Leader-based Multi-agent Consensus: Robust Control Design and Stability Analysis

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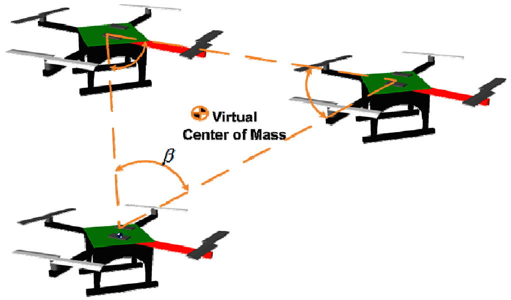

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- Uncertain Multi-agent systems
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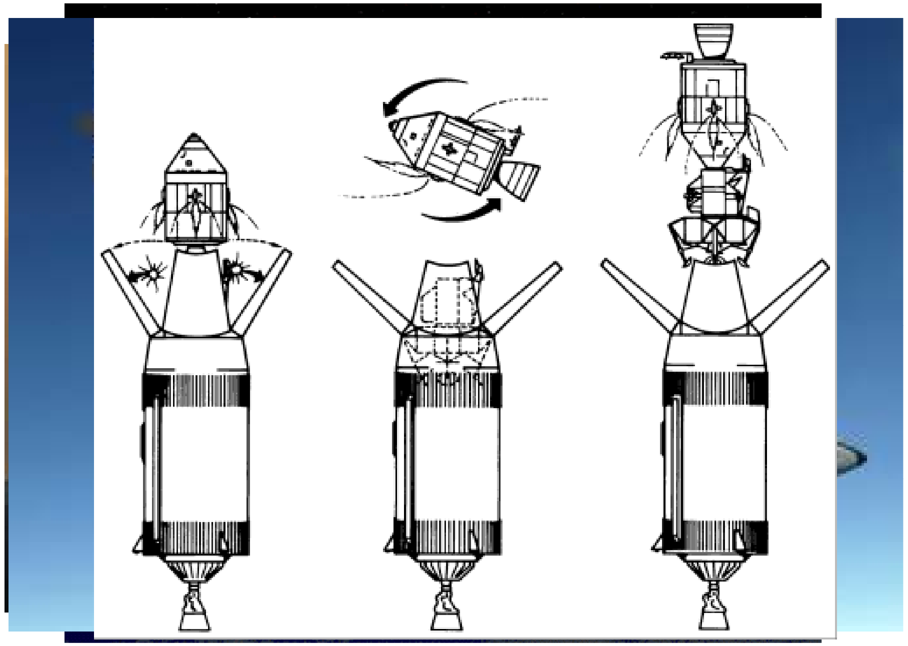
Multi-agent Consensus Applications

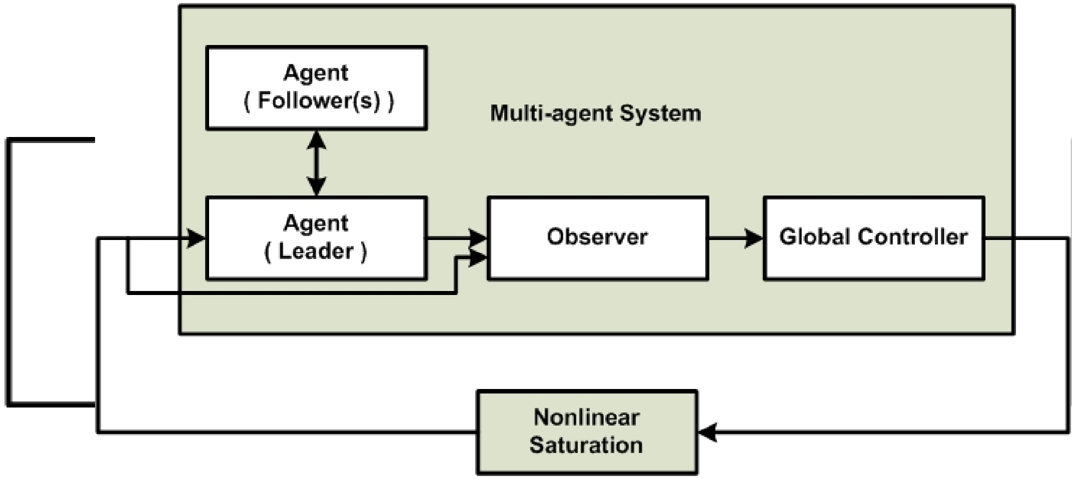
- Multi-agent Consensus has many applications

- **Civil:**
  - Reinforce border security and surveillance.
  - Heavy weight transportation.
  - Ensure the best coverage with minimal resources.
- **Military:**
  - Terrestrial Planet Finder (NASA).
  - Aerial Refueling.
  - Spacecraft docking maneuver.



Multi-agent scenario

- Common scenario for multi-agent consensus.
  - Information flow topology based on nearest neighbor rules.
  - Perfect communication.
- Basic leader-based multi-agent consensus scenario
  - Single or multiple leader
  - Information flow topology
  - Packet loss
  - Time delays
- Leader-based multi-agent scenario
  - Forced consensus



Kinematic Model

- **Let us consider a group of N agents with single integrator dynamics**

$$\dot{x}_i(t) = \overline{u}_i(t) \qquad \forall i = 1,.., N$$

- **Multi-agent consensus algorithm**

$$\overline{u}_i(t) = -\sum_{j \in N_i} (x_i(t) - x_j(t))$$

- **Forced consensus algorithm**

$$\overline{u}_i(t) = -\sum_{j \in N_i} (x_i(t) - x_j(t)) + \mathrm{b_i u_i(t)}$$

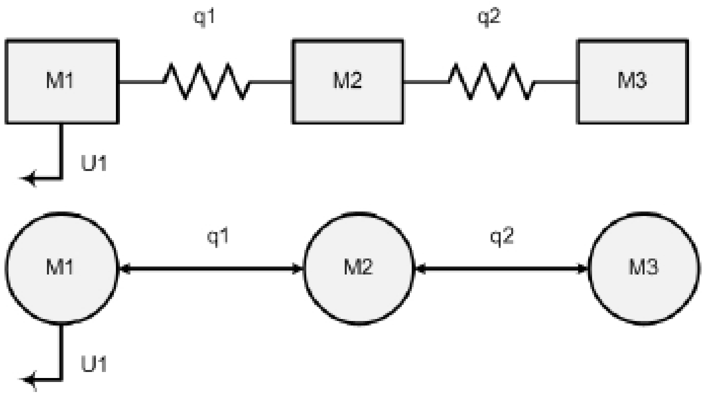
$$\begin{aligned} b_i &= 0 \quad \text{the } i^{th}\text{-agent is follower} \\ b_i &= 1 \quad \text{the } i^{th}\text{-agent is leader} \end{aligned}$$

- **Forced consensus system**

$$\begin{aligned} \dot{\mathbf{x}}(t) &= -L\mathbf{x}(t) + \mathbf{Bu}(\mathrm{t}) \\ \mathbf{y}(\mathrm{t}) &= \mathbf{C}^{\mathrm{T}}\mathbf{x}(\mathrm{t}) \end{aligned} \qquad (4)$$

Origin of Uncertainty in M-A Systems

- Consider the following mechanical system



$$\dot{\mathbf{x}}(t) = -L(\mathbf{q})\mathbf{x}(t) + \mathbf{B}\mathbf{u}(\mathrm{t})$$
$$\mathbf{y}(\mathrm{t}) = \mathbf{C}^{\mathrm{T}}\mathbf{x}(\mathrm{t})$$

$$\mathbf{Q}(t) = \{\mathbf{q} = [q_0 \quad \cdots \quad q_{n-1}] : q_i^- \le q_i \le q_i^+\}$$

- Similarity transformation

$$\dot{\eta}_i(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -q_0 & -q_1 & -q_2 & -q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{u(t)}$$

$$\mathrm{y(t)} = \mathrm{C}^{\mathrm{T}}\eta(\mathrm{t})$$

- System characteristic equation
- Transfer function
- Eigenvalues
- Eigenvectors

Controllability and Observability

Proposition 1: Consider the multiple agent system whose evolution is described by (4). This system is not observable if there exist a right eigenvector ω_i of $\mathcal{L}$ such that $\mathbf{C}^T\omega_i = 0$.

Proposition 2: Consider the multiple agent system whose evolution is described by (4). This system is uncontrollable if there exist an eigenvector $\mathbf{v}_i$ of $\mathcal{L}^T$ such that $\mathbf{v}_i^T\mathbf{B} = 0$.

Lemma 1: The center of mass of the multi-agent system (4) corresponds to controllable and observable modes.

R. Lozano, Mark W. Spong, J.A. Guerrero, N. Chopra, « Controllability and Observability of Leader-based Multi-Agent Systems », in *IEEE Conference on Decision and Control (CDC 2008)*, Cancun, México, 2008.

Observer Design

- Due to the nature of information flow between agents, full state is in general not available.
- In order to obtain the full state we propose a simple full order observer of the form

$$\dot{x} = Ax + Bu(\hat{x})$$

$$\dot{\hat{x}} = \overline{L}Cx + (A - \overline{L}C)\hat{x} - u(\hat{x})$$

$$y = Cx$$

Forced Robust Consensus Design

- Assuming controllabiliy and observability from the input and output of the leader.

$$\Sigma_{un} \triangleq \begin{cases} \dot{\eta}(t) = \mathbf{A}(\mathbf{q}^-)\eta(t) + \mathbf{Bu}(t) + \mathbf{B}\Gamma(r)\eta(t) \\ \mathbf{y} = \mathbf{C}\eta(t) \end{cases}$$

$$\Sigma_{nom} \triangleq \begin{cases} \dot{\eta}(t) = \mathbf{A}(\mathbf{q}^-)\eta(t) + \mathbf{Bu}(t) \\ \mathbf{y}(t) = \mathbf{C}\eta(t) \end{cases}$$

- With control $u = -B^T S \eta(t)$
- Optimal Control Approach

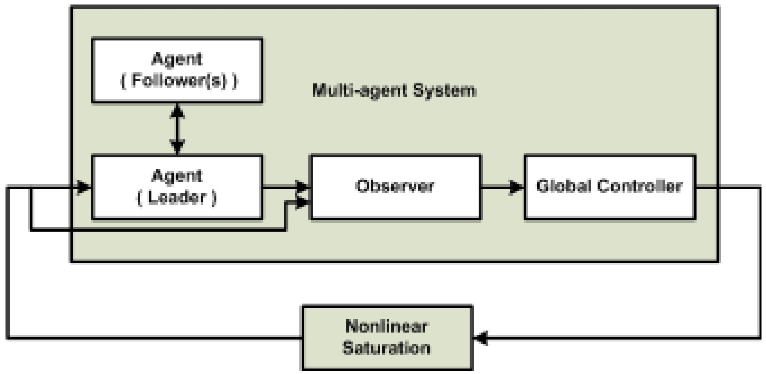
$$V(\eta) = \min_{u \in \mathbb{R}} \int_0^\infty (\eta^T(t)\mathbf{F}\eta(t) + \eta^{\mathbf{T}}(t)\eta(t) + \mathbf{u}^T(t)\mathbf{u}(t))dt$$

J.A. Guerrero, G. Romero, R. Lozano, "Robust Consensus Tracking of Leader-Based Multi-Agent System ", *in IEEE American Control Conference (ACC 2010)*, Baltimore, 2010.

Robust Stability Analysis

- **Consider the multi-agent system**

$$\begin{aligned} \dot{\mathbf{x}}(t) &= -L(\mathbf{q})\mathbf{x}(t) + \mathbf{B}(\mathbf{q})\mathbf{u}(\mathrm{t}) \\ \mathbf{y}(\mathrm{t}) &= \mathbf{C}^{\mathrm{T}}(\mathbf{q})\mathbf{x}(\mathrm{t}) \\ \mathbf{u}(\mathrm{t}) &= -\psi(\mathrm{t}, \mathrm{y}) \end{aligned} \qquad (11)$$



- **Transfer Function**

$$G(s,\mathbf{q}) = \mathbf{C}^{\mathrm{T}}(\mathbf{q})(sI - (-L(\mathbf{q})))^{-1}\mathbf{B}(\mathbf{q})$$

$$G(s,\mathrm{q}) = \frac{n(s,q)}{d(s,q)} = \frac{\sum_{i=0}^{m} a_i(\mathrm{q})s^i}{\sum_{i=0}^{n} b_i(\mathrm{q})s^i} \qquad \forall\ \mathrm{q} \in Q$$

Lemma 1: Consider the system (11), where $\psi(t,y)$ satisfies the sector $[0,k]$ condition. Then, the system (11) is robustly absolutely stable if $z(s,\mathrm{q}) = 1 + kG(s,\mathrm{q})$ is SPR for all $\mathrm{q} \in Q$.

Robust Stability Analysis

Theorem 1: A polynomic plant is robustly SPR if and only if $G(s,\mathbf{q})$ is stable for some $\mathbf{q} \in Q$ and:

1) $h(\omega,\mathbf{q})$ is positive for all $\mathbf{q} \in Q$ and $\omega \in (0,\infty)$.
2) $g(\omega,\mathbf{q})$ is positive for all $\mathbf{q} \in Q$ and $\omega \in (0,\infty)$.

where

$$h(\omega,\mathbf{q}) = |d(j\omega,\mathbf{q})|^2 \quad (13)$$

$$g(\omega,\mathbf{q}) = n(j\omega,\mathbf{q})d(-j\omega,\mathbf{q}) + n(-j\omega,\mathbf{q})d(j\omega,\mathbf{q}) \quad (14)$$

Sign Decomposition

- ***Definition:*** *Let f : $R^n \to R$ be a continuous function and $Q \subset P \subset Rn$ a convex subset, $f(\cdot)$ has sign decomposition in Q if there exist two bounded non-growing functions $f_n(\cdot) \geq 0$, $f_p(\cdot) \geq 0$, that $f(q)= f_p(q) - f_n(q)$ for all $q \in Q$.*

- ***Theorem [Elizondo C.]:*** *Let f : $Rn \to R$ be a continuous function with sign decomposition in Q such $Q \subset P \subset R^n$ is a box with minimum and maximum Euclidian vertices vmin, vmax then f(q) is lower and upper bounded by $f_p(vmin) - f_n(vmax)$ and $f_p(vmqx) - f_n(vmin)$ respectively.*

Sign Decomposition

- ***Theorem [Elizondo C.]:*** *Let f : $R^n \rightarrow R$ be a continuous function with sign decomposition in Q such $Q \subset P \subset R^n$ is a box with minimum and maximum Euclidian vertices vmin, vmax then the function f(q)>0 in Q if and only if there exist some Λ_j box sets, such that*
- $Q = \bigcup_j \Lambda^j$ *and* $\bar{f} = f_p(v_{\min}) - f_n(v_{\max})$
- *are greater than zero for each one Λ_j*

Sign Decomposition

- Minimized functions:

$$\begin{aligned} h_{\min}(\omega) &= h_p(\omega, q^-) - h_n(\omega, q^+) \\ g_{\min}(\omega) &= g_p(\omega, q^-) - g_n(\omega, q^+) \end{aligned}$$

- The next conditions are satisfied:

$$\begin{aligned} h_{\min}(\omega) &\leq h(\omega, q) \\ g_{\min}(\omega) &\leq g(\omega, q) \end{aligned}$$

$$\forall\ \omega \in (0, \infty);\ q \in Q$$

- Definition of new sets:

$$\mathcal{V}_h = [\omega_h^-, \omega_h^+] \times Q;\ \mathcal{V}_g = [\omega_g^-, \omega_g^+] \times Q$$

Robust Stability Analysis

Proposition 3: The fictitious transfer function $z(s,q) = 1 + kG(s,q)$ is robustly SPR if

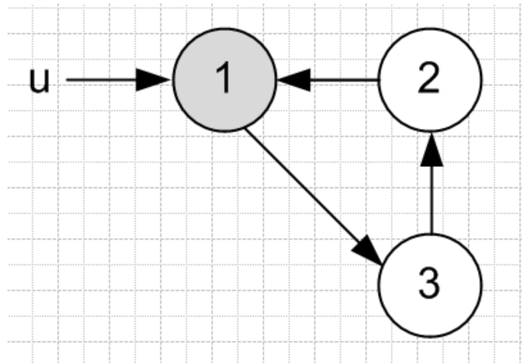
1) For all $\Lambda_h^j \in \mathcal{V}_h$, $\bar{h}_j = h_p(v_{\min}^j) - h_n(v_{\max}^j)$ is greater than zero for each $\bar{h}_j$.
2) For all $\Lambda_g^j \in \mathcal{V}_g$, $\bar{g}_j = g_p(v_{\min}^j) - g_n(v_{\max}^j)$ is greater than zero for each $\bar{g}_j$.

Corollary 1: Consider the multi-agent system shown in Figure 3, where $\psi(t,y)$ satisfies the sector $[0,k]$ condition. Then, the multi-agent system is robustly absolutely stable if $z(s,\mathbf{q}) = 1 + kG(s,\mathbf{q})$ is robustly SPR for all $\mathbf{q} \in Q$.

Example 1: Cyclic Topology

- **Consider a 3-agent system with cyclic topology**

$$G(s,q) = \frac{s^2 + s + 1}{s^3 + q_1 s^2 + q_2 s + 1} \qquad q_1 = [3.2168, 3.8168] \quad q_2 = [3.8591, 4.4591]$$



- **The ficticious TF**

$$z(s,q) = \frac{s^3 + (q_1 + 1)s^2 + (q_2 + 1)s + 2}{s^3 + q_1 s^2 + q_2 s + 1}$$

- **Positive and neagive parts**

$$h_p(\omega,q) = \omega^3 + q_1\omega^2 + q_2\omega + 1$$

$$h_n(\omega,q) = 2q_2^2\omega^2 + 2q_1^2\omega$$

$$g_p(\omega,q) = 2\omega^3 + (2q_1^2 + 2q_1)\omega^2 + (2q_2^2 + 2q_2)\omega + 4$$

$$g_n(\omega,q) = (4q_2 + 2)\omega^2 + (6q_1 + 2)\omega$$

- **Minimized functions**

$$h_{\min}(\omega) = \omega^3 - 29.4176\ \omega^2 - 14.2433\ \omega + 1$$

$$g_{\min}(\omega) = 2\omega^3 + 7.2932\ \omega^2 + 12.5873\ \omega + 1$$

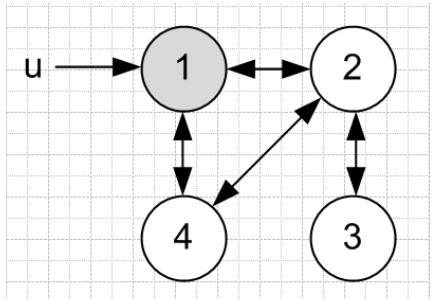
- **Using sign decomposition method:** $h(\omega,q) > 102.2151$

$$h(\omega,q) > 102.2151$$

Example 2: Balanced Graph Topology

- **Consider a 4-agent system with balanced topology**

$$G(s,q)=\frac{s^3+s^2+s+1}{s^4+q_1s^3+q_2s^2+q_3s+0.31262}$$



$q_1=[7.2489\ ,8.8489\ ]$

$q_2=[17.3538\ ,21.1538\ ]$

$q_3=[11.3221\ ,13.7221\ ]$

- **The ficticious TF**

$$z(s,q)=\frac{s^4+(q_1+1)s^3+(q_2+1)s^2+(q_3s+1)+0.3162}{s^4+q_1s^3+q_2s^2+q_3s+1}$$

- **Positive and neagive parts**

$$h_p(\omega,q)=\omega^4+(2q_1q_3+q_2^2+0.62)\omega^2+0.0961$$

$$h_n(\omega,q)=(2q_2+2q_1^2)\omega^3+(q_3^2+0.62q_2)\omega$$

$$g_p(\omega,q)=2\omega^4+(2q_1^2+2q_1)\omega^3+(2q_2^2+2q_2)\omega^2+(2q_3^2+2q_3)\omega+0.8122$$

$$g_n(\omega,q)=2(q_2+1)\omega^3+2(q_1+q_3+2q_1q_3)\omega^2+2(1.62q_2+0.31)\omega$$

- **Minimized functions**

$$h_{\min}(\omega)=\omega^4+465.9199\ \omega^3-120.6106\ \omega^2-201.4114\ \omega+0.091$$

$$g_{\min}(\omega)=2\omega^4+32.9757\ \omega^3+109.4124\ \omega^2+209.8658\ \omega+0.8122$$

- **Using sign decomposition method:**

$$h(\omega,q)>53019536$$

Conclusions and Future Work

- A method to verify the robust absolute stability property for multi-agent systems has been developed.
- The systems is transformed into a Lur’e system.
  - Transforms the original problem of robust stability into a verifying the positivity of a multivariable polinomial
- Sign decomposition method has been used to verify the positivity of a multivariable polinomial.
- The results were validated in simulations
- Independent uncertainty has been considered in examples, however, this method is also valid for systems with polynomic uncertainty structure.

Thank you for your attention

Questions?