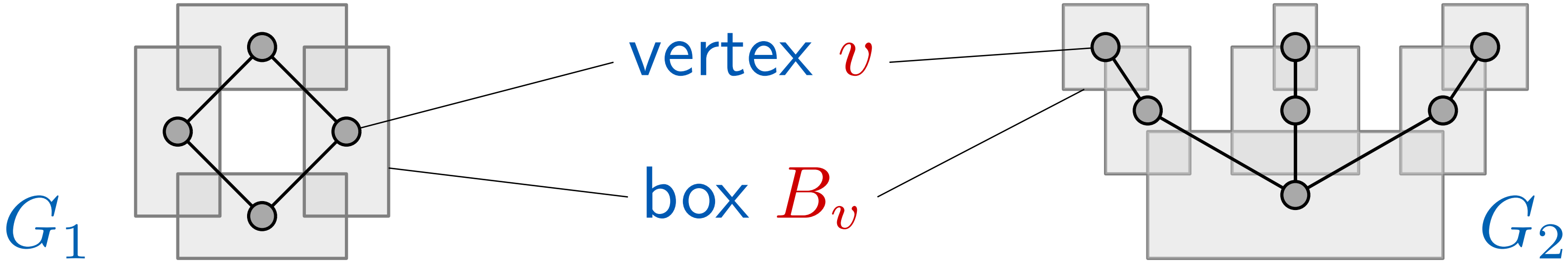

$$v \in V(G) \iff B_v = I_v^1 \times \cdots \times I_v^k \subset \mathbb{R}^k$$

$$uv \in E(G) \iff B_u \cap B_v \neq \emptyset$$

boxicity

$\mathrm{box}(G) = \min\{k :$ boxes in $\mathbb{R}^k$ suffice$\}$

Thm. (Roberts 1969)
$\mathrm{box}(G) \leq k$ if and only if $G = I_1 \cap \cdots \cap I_k$
for some interval graphs $I_1, \ldots, I_k$.

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Obs. (Cozzens-Roberts 1983)

$$G = I_1 \cap \cdots \cap I_k \quad \Longleftrightarrow \quad G^c = I_1^c \cup \cdots \cup I_k^c$$

$\Longrightarrow \mathrm{box}(G) = \min\{k : G^c$ union of k cointerval graphs$\}$

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$\Longrightarrow \mathrm{box}(G) = c_g^{\mathcal{C}}(G^c)$ where $\mathcal{C} = \{$cointerval graphs$\}$

global, local and folded boxicity?

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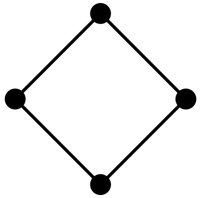
$\Longrightarrow$ $\mathrm{box}(G) = c_g^{\mathcal{C}}(G^c)$ where $\mathcal{C} = \{$cointerval graphs$\}$

Question: What is local and folded boxicity?

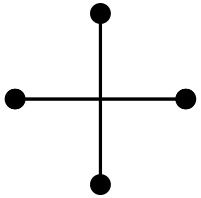
$$\mathrm{box}_\ell(G) = c_\ell^{\mathcal{C}}(G^c) \qquad \mathrm{box}_f(G) = c_f^{\mathcal{C}}(G^c)$$

bad news

- $\mathcal{C}$ = {cointerval graphs} is not union-closed !



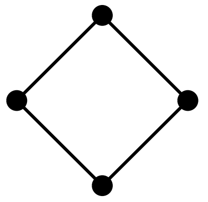
$\notin \mathcal{I} \quad \iff$



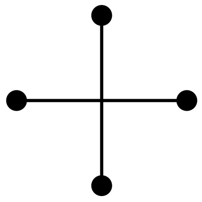
$\notin \mathcal{C}$

$\Longrightarrow$ not necessarily $\mathrm{box}_f(G) \leq \mathrm{box}_\ell(G)$

- $\mathcal{C} = \{$cointerval graphs$\}$ is not union-closed !



$\notin \mathcal{I} \qquad \Longleftrightarrow$



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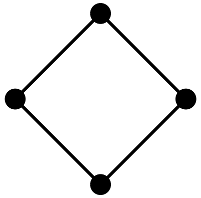
$\Longrightarrow$ not necessarily $\operatorname{box}_f(G) \leq \operatorname{box}_\ell(G)$

Prop. (Bläsius-Stumpf-U. 2016+)

$$\operatorname{box}_f(G) = \begin{cases} 1 & \text{if } G \in \mathcal{I} \\ \infty & \text{otherwise} \end{cases}$$

bad news

▷ $\mathcal{C} = \{$cointerval graphs$\}$ is not union-closed !



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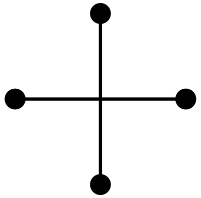

$\notin \mathcal{C}$

$\implies$ not necessarily $\mathrm{box}_f(G) \leq \mathrm{box}_\ell(G)$

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▷ consider $\overline{\mathcal{C}} = \{$vertex-disjoint unions of cointerval graphs$\}$

$$\overline{\mathrm{box}}(G) = c_g^{\overline{\mathcal{C}}}(G^c) \qquad \overline{\mathrm{box}}_\ell(G) = c_\ell^{\overline{\mathcal{C}}}(G^c) \qquad \overline{\mathrm{box}}_f(G) = c_f^{\overline{\mathcal{C}}}(G^c)$$

too many

- ▷ $\mathcal{C} = \{$cointerval graphs$\}$

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$\Leftrightarrow$ $I_j \neq \mathbb{R}$ for at most k indices $j \in \{1, \ldots, d\}$.

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$\Leftrightarrow$ G intersection graph of 1-local boxes in $\mathbb{R}^d$.

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local boxicity

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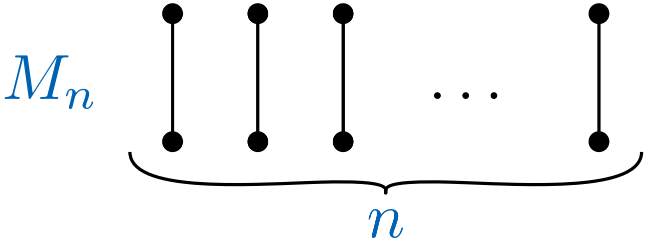
- For $G = M_n^c$ we have
  $$\mathrm{box}_\ell(G) = \overline{\mathrm{box}}(G) = 1 \quad \text{and} \quad \mathrm{box}(G) = n.$$



Proof. $M_2 \notin \mathcal{C}$. ☐

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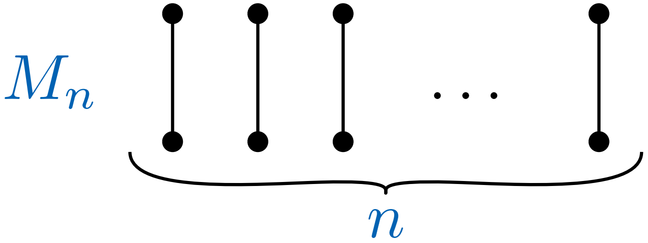

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Thm. (Bläsius-Stumpf-U. 2016+)
For $G = L(K_n)^c$ we have
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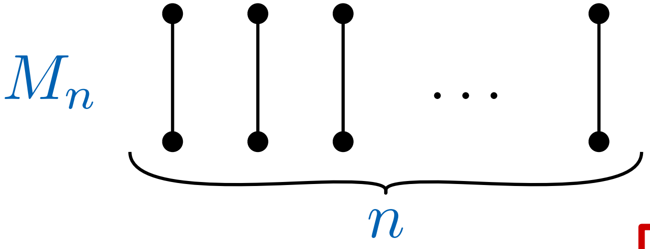

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Note: M_3^c is planar!

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Que. Maximum $\operatorname{box}_\ell(G)$ / $\overline{\operatorname{box}}(G)$ / $\operatorname{box}(G)$ if G is planar?

graph class	$\operatorname{box}_\ell(G)$	$\overline{\operatorname{box}}(G)$	$\operatorname{box}(G)$
outerplanar	2	2	2 [Scheinerman '84]
planar bipartite	2	2	2 [Hartman-Newman-Ziv '91]
planar	?	?	3 [Thomassen '86]
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Open Do we have for planar G that $\mathrm{box}_\ell(G) \leq 2$?

Interval Graphs

Several Intervals per Vertex

- ▷ interval, track and local track number
- ▷ planar graphs

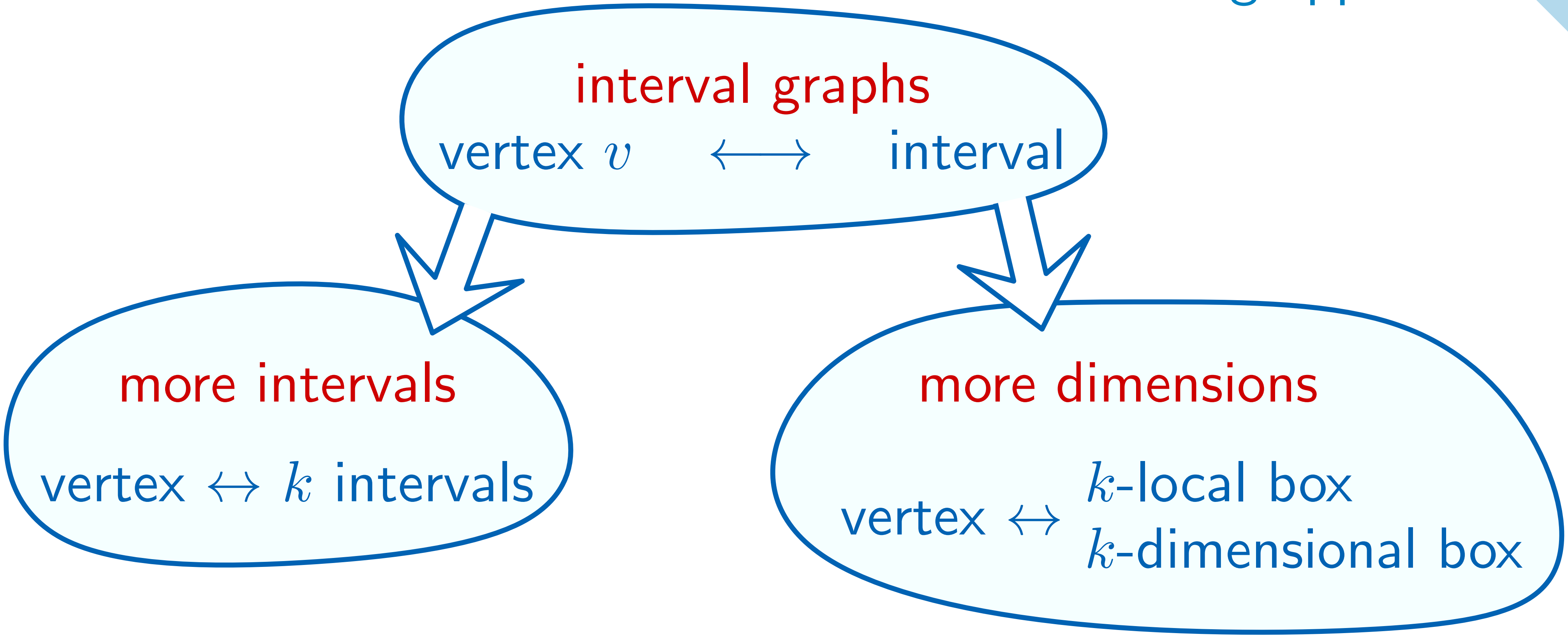
General Covering Parameters

Higher-Dimensional Box per Vertex

- ▷ boxicity, local and union boxicity
- ▷ planar graphs

▷ **Combining Approaches** ◁

combining approaches



interval graphs

vertex v $\longleftrightarrow$ interval

more intervals

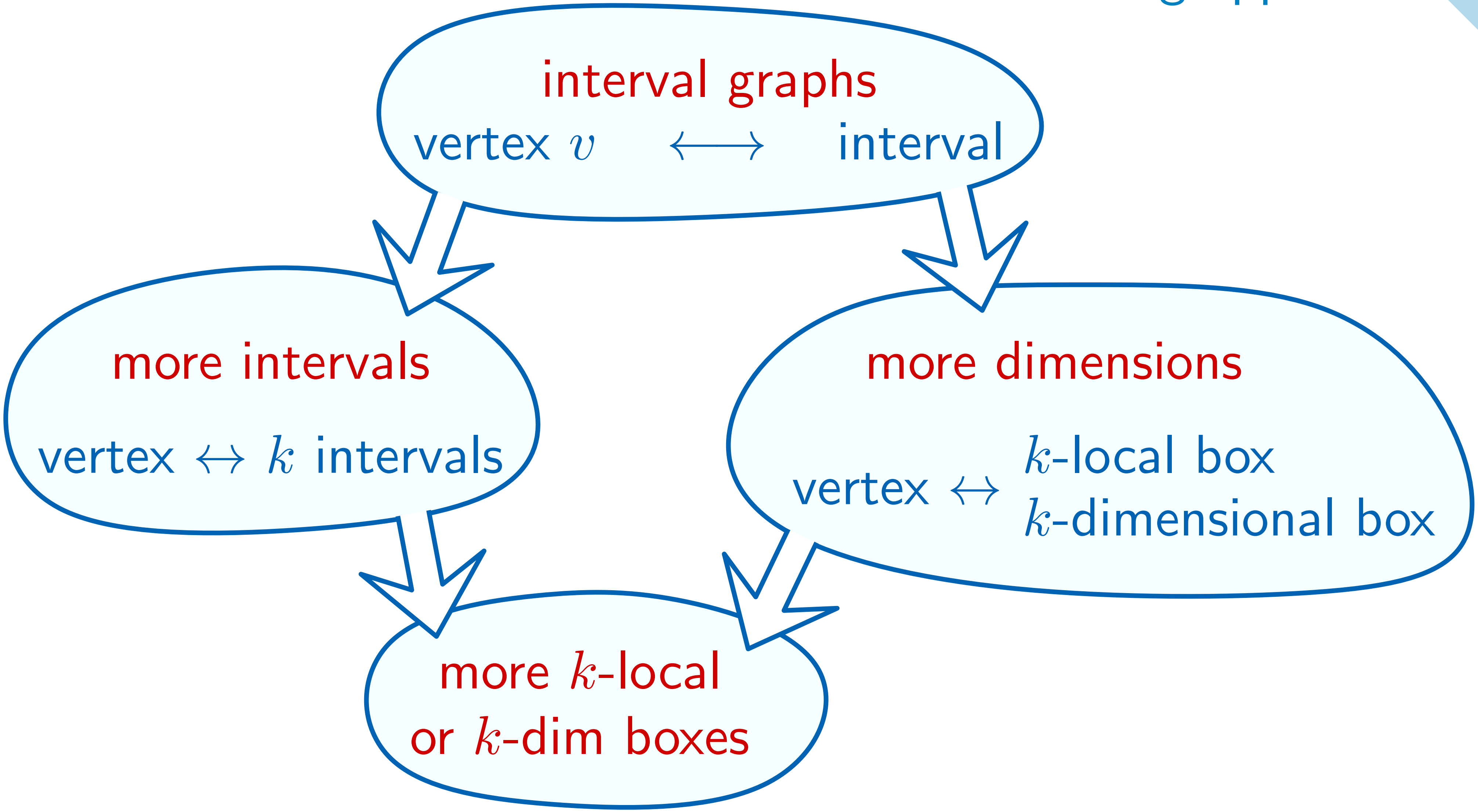
vertex $\leftrightarrow$ k intervals

more dimensions

vertex $\leftrightarrow$ k-local box

k-dimensional box

combining approaches



interval graphs

vertex v $\longleftrightarrow$ interval

more intervals

vertex $\leftrightarrow$ k intervals

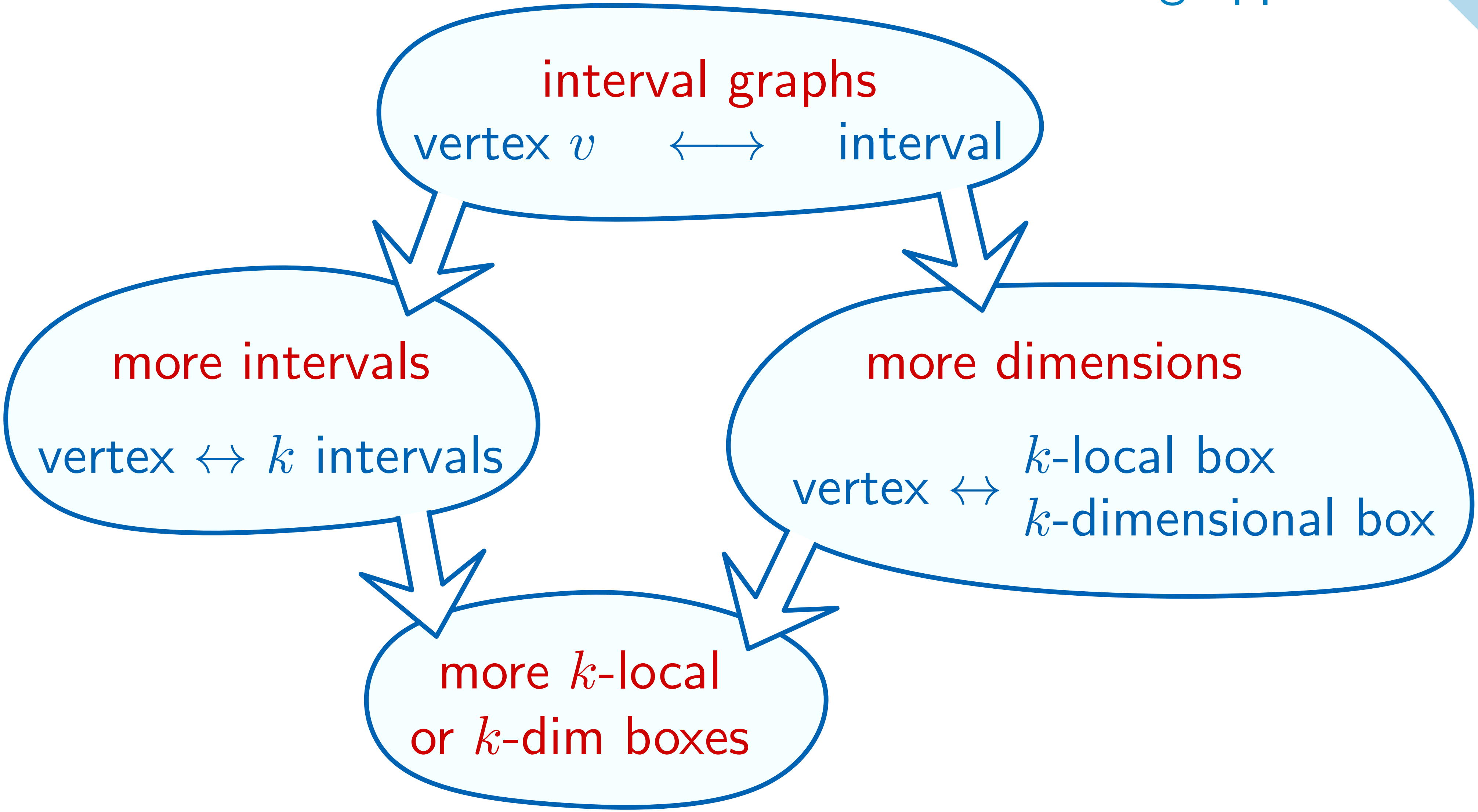
more dimensions

vertex $\leftrightarrow$ k-local box

k-dimensional box

more k-local

or k-dim boxes



interval graphs

vertex v $\longleftrightarrow$ interval

more intervals

vertex $\leftrightarrow$ k intervals

more dimensions

vertex $\leftrightarrow$ k-local box
 k-dimensional box

more k-local or k-dim boxes

Thm. (Thomassen 1986)
Every planar graph is 2-folded contact graph of 2-dimensional boxes.

summary – open problems

- ▷ interval number $i(G)$, local track number $t_\ell(G)$, track number $t(G)$

Open Do we have for planar G that $t_\ell(G) \leq 3$?

- ▷ boxicity $\mathrm{box}(G)$, union boxicity $\overline{\mathrm{box}}(G)$, local boxicity $\mathrm{box}_\ell(G)$

Open Do we have for planar G that $\mathrm{box}_\ell(G) \leq 2$?

- ▷ k-folded, k-local, k-global covers with j-local and j-dimensional boxes

Open Arbitrary separation for every k and j?

summary

$\mathcal{H} = \{$nice graphs$\}$ (guest graphs)	graph $G \notin \mathcal{H}$	question
$\mathcal{B} = \{$bipartite graphs$\}$	any	$c_\ell^{\mathcal{B}}(G) = c_g^{\mathcal{B}}(G)$
	K_n-minor free	$c_\ell^{\mathcal{B}}(G) \leq \log(n)$
$\{$outerplanar graphs$\}$	K_n	$c_f^{\mathcal{H}}(G), c_\ell^{\mathcal{H}}(G), c_g^{\mathcal{H}}(G)$
	planar	$c_f^{\mathcal{H}}(G) \leq 2$
$\mathcal{I} = \{$interval graphs$\}$		$c_\ell^{\mathcal{I}}(G) \leq 3$
	any	$c_g^{\mathcal{H}}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$
$\{$complete bipartite graphs$\}$	any	$\chi(G) \leq f(c_f^{\mathcal{H}}(G))$
$\mathcal{P} = \{$planar graphs$\}$	$c_\ell^{\mathcal{P}}(G) \leq m$	$\chi(G)$

summary

class* $\mathcal{H}$	$c_g^{\mathcal{H}}(G)$	$c_\ell^{\mathcal{H}}(G)$	$c_f^{\mathcal{H}}(G)$
K_2	NPC	P	
stars	NPC	P	
trees	P		
bipartite	NPC	NPC	P
interval graphs	NPC	NPC	NPC
planar	NPC	open	NPC
outerplanar	open	open	open

* closure under disjoint unions

decreasing difficulty?