Computing the Geometric Intersection Number of Curves

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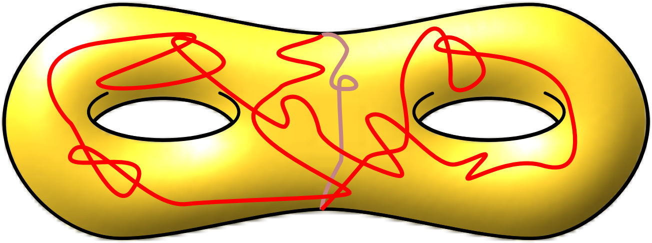
3 February 2016

Problem



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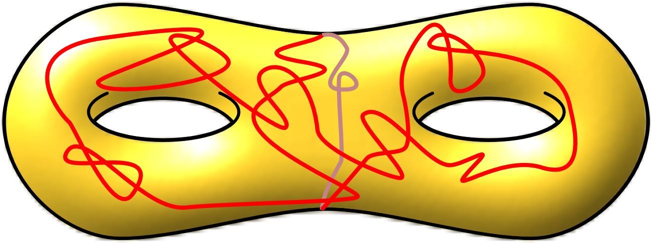

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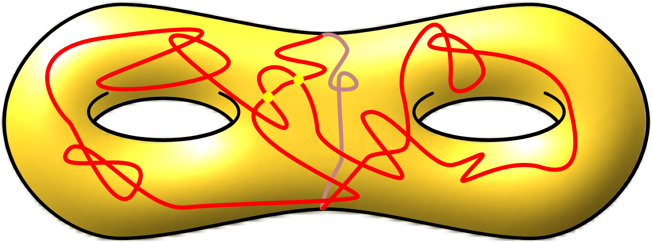

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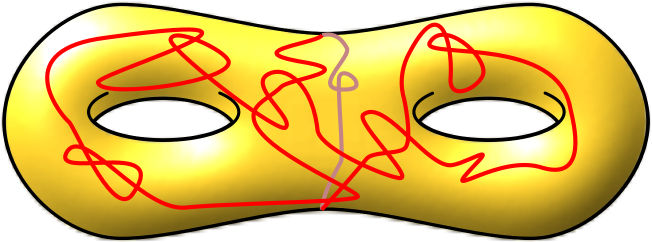

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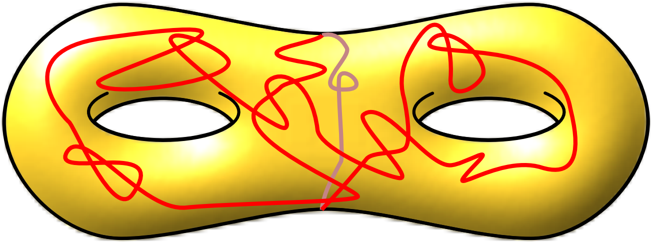

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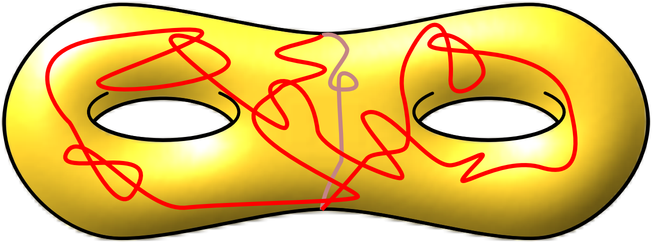

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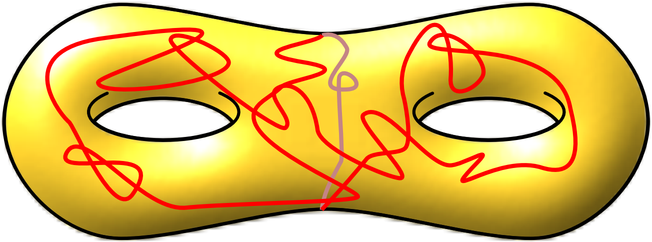

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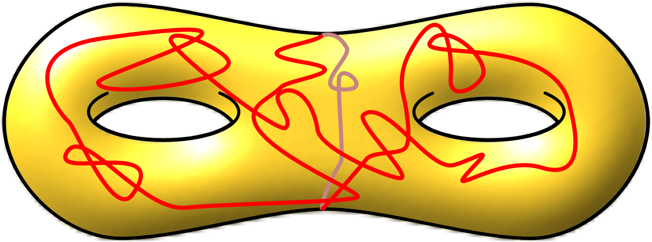

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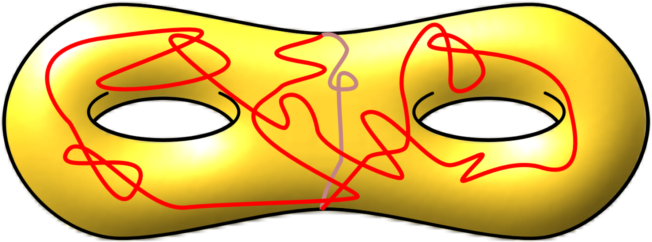

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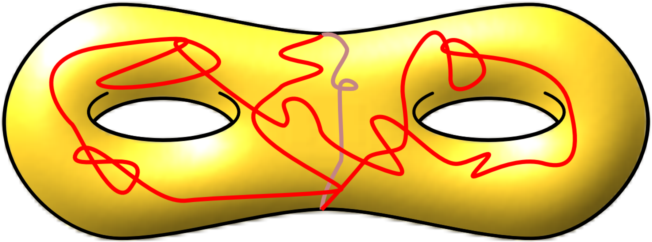

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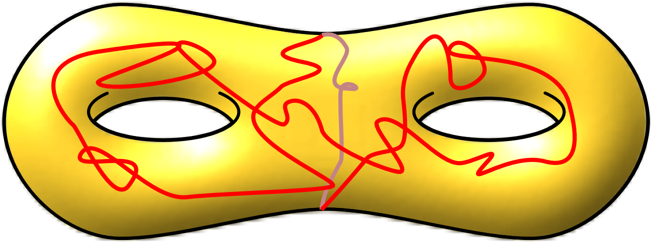

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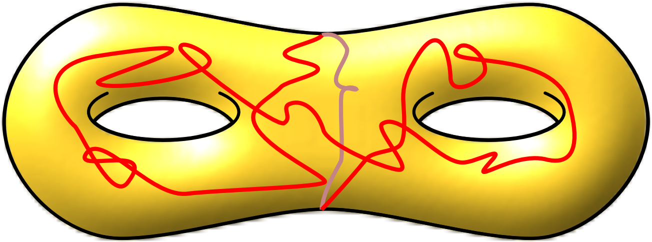

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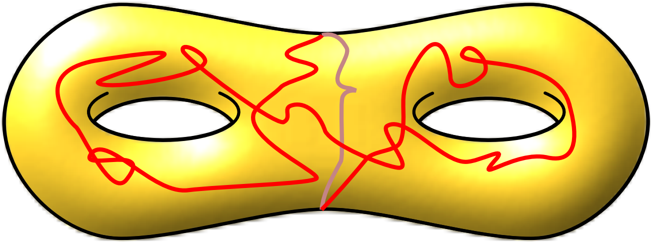

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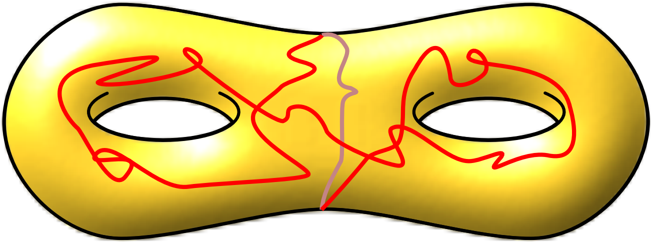

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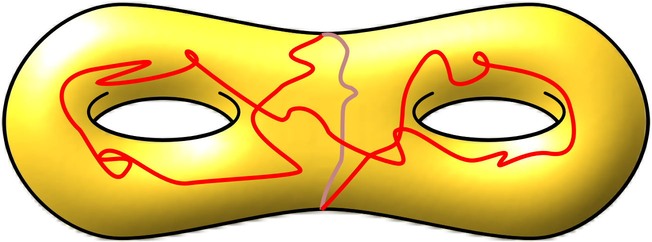

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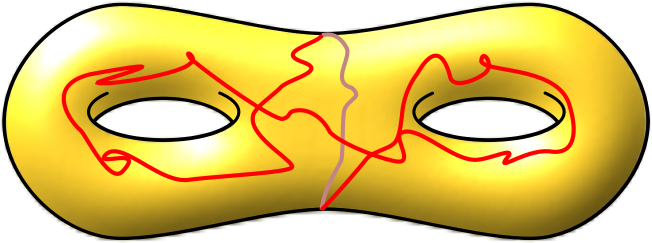

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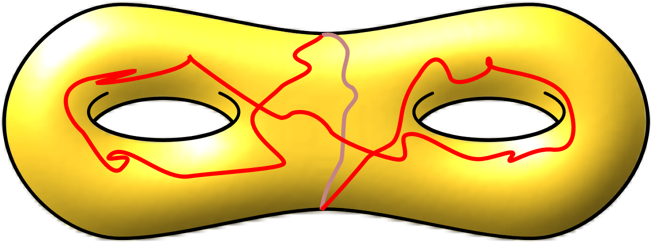

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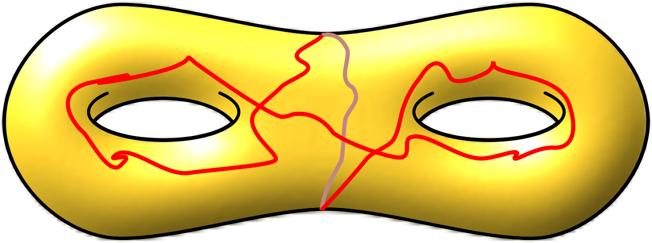

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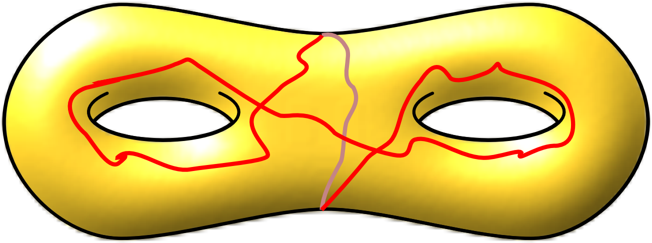

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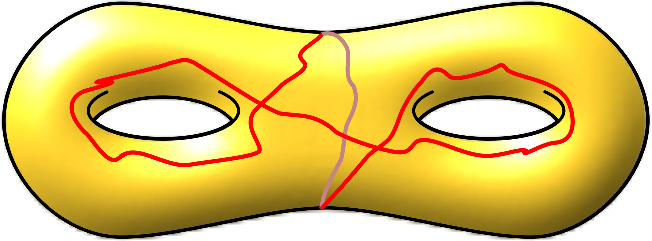

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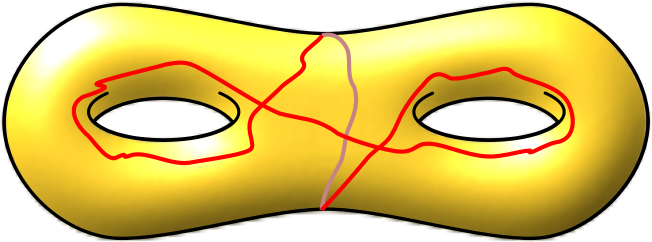

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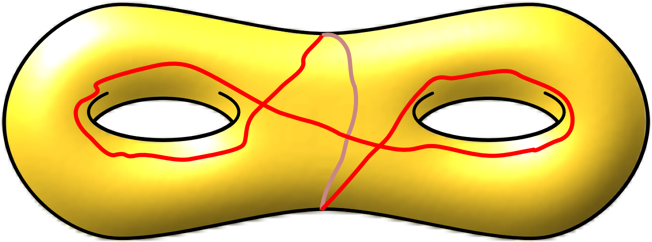

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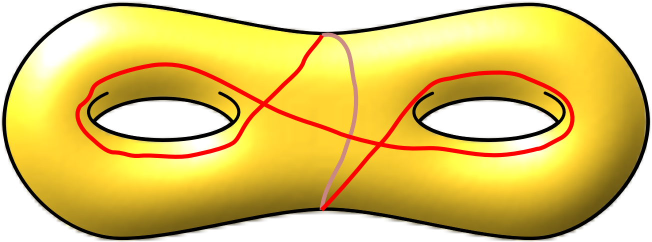

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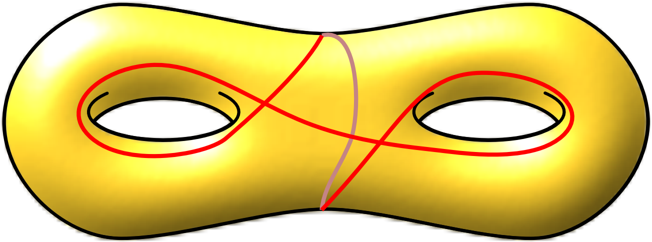

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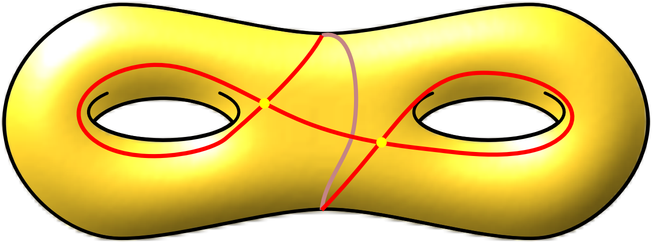

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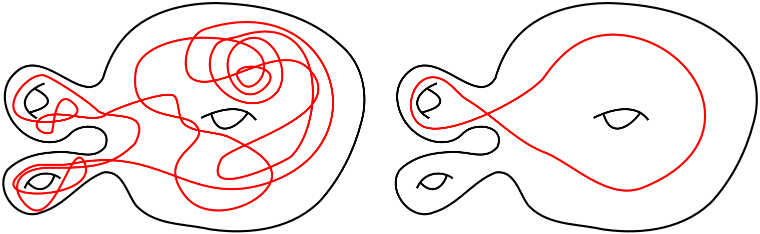


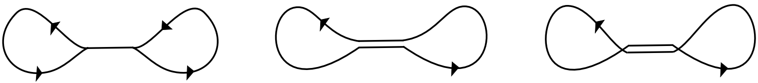
(a) Number of crossings: too many!

(b) Number of crossings: 1 → optimal

Three problems:

- ➯ Deciding if a curve can be made simple by homotopy.
- ➯ Finding the minimum possible number of self-intersections.
- ➯ Finding a corresponding minimal representative.

INPUT:
A combinatorial surface of complexity n.
A closed walk of length l.

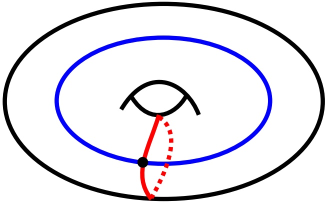


A choice of edge orders leads to a generic perturbation of the curve.

Discrete vs. Continuous

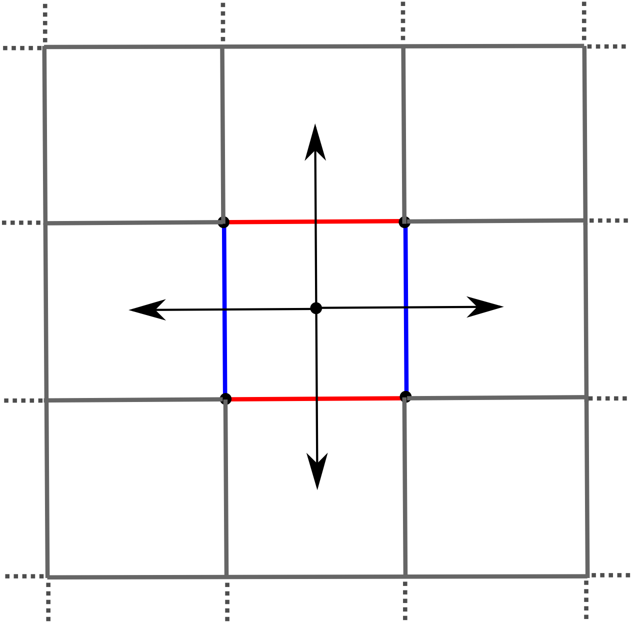
- Each equivalence class of curves can be described by a closed walk.
- Each minimal configuration can be realized by a closed walk with appropriate orders on the edges.

Classical Topology



$$\pi_1(torus) = \{\alpha^n \cdot \beta^m, n, m \in \mathbb{Z}\}$$

Classical Topology



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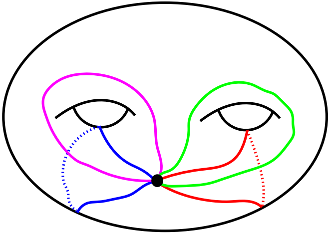

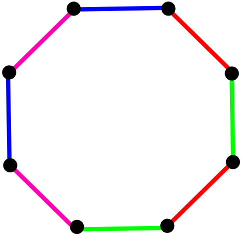

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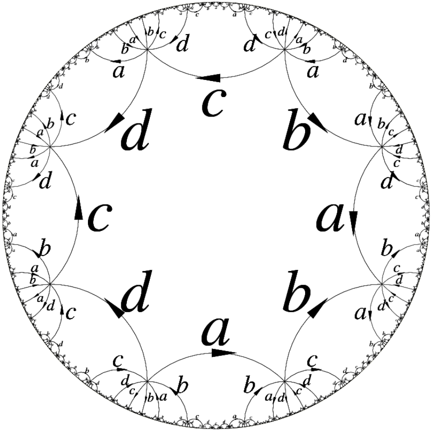

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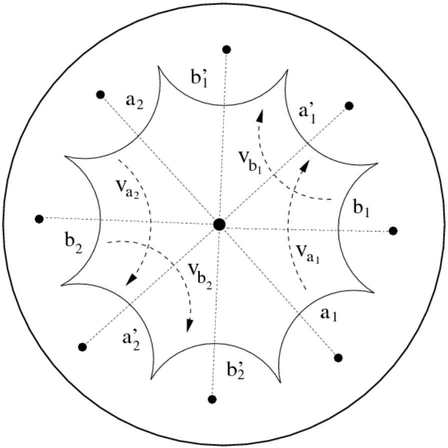

An Old Problem

- **Poincaré**, 5ème complément analysis situs (1905)
  We can take advantage of hyperbolic metric.
- **Reinhart**, Algorithms for jordan curves on compact surfaces (1962)
- **Chillingworth**, Simple closed curves on surfaces (1969)
- **Birman and Series**, An algorithm for simple curves on surfaces (1984)
- **Cohen and Lustig**, Paths of geodesics and geometric intersection numbers (1987)
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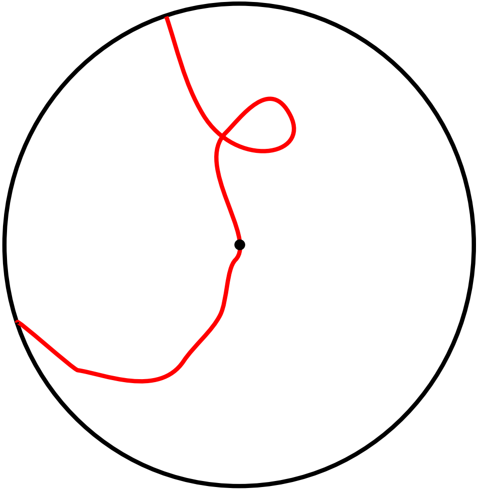

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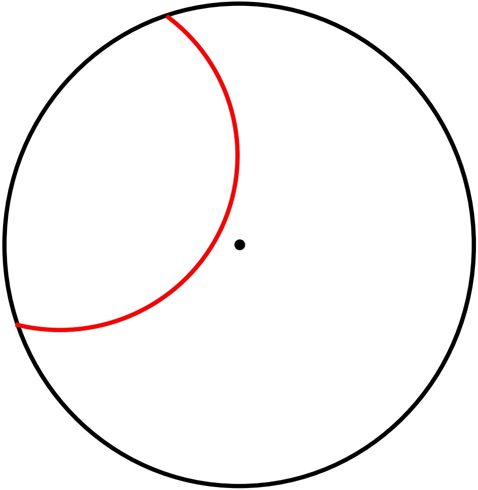

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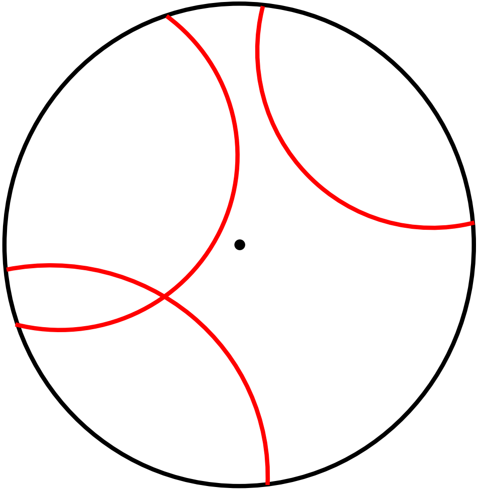

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  It can be made a finite algorithm.
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  We can also decide of simplicity by using winding numbers.
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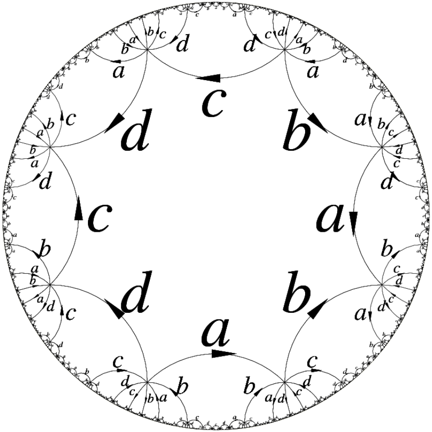

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- **Cohen and Lustig**, Paths of geodesics and geometric intersection numbers (1987)
  It leads to an efficient algorithm to compute geometric intersection numbers.
- **Gonçalves et al.**, An algorithm for minimal number of (self-)intersection points of curves on surfaces (2005)

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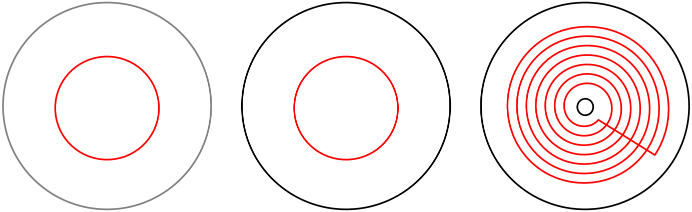
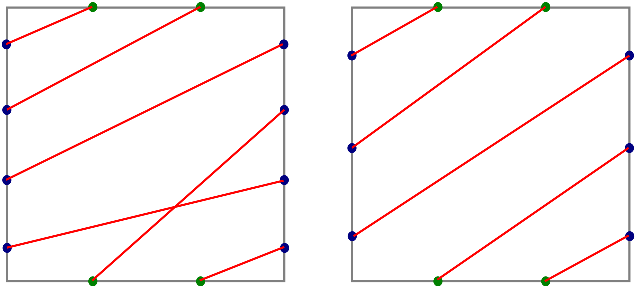
  It becomes more algebraic and less algorithmic.

Easy Cases

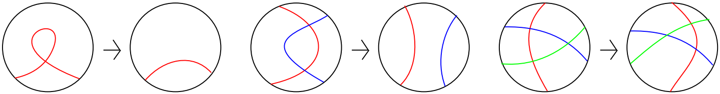
Euler characteristic: $\chi(\Sigma) = 2 - 2g - b$





de Graaf and Schrijver's Algorithm

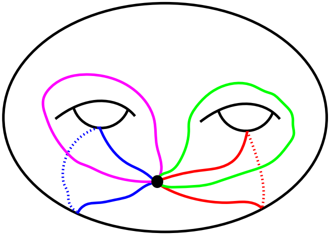
Reidemeister moves:

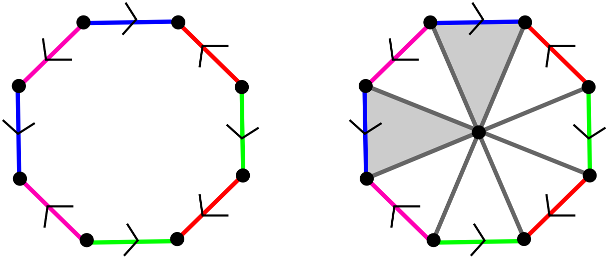


de Graaf and Schrijver (1997)

Every curve can be made minimally crossing by Reidemeister moves.

System of Quads

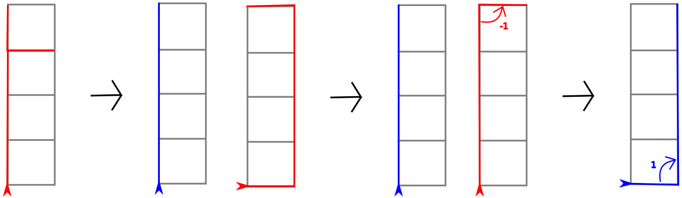




Canonical Representative

Canonical representative:

- Choose a shortest representative and push it to the right as much as possible.



Erickson and Whittlesey (2013)

The canonical representative of a curve is its only representative with no spur, no bracket, no angle -1 and not all angles -2.

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Erickson and Whittlesey (2013)

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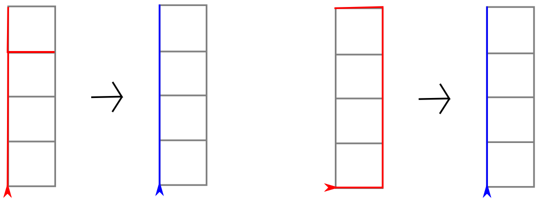
Lazarus and Rivaud (2012)

After $O(n)$ precomputation time, one can compute the canonical representative of a curve of length ℓ in $O(\ell)$ time. Its length is at most 2ℓ.

Geodesic Representative



Geodesics

A curve in a system of quads has minimal length if and only if it has no spurs nor brackets.

Discrete Curvature

We assign to all the angles the measure θ.
We define the discrete curvature:

- for vertices, $\kappa(v) = 1 - \frac{deg(v)}{2} + \sum_{a \in v} \theta$.
- for faces, $\kappa(f) = 1 - \sum_{a \in f} \theta$.

Combinatorial Gauss-Bonnet

$$\sum_v \kappa(v) + \sum_f \kappa(f) = \#V - 2\frac{\#E}{2} + \sum_a \theta + \#F - \sum_a \theta$$

$$= \#V - \#E + \#F = \chi$$

Spurs and Brackets

Van Kampen (1933), Lyndon (1966)

A cycle C is contractible if and only if there is a disk diagram whose boundary cycle is sent to C.

Gersten and Short (1990)

A nontrivial contractible closed curve on a system of quads must have either a spur or four brackets.

Let set $\theta = \dfrac{1}{4}$, so $\forall f, \kappa(f) = 0$.

$\sum_v \kappa(v) = 1$
$= \sum_{interior\ v}(1 - \frac{deg(v)}{4}) + \sum_{boundary\ v}(1 - \frac{deg(v)}{4} - \frac{1}{4})$

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Proposition

A nontrivial contractible closed curve on a system of quads which admits a disk diagram with at least one interior vertex must have either a spur or five brackets.

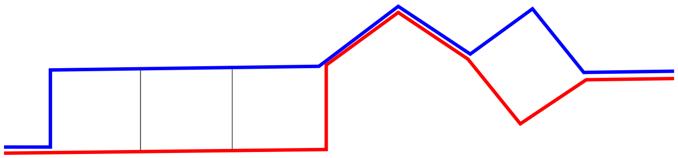
Properties of the Canonical Form

Lemma

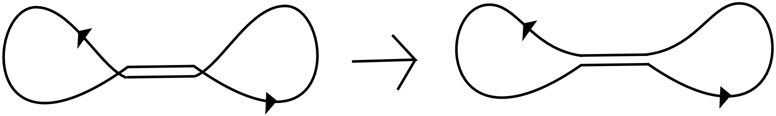
A curve in canonical form has no monogon.

Lemma

A curve in canonical form has only **quasi-flat** bigons.

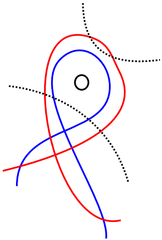


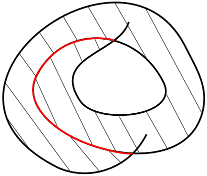
Monogons and Bigons



Hass and Scott (1985)

If a curve has excess self-intersections then it has a **singular** monogon or a **singular** bigon.





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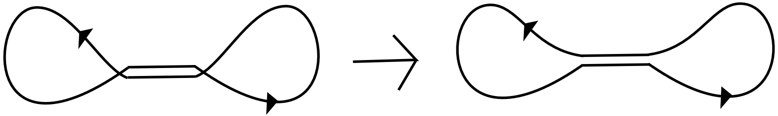

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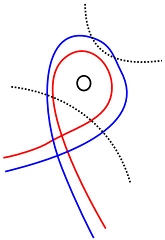
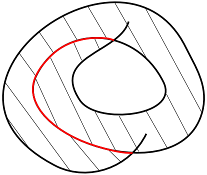
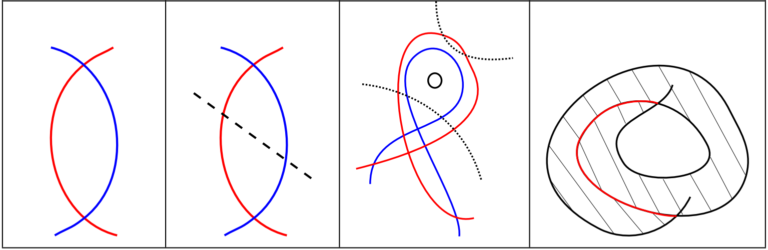




Bigon’s Fauna

Innermost	Embedded	Singular	Weak



Remark: Any kind of bigon can be quasi-flat or not, this is related to the actual immersion in the system of quads!

Computing a Minimal Representative

Algorithm for **primitive** curves:

- ➯ Compute a minimal length representative.
- ➯ Choose a random order on the edges.
- ➯ Look for a singular bigon ($O(\ell^2)$):
  - If there is one, exchange its two paths to remove intersections.
  - Repeat until there is no more singular bigon.

Theorem

This algorithm computes a minimal representative in $O(\ell^4)$ time.

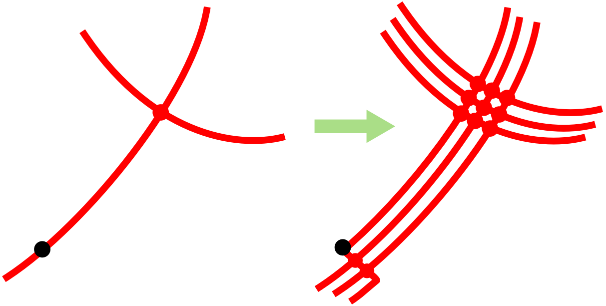
Non-Primitive Curves

Formula

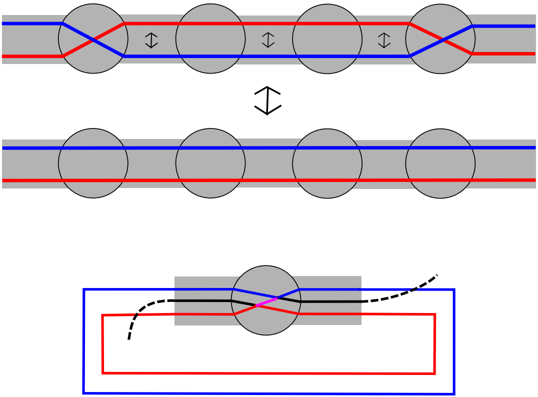
Let c be a curve, $p \in \mathbb{N}^*$. Then,

$$i(c^p) = p^2 \cdot i(c) + p - 1$$

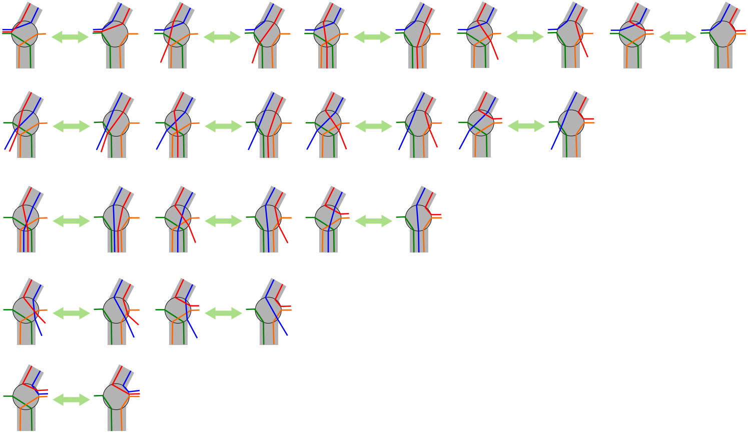
where $i(x)$ is the geometric intersection number of curve x.



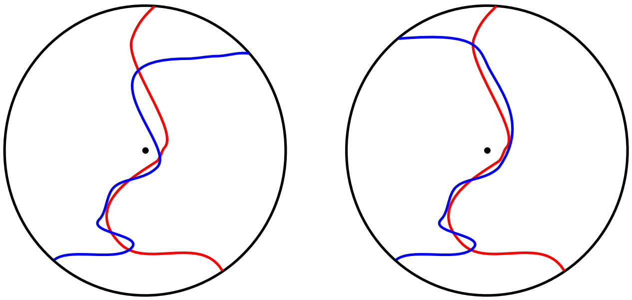
Bigon Removing



Bigon Removing



Maximal Bigons



Proposition

The the number of **crossing** maximal bigons is the same for all representative of a same homotopy class.

Main Result

Theorem

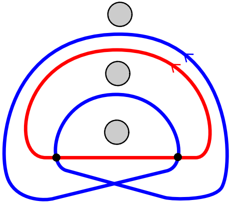
Given a curve c of length ℓ on a surface of complexity n, one can compute the geometric intersection number of c in $O(n+\ell^2)$ time.

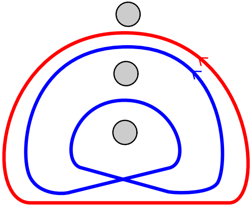
- We first compute a minimal length representative of c in $O(n+\ell)$ time.
- We compute the number of crossing maximal bigons of that representative.

Lemma

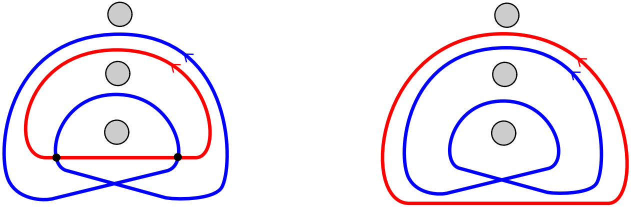
The maximal bigons form a partition of the pairs of edges of a given curve. So we can compute the number of **crossing** maximal bigons in $O(\ell^2)$ time.

System of Curves



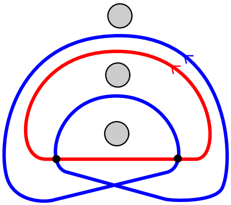


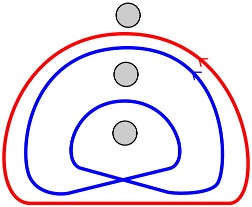
System of Curves



➯ Hass and Scott for singular bigons does not hold.

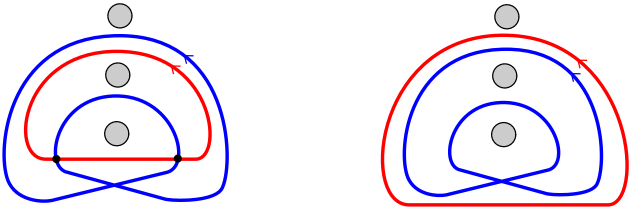
System of Curves





- Hass and Scott for singular bigons does not hold.
- The computation of the minimal representative for a single curve cannot be extended to two curves.

System of Curves



- Hass and Scott for singular bigons does not hold.
- The computation of the minimal representative for a single curve cannot be extended to two curves.
- However, the number of crossing maximal bigons is still an invariant.

Summary

k = number of curves
b = number of boundaries

Before	Number	Representative
$k = 1, b > 0$	$O((g\ell)^2)$ CL (1987)	$O((g\ell)^4)$ A (2009)
$k = 2, b > 0$	$O((g\ell)^2)$ CL (1987)	? dGS (1997)
$k = 1, b = 0$	? L (1987)	? dGS (1997)
$k = 2, b = 0$	? L (1987)	? dGS (1997)

Now	Number	Representative
$k = 1$	$O(\ell^2)$	$O(\ell^4)$
$k = 2$	$O(\ell^2)$	?

Thank you