Leader-based Multi-agent Consensus: Robust Control Design and Stability Analysis

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Abstract— **In this paper, the robust absolute stability property of multi-agent systems by transforming the original system into a Lur'e system is presented. Different configurations of multi-agent systems are considered, which they include parametric uncertainty. These systems are primarily stabilized by applying robust design techniques to further verify the absolute stability property of the closed loop system, taking into account the parametric uncertainty. To solve the problem of robust absolute stability, the strict positive realness property (SPR) of a fictitious transfer function needs to be verified. Sufficient conditions to verify the robust absolute stability property of leader-based multi-agent systems with parametric uncertainty using a sign decomposition technique that guarantees the positivity of multivariable polynomials are presented. Therefore, the aim of this work is to present a method to analyze the robust absolute stability of uncertain leader-based multi-agent systems.**

I. INTRODUCTION

Multi-agent consensus and coordination has attracted much attention in the recent years. Multi-agent consensus has important applications including swarming [1], [2], flocking [3]-[5], coordination and formation of aerial, ground or underwater vehicles [6], [7], [8], etc. In the last few years, different approaches for coordination of multiple autonomous robot systems have been developed such as Leader/Follower [9], [10], Virtual Structure [11], [12], and Behavioral Control [13], [14].

The distributed nature of a multi-agent (multi-robot) system implies the need of information sharing among agents. Generally, information flow among agents is modeled using graphs where every agent in the system is considered as a node in the graph. By using this technique several control strategies have been developed, *e.g.*, [3], [4] and [5] among others. A mechanical approach for multi-agent systems has been developed in [15] and [16]. This approach uses a passivity approach to decompose the system into two passive subsystems called "shape" and "lock". The shape subsystems maintains the formation of the group while the lock subsystem represents the translational dynamics of the group. Another interesting approach has been adopted in [17] where the multi-agent system is modeled as a bilateral teleoperation system. The authors provide results to achieve bilateral teleoperation one-master to multiple-slaves.

Recently, the attention has been focused on the dynamic interaction among agents, *i.e.* the information topology may change dynamically. For instance, communication links may be affected by disturbances, communication range limitations, data corruption, packet lost, etc. Another phenomena that has attracted the attention of many researchers is the noise present in the sensors. In [18] and [19], the authors analyze the consensus problem considering additive noise disturbances.

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The literature has shown that we can model multi-agent systems using a mechanical approach where every agent is represented by a mass and every communication link is modeled as a damper and/or spring interconnecting any two agents. In this case, we consider uncertainty in the damping and/or spring coefficient(s) as shown in Figure 1. On the other hand, we can assign a confidence index to the communication links depending on the reliability of the communication link as discussed in [20].

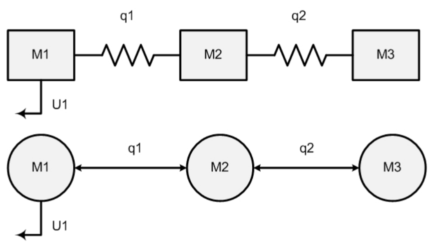

Fig. 1. Uncertainty in multi-agent systems

A common scenario of a leader-based multi-robot system includes a group of agents sharing information as shown in Figure 2. It can be observed that this scenario considers a

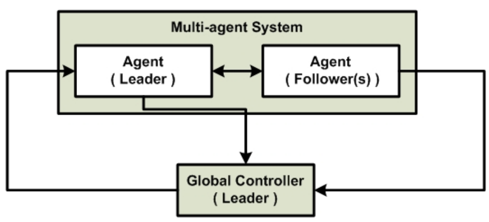

Fig. 2. Leader-based multi-agent system

group of robots acting as followers and at least one agent acting as the leader of the platoon. We notice that the leader is the only robot that can have access to external references. The state of the art in multi-robot system has shown that the authors consider either time delay, noise, packet drops or combinations of them. In the particular case of multiple mobile robot systems, it is quite often to find ourselves facing the problem that all actuator have physical limits. Therefore, the control action is saturated to avoid damages to the electronic and/or mechanical devices used for control. Then, we propose to analyze the robust absolute stability of the multi-robot system shown in Figure 3.

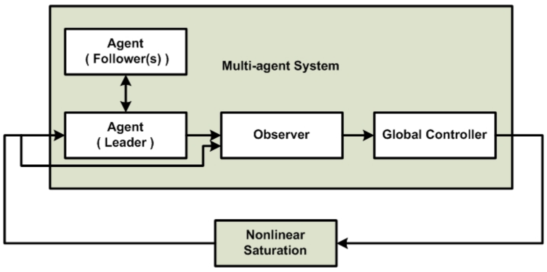

Fig. 3. Leader-based multi-agent system with parametric uncertainty

The proposed scheme presents the form of a Lur'e problem which consists in determining the asymptotic stability (or absolute stability) property of a nonlinear system that is a feedback connection of a linear part and a nonlinear element that belongs to a sector $[0, k]$, for more details see [21].

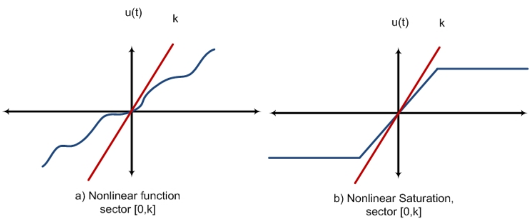

Fig. 4. Nonlinear function inside a sector $[0, k]$

We remark that there are many functions for which the nonlinearity do not belong to the sector $[0, k]$. However, we limit our study to nonlinear functions that belong to a sector $[0, k]$ as the one shown in Figure 4-b.

The present work addresses the robust absolute stability of a leader-based multi-robot system over a wireless network considering uncertainty in the communication links. First, a robust optimal control method to design a controller that robustly stabilize a forced consensus to a given reference for all possible uncertainty q is presented. Secondly, a robust strict positive realness (SPR) approach to verify that the resulting closed loop system is robustly absolutely stable is developed. Due to the distributed nature of multi-agent systems, a Luenberger observer for the multi-agent system is developed . The main contribution of this work is to provide a methodology to analyze the robust absolute stability of leader-based multi-agent systems considering uncertainties in the communication links and a saturated control action.

This paper is organized as follows: section II presents some preliminary definitions and results on graph theory, wireless networks and robust control design. In section III, the stability analysis for multi-agent systems with parametric polynomic uncertainty is developed. The robust polynomial positivity approach will be considered in order to verify the robust absolute stability of the system. In section IV, simulation results are presented. Finally, the conclusions are provided in section V.

II. UNCERTAIN MULTI-AGENT SYSTEMS

In order to model the interactions among robots(agents), a graph-based theoretical approach will be considered. A graph $\mathcal{G}$ is a pair $\mathcal{G}(\mathcal{N}, \mathcal{E})$ consisting of a set of nodes $\mathcal{N} = \{n_i \colon n_i \in \mathcal{N}, \forall i = 1, ..., n\}$ together with their inter-connections $\mathcal{E}$ on $\mathcal{N}$ [24]. Each pair (n_1, n_2) is called an edge $e \in \mathcal{E}$. An undirected graph is one where nodes i and j can get information from each other. In a directed graph or simply digraph, the i^{th} node can get information from the j^{th} node but not necessarily vice versa. Thus, information flow between robots can be modeled as a directed graph but also as an undirected graph. More complex graphs can be used to model multi-robot systems including switching graphs and weighted graphs. The interaction among robots in a multiple mobile robot system over a wireless network can be modeled using a switching graph while a multiple mobile robot system over a wireless network with uncertainty on the communication links can be modeled using weighted graphs.

Let us consider the kinematic model for a group of robots as follows

$$\dot{x}_i(t) = \bar{u}_i(t) \qquad \forall i = 1, .., n; \tag{1}$$

with multi-robot consensus achieved using the following algorithm

$$\bar{u}_i(t) = -\sum_{j \in \mathcal{N}_i} (x_i(t) - x_j(t)) \tag{2}$$

where $\bar{u}_i$ is the control input and $\mathcal{N}_i$ is the set of vehicles transmitting their information to the $i^{th}-$robot. We remark that the control law (2) ensures the consensus agreement in the sense of

$$\lim_{t \to \infty} |x_i(t) - x_j(t)| = 0$$

Assuming a leader-follower approach, we consider the forced consensus algorithm

$$\bar{u}_i(t) = -q_{ij} \sum_{j \in \mathcal{N}_i} (x_i(t) - x_j(t)) + b_i u_i(t) \tag{3}$$

where b_i is defined as follows

$$\begin{aligned} b_i &= 0 \quad \text{the } i^{th}\text{-agent is follower} \\ b_i &= 1 \quad \text{the } i^{th}\text{-agent is leader} \end{aligned}$$

Then, the multi-robot system (1) with forced consensus algorithm (3) can be written as

$$\dot{\mathbf{x}}(t) = -\mathcal{L}(q)\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{4}$$
$$\mathbf{y}(t) = \mathbf{C}^T\mathbf{x}(t)$$

where $\mathcal{L}(q)$ is the laplacian matrix of the information graph. We assume that the information graph includes a spanning tree with root at the leader. For the i^{th} row of $\mathcal{L}(q)$, the entries $l_{ij} = -q_{ij}$ for $i \neq j$ correspond to the confidence indexes (gain) multiplying the signals from other robots coming to the $i^{th}-$robot. For the i^{th} column of $\mathcal{L}(q)$, the entries $l_{ji} = -q_{ji}$ for $i \neq j$ correspond to the confidence indexes (gains) multiplying the signals going out of the $i^{th}-$robot towards the other robots.

A leader-follower approach for multi-robot systems (4) will be adopted, then a similarity transformation can be applied such that the multi-robot system can be represented as follows

$$\dot{\eta}(t) = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 1 \\ -q_0 & -q_1 & \dots & -q_{n-1} \end{bmatrix} \eta(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix} \mathbf{u}(t) \qquad (5)$$
$$\mathbf{y} = \mathbf{C}^T \eta(t)$$

where $\eta \in \mathbb{R}^n$, $b \in \mathbb{R}$ and $u \in \mathbb{R}$ are the state variables and control inputs respectively. It should be noticed that under a similarity transformation the system's characteristic equation, transfer function, eigenvalues, eigenvectors are all preserved.

Remark 1: It is worth to mention that the controllability and observability properties of the leader-based multi-robot system must hold in order to apply the similarity transformation. If the multi-robot system is controllable and observable from the input and output of the leader, then we can apply a similarity transformation to the original system to develop a robust controller for the transformed system. Once the controller has been obtained, an inverse transformation can be used to obtain the controller that robustly stabilizes the forced consensus.

The following results resume the controllability and observability conditions for leader-based multi-agent systems.

Proposition 1: Consider the multiple agent system whose evolution is described by (4). This system is not observable if there exist a right eigenvector ω_i of $\mathcal{L}$ such that $\mathbf{C}^T \omega_i = 0$.

Proposition 2: Consider the multiple agent system whose evolution is described by (4). This system is uncontrollable if there exist an eigenvector $\mathbf{v}_i$ of $\mathcal{L}^T$ such that $\mathbf{v}_i^T \mathbf{B} = 0$.

Remark 2: Recall that for $i = 1$ we have $v_1 = \omega_1$ and therefore $\mathbf{C}^T \omega_1 \neq 0$ and $\mathbf{v}_1^T \mathbf{B} \neq 0$. Thus the mode corresponding to $(\lambda_1, \mathbf{v}_1)$ is controllable and the mode corresponding to (λ_1, ω_1) is observable. If for $i = 2, .., n$ there exists a $\mathbf{C}^T \omega_i = 0$ or $\mathbf{v}_i^T \mathbf{B} = 0$, such mode is not observable or not controllable respectively. Nevertheless, such modes are asymptotically stables and converges to zero.

For more details on controllability and observability properties of leader-based multi-agent system, see [25].

A. Forced Consensus Control Design

Let us assume that the multi-robot system is controllable and observable from the input and output of the leader. Then, we will show that there is an optimal feedback that robustly stabilizes the forced coordination of multiple agent systems to a given reference. A combination of the Lyapunov method and the LQR approach are used to find the robust control law which guarantee the stability property and the system robustness. To do this, we consider the system (5) where $q_i \in \mathcal{Q}$ are the parametric uncertain values. $\mathcal{Q}$ is a set that represents the parametric uncertainty defined as

$$\mathcal{Q} = \left\{ \mathbf{q} = \begin{bmatrix} q_0 & \cdots & q_{n-1} \end{bmatrix} : q_i^- \leq q_i \leq q_i^+ \right\} \qquad (6)$$

Let us consider the following representation of the multi-agent system (5) as in [26], [27]

$$\Sigma_{nom} \triangleq \begin{cases} \dot{\eta}(t) = \mathbf{A}(\mathbf{q}^-)\eta(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\eta(t) \end{cases} \qquad (7)$$

$$\Sigma_{un} \triangleq \begin{cases} \dot{\eta}(t) = \mathbf{A}(\mathbf{q}^-)\eta(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{B}\Gamma(r)\eta(t) \\ \mathbf{y} = \mathbf{C}\eta(t) \end{cases} \qquad (8)$$

where:

$$\Gamma(r) = \begin{bmatrix} r_0 & \cdots & r_{n-1} \end{bmatrix}$$

$$0 \leq r_i \leq r_i^+ = \frac{q_i^+ - q_i^-}{b}$$

Σ_{nom} represents the nominal system and Σ_{un} represents the uncertain system. Consider the system (8), and the following control law

$$\mathbf{u}(t) = -\mathbf{B}^T \mathbf{S} \eta(t) \qquad (9)$$

where:

$$\mathbf{S}\mathbf{A}(\mathbf{q}^-) + \mathbf{A}(\mathbf{q}^-)^T \mathbf{S} + \mathbf{F} + \mathbf{I} - \mathbf{S}\mathbf{B}\mathbf{B}^T\mathbf{S} = 0, \mathbf{S} > 0 \qquad (10)$$

The $\mathbf{F}$ matrix is defined in such a way that the following condition is satisfied:

$$\Gamma(r)^T \Gamma(r) \leq \mathbf{F} \qquad \forall r_i \in [0, r_i^+]$$

To find the solution of the LQR optimal control problem for the Σ_{nom} system, the following cost functional is considered:

$$V(\eta) = \min_{u \in \mathbb{R}} \int_0^\infty (\eta^T(t)\mathbf{F}\eta(t) + \eta^{\mathbf{T}}(t)\eta(t) + \mathbf{u}^T(t)\mathbf{u}(t))dt$$

It is possible to verify that the proposed control law (9) corresponds to the solution of the LQR optimal control problem for the Σ_{nom} system (7), considering the cost functional $V(\eta)$, and the relative weights matrices $\mathbf{Q} = \mathbf{F} + \mathbf{I}$ and $R = 1$. Obviously, the above control law stabilizes the nominal system Σ_{nom}. A proof that the same control law also stabilizes the uncertain system Σ_{un} can be found in [27].

It is worth to mention that the robust forced consensus for the original multi-robot system can be obtained by applying a simple inverse transformation.

B. Observer design

It is worth to mention that a multi-agent system is a decentralized system where every agent obtains information from its neighbors. We remark that the global controller (forced consensus) is computed only at the leader and therefore, full state is needed in order to compute the coordination control (9). In order to obtain the full state from the input and output of the leader, we propose a Luenberger observer of the form:

$$\begin{aligned} \dot{x} &= Ax - Bu(\hat{x}) \\ \dot{\hat{x}} &= \bar{L}y + (A - \bar{L}C)\hat{x} - Bu(\hat{x}) \\ y &= Cx \end{aligned}$$

where x is the state vector, $\hat{x}$ is the observed state vector, $\bar{L}$ is the Luenberger gain vector.

III. ROBUST STABILITY ANALYSIS

Note that the robust controller has been developed without taking into account the saturation function in the input. Then, the robust absolute stability analysis is presented next.

A. Robust Absolute Stability

Let us consider the multi-agent system, shown in Figure 3, with the following state space representation

$$\begin{aligned} \dot{\mathbf{x}}(t) &= -\mathcal{L}(\mathbf{q})x(t) + \mathbf{B}(\mathbf{q})\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}^T(\mathbf{q})\mathbf{x}(t) \\ \mathbf{u} &= -\psi(t, y) \end{aligned} \tag{11}$$

with uncertainty in the communication links represented by $\mathbf{q}$, and $\psi(t, y)$ is a memoryless nonlinear function contained in a region called sector $[0, k]$. Remark that the system (11) can also be represented by a transfer function $G(s, \mathbf{q}) = \mathbf{C}(\mathbf{q})(s\mathbf{I} - (-\mathcal{L}(\mathbf{q})))^{-1}\mathbf{B}(\mathbf{q})$ defined as

$$G(s, \mathbf{q}) = \frac{n(s, q)}{d(s, q)} = \frac{\sum_{i=0}^{m} a_i(\mathbf{q})s^i}{\sum_{i=0}^{n} b_i(\mathbf{q})s^i} \qquad \forall\ \mathbf{q} \in \mathcal{Q} \tag{12}$$

where $a_i(\mathbf{q})$ and $b_i(\mathbf{q})$ are polynomic functions of $\mathbf{q}$.

In order to verify the robust absolute stability of the system (11) with state feedback control (9) we introduce the following Lemma [23]:

Lemma 1: Consider the system (11), where $\psi(t, y)$ satisfies the sector $[0, k]$ condition. Then, the system (11) is robustly absolutely stable if $z(s, \mathbf{q}) = 1 + kG(s, \mathbf{q})$ is SPR for all $\mathbf{q} \in \mathcal{Q}$.

It is important to note that this condition implies verifying the robust SPR condition of the fictitious transfer function $z(s, \mathbf{q})$. Hence, the robust absolute stability problem is transformed to determine if $z(s, \mathbf{q})$ is robustly SPR. Additionally, when the relative degree of the polynomic plant $G(s, \mathbf{q})$ is known to be equal to 1, we assume that the $z(s, \mathbf{q})$ is also a polynomic plant with relative degree equal to zero.

It is clear that the polynomic plant defined in (12) represents an infinite number of transfer functions; therefore, the problem of robust strict positive realness is transformed into a problem where the positivity of two multivariable polynomic functions is verified by using the following result [23], [28]

Theorem 1: A polynomic plant is robustly SPR if and only if $G(s, \mathbf{q})$ is stable for some $\mathbf{q} \in \mathcal{Q}$ and:

1) $h(\omega, \mathbf{q})$ is positive for all $\mathbf{q} \in \mathcal{Q}$ and $\omega \in (0, \infty)$.
2) $g(\omega, \mathbf{q})$ is positive for all $\mathbf{q} \in \mathcal{Q}$ and $\omega \in (0, \infty)$.

where

$$h(\omega, \mathbf{q}) = |d(j\omega, \mathbf{q})|^2 \tag{13}$$

$$g(\omega, \mathbf{q}) = n(j\omega, \mathbf{q})d(-j\omega, \mathbf{q}) + n(-j\omega, \mathbf{q})d(j\omega, \mathbf{q}) \tag{14}$$

To verify the positivity of these functions, a sign decomposition method will be used. This method analyzes the positivity of a multivariable real polynomic function $f(\cdot)$ by its decomposition into its positive $f_p(\cdot)$ and negative $f_n(\cdot)$ parts, see [22]. Then, to verify if a function with sign decomposition is positive, it is needed to verify that $\bar{f} = f_p(\upsilon_{min}) - f_n(\upsilon_{max})$ is greater than zero, *i.e.* the function $f(\cdot)$ is positive if the difference between its positive part evaluated at its minimum euclidian value υ_{min} and its negative part evaluated at its maximum euclidian value υ_{max} is be greater than zero.

We remark that full knowledge of the low and high boundaries of the uncertainty is needed, and as it was seen in Theorem 1, ω is an unbounded parameter. Therefore, the limits for the bound of ω can be found by using the next function:

$$h_{min}(\omega) = h_p(\omega, \mathbf{q}^-) - h_n(\omega, \mathbf{q}^+) \tag{15}$$

$$g_{min}(\omega) = g_p(\omega, \mathbf{q}^-) - g_n(\omega, \mathbf{q}^+) \tag{16}$$

where $h_p(\cdot)$, $h_n(\cdot)$, $g_p(\cdot)$ and $g_n(\cdot)$ are the negative and positive parts of $h(\omega, \mathbf{q})$ and $g(\omega, \mathbf{q})$ respectively. $\mathbf{q}^- = [\ q_0^- \ \cdots \ q_{n-1}^-\ ]^T$ and $\mathbf{q}^+ = [\ q_0^+ \ \cdots \ q_{n-1}^+\ ]^T$.

From (15)-(16), it is clear that the the following condition is satisfied

$$h_{min}(\omega) \le h(\omega, \mathbf{q});\ \forall \omega \in (0, \infty); \mathbf{q} \in \mathcal{Q}$$
$$g_{min}(\omega) \le g(\omega, \mathbf{q});\ \forall \omega \in (0, \infty); \mathbf{q} \in \mathcal{Q}$$

Due to the shape of $h_{min}(\omega)$ and $g_{min}(\omega)$ it is possible to find the minimum ω^- and maximum ω^+ values so that the functions may take negative values and thus, the function have the possibility to be negative only when they are inside these intervals. Usually, the limits for the ω parameter in both functions correspond to some roots of $h_{min}(\omega)$ and $g_{min}(\omega)$ respectively, and can be gotten graphically, for more details see [23]. Now, it is possible to define the following sets

$$\mathcal{V}_h = [\omega_h^-, \omega_h^+] \times \mathcal{Q};\ \mathcal{V}_g = [\omega_g^-, \omega_g^+] \times \mathcal{Q} \tag{17}$$

which can be divided into j-subsets

$$\mathcal{V}_h = \bigcup_j \Lambda_h^j;\ \mathcal{V}_g = \bigcup_j \Lambda_g^j \tag{18}$$

Then, in order to verify the Lemma 1, we introduce the following result

Proposition 3: The fictitious transfer function $z(s, q) = 1 + kG(s, q)$ is robustly SPR if

1) For all $\Lambda_h^j \in \mathcal{V}_h$, $\bar{h}_j = h_p(v_{\min}^j) - h_n(v_{\max}^j)$ is greater than zero for each $\bar{h}_j$.

2) For all $\Lambda_g^j \in \mathcal{V}_g$, $\bar{g}_j = g_p(v_{\min}^j) - g_n(v_{\max}^j)$ is greater than zero for each $\bar{g}_j$.

Now, the result on robust absolute stability for leader-based multi-agent systems will be presented next

Corollary 1: Consider the multi-agent system shown in Figure 3, where $\psi(t, y)$ satisfies the sector $[0, k]$ condition. Then, the multi-agent system is robustly absolutely stable if $z(s, \mathbf{q}) = 1 + kG(s, \mathbf{q})$ is robustly SPR for all $\mathbf{q} \in \mathcal{Q}$.

IV. EXAMPLES

To illustrate the application of the previous results, three cases of a leader-based multi-agent system with a topology of information exchange that is controllable and observable from the input and output of the leader are presented, see Figure 5.

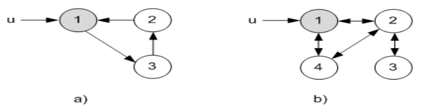

Fig. 5. Topologies of Information Exchange: a) cyclic, b) balanced.

A. Cyclic Topology

This multi-agent system is one of the most frequently used in coordination and consensus on multi-agent systems. Also, this scheme is one of the most easily implementable multi-robot system.

A 3-agents system with cyclic topology of information exchange is given by

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{19}$$

The multi-agent system in terms of a canonical form is given by

$$\dot{\eta}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -3 \end{bmatrix} \eta(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

where

$$\eta(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} x(t) \tag{20}$$

Now, we will consider a $[10\%, 10\%, 10\%]$ uncertainty in the last row coefficients of the state matrix. As a result of this assumption, the following matrix is obtained

$$A(q) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -q_0 & -q_1 & -q_2 \end{bmatrix} \tag{21}$$

where $q_0 \in [0, 0]$, $q_1 \in [-3.3, -2.7]$ and $q_2 \in [-3.3, -2.7]$.

Despite the fact that the structure of the uncertainty in (21) may be multilineal, polynomial, etc., it is always possible to lump the uncertainty such that uncertainty structure in (21) becomes independent interval uncertainty. Then,

$$A(q^-) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3.3 & -3.3 \end{bmatrix}$$

$$\Gamma^T(r) = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$$

where $r_1 \in [0, 0]$; $r_2 \in [0, 0.6]$; $r_3 \in [0, 0.6]$.

The F matrix is given by:

$$F = \begin{bmatrix} 0.00 & 0.00 & 0.00 \\ 0.00 & 0.36 & 0.36 \\ 0.00 & 0.36 & 0.36 \end{bmatrix}$$

Then, the optimal control law is given by

$$K = \begin{bmatrix} 1.0000 & 1.1519 & 0.5168 \end{bmatrix} \tag{22}$$

After a simple inverse transformation of the optimal control law (22), the resulting optimal control law is applied to the multi-agent system (19).

Now, let us consider that the input is saturated and the saturation function satisfies the sector condition. We will consider the case of robust absolute stability of the multi-agent system as follows. First, let us consider the transfer function of the above system

$$G(s, q) = \frac{s^2 + s + 1}{s^3 + q_1 s^2 + q_2 s + 1} \tag{23}$$

where $q_1 \in [3.2168, 3.8168]$ and $q_2 \in [3.8591, 4.4591]$

The fictitious transfer function has the following form:

$$z(s, q) = \frac{s^3 + (q_1 + 1)s^2 + (q_2 + 1)s + 2}{s^3 + q_1 s^2 + q_2 s + 1} \tag{24}$$

where the positive and negative parts of $h(\omega, q)$ and $g(\omega, q)$ functions are the following

$$h_p(\omega, q) = \omega^3 + q_1^2\omega^2 + q_2^2\omega + 1$$
$$h_n(\omega, q) = 2q_2^2\omega^2 + 2q_1^2\omega$$
$$g_p(\omega, q) = 2\omega^3 + (2q_1^2 + 2q_1)\omega^2 + (2q_2^2 + 2q_2)\omega + 4$$
$$g_n(\omega, q) = (4q_2 + 2)\omega^2 + (6q_1 + 2)\omega$$

Using the equations defined in (15)-(16) we get:

$$h_{\min}(\omega) = \omega^3 - 29.4176\omega^2 - 14.2433\omega + 1 \tag{25}$$
$$g_{\min}(\omega) = 2\omega^3 + 7.2932\omega^2 + 12.5873\omega + 1 \tag{26}$$

On one hand, it is clear that the $g_{min}(\omega)$ function is positive, because all of its coefficients are positive and the frequency ω belongs to the interval $(0, \infty)$. In the other hand from $h_{min}(\omega)$ function it is possible to get the interval $[0.062, 29.8929]$. Then, the set $\mathcal{V}$ is defined as

$$\mathcal{V} = [0.062, 29.8929] \times [3.2168, 3.8168] \times [3.8591, 4.4591]$$

Using the sign decomposition method we get

$$h(\omega, q) > 102.2151 \tag{27}$$

From (26) and (27) and applying the Proposition 3 and Corollary 1, we can conclude that the 3-agent system with cyclic topology of information exchange is robustly absolutely stable.

B. Balanced Graph Topology

A 4-agents system with balanced topology of information exchange is given by

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \quad (28)$$

The multi-agent system in terms of the canonical form is given by

$$\dot{\eta}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -12 & -19 & -8 \end{bmatrix} \eta(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

with the following transformation

$$\eta(t) = \begin{bmatrix} 3 & 5 & 1 & 0 \\ 3 & 4 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 9 & 6 & 1 \end{bmatrix} x(t) \quad (29)$$

Considering a $[10\%, 10\%, 10\%, 10\%]$ uncertainty in the last row coefficients of the state matrix. As a result of this assumption, the following matrix is obtained

$$A(q) \quad = \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -q_0 & -q_1 & -q_2 & -q_3 \end{bmatrix} \quad (30)$$

where $q_0 \in [0, 0]$, $q_1 \in [10.8, 13.2]$, $q_2 \in [17.1, 20.9]$ and $q_3 \in [7.2, 8.8]$.

Thus, the optimal control law is given by

$$K = \begin{bmatrix} -0.1445 & 0.1843 & -0.0963 - 0.0489 \end{bmatrix} \quad (31)$$

After a simple inverse transformation of the optimal control law (31), the resulting optimal control law is applied to the multi-agent system (28).

Now, let us consider that the input is saturated and the saturation function satisfies the sector condition. We will consider the case of robust absolute stability of the multi-agent system as follows. The transfer function of the above system is as follows

$$G(s, q) = \frac{s^3 + s^2 + s + 1}{s^4 + q_1 s^3 + q_2 s^2 + q_3 s + 0.3162} \quad (32)$$

where $q_1 = [7.2489, 8.8489]$, $q_2 = [17.3538, 21.1538]$ and $q_3 = [11.3221, 13.7221]$

The fictitious transfer function has the following form:

$$z(s, q) = \frac{s^4 + (q_1 + 1)s^3 + (q_2 + 1)s^2 + (q_3 + 1)s + 0.3162}{s^4 + q_1 s^3 + q_2 s^2 + q_3 s + 0.3162} \quad (33)$$

where the positive and negative parts of $h(\omega, q)$ and $g(\omega, q)$ functions are the following

$$\begin{aligned} h_p(\omega, q) =& \omega^4 + (2q_1 q_3 + q_2^2 + 0.62)\omega^2 + 0.0961 \\ h_n(\omega, q) =& (2q_2 + q_1^2)\omega^3 + (q_3^2 + 0.62q_2)\omega \\ g_p(\omega, q) =& 2\omega^4 + 2(q_1^2 + q_1)\omega^3 + 2(q_2^2 + q_2)\omega^2 \\ &+ 2(q_3^2 + q_3)\omega + 0.8122 \\ g_n(\omega, q) =& 2(q_2 + 1)\omega^3 + 2(q_1 + q_3 + 2q_1 q_3)\omega^2 \\ &+ 2(1.62q_2 + 0.31)\omega \end{aligned}$$

Using the equations defined in (15)-(16) we get:

$$h_{\min}(\omega) = \omega^4 + 465.9199\omega^2 - 120.6106\omega^2 \quad (34)$$
$$- 201.4114\omega + 0.091 \quad (35)$$
$$g_{\min}(\omega) = 2\omega^4 + 32.9757\omega^3 + 109.4124\omega^2 \quad (36)$$
$$+ 209.8658\omega + 0.8122 \quad (37)$$

In the one hand, it is clear that the $g_{min}(\omega)$ function is positive, because all of its coefficients are positive and the frequency ω belongs to the interval $(0, \infty)$. In the other hand from $h_{min}(\omega)$ function it is possible to get the interval $[0.0005, 0.7985]$. Then, the set $\mathcal{V}$ is defined as

$$\begin{aligned} \mathcal{V} =& [0.0005, 0.7985] \times [7.2489, 8.8489] \\ &\times [17.3538, 21.1538] \times [11.3221, 13.7221] \end{aligned} \quad (38)$$

Using the sign decomposition method we get

$$h(\omega, q) > 53019536.0 \quad (39)$$

From (37) and (39) and applying the Proposition 3 and Corollary 1, we can conclude that the 4-agent system with balanced topology of information exchange is robustly absolutely stable.

V. CONCLUSIONS

We presented a method to verify the robust absolute stability property for multi-agent systems. This method consisted in transforming the original system into a Lur'e system. Then, transforming the original problem of robust stability into one where the positivity of a multivariable polynomial using the sign decomposition technique is verified. The theoretic results were illustrated with different topologies of information exchange for multi-agent systems. It is worth to mention that, although in this paper, the examples were considering independent uncertainty, the method is also valid for systems with polynomic uncertainty structure.

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