Characterization of Parallel Manipulator Available Wrench Set Facets

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Abstract. The available wrench set is the set of wrenches that can be generated at the platform of a parallel manipulator. In a given configuration, this set is known to be a convex polytope and, recently, it has been pointed out that it is in fact a particular type of convex polytope called a zonotope. This paper deals with the case of parallel manipulators having as many or more actuators than degrees of freedom and discusses a characterization of the facets of the available wrench set.

Key words: Parallel manipulators, actuation redundancy, available wrench set.

1 Introduction

The determination of the available wrench set is a useful tool for parallel manipulator analysis and design. Notably, in the case of parallel cable-driven manipulators, the following relevant use of the available wrench set is proposed in [1]. Being given a wrench set T that the cable-driven manipulator is required to generate in order to achieve the tasks assigned to it, determine the available wrench set A and test if $T \subseteq A$, in which case T is feasible. This methodology proposed for cable-driven manipulator can obviously be applied to other types of parallel manipulator.

The available wrench set appears in numerous previous works and is known to be a convex polytope [2], [3]. In the case of serial manipulators, a set with a similar geometry is, for example, studied in [4] where it is called manipulability polytope. Recently, Bouchard et al [5] pointed out that the available wrench set is a particular type of convex polytope called a *zonotope*. Based on specific properties of zonotopes, a simple method referred to as the hyperplane shifting method is introduced in [5]. This method provides a representation of the available wrench set as the set of solutions to a finite system of linear inequalities. By means of such a representation, it is usually straightforward to test whether or not a given required wrench set T is fully included in the available wrench set A.

The contribution of this paper is to provide a proof (Section 4) which justifies the hyperplane shifting method. This proof is mainly based on a characterization of zonotope facets which appears in [6] but without details. This lack of details explains probably why this characterization of the facets of a zonotope seems to have been overlooked in [5]. Moreover, the proof provided in the present paper leads directly to an improved version of the hyperplane shifting method (Section 5). The contribution of the paper is limited but it is nevertheless hoped to be useful in pointing out a clear characterization of the available wrench set facets, thereby complementing the work presented in [5].

2 Available Wrench Set Definition

Let us consider an n-degree-of-freedom parallel manipulator having m actuators ($m \geq n$). In a given configuration, the vector of actuator forces/torques τ is usually mapped to the mobile platform wrenches $\mathbf{f}$ (combination of a force and a moment) according to the following linear relationship [3]

$$\mathbf{W}\tau = \mathbf{f} \tag{1}$$

where $\mathbf{W}$ is an $n \times m$ matrix called the wrench matrix in this paper. Its ith column is denoted $\mathbf{w}_i$. Note that, in the remainder of the paper, the space of mobile platform wrenches is considered to be an affine space. Hence, it may be judged necessary to modify Eq. (1) so as to avoid physical inconsistencies, *i.e.*, in order to avoid adding variables with different physical units in the case of parallel manipulators with mixed translational and rotational degrees of freedom.

The limited force/torque capabilities of the actuators imply that each component τ_i of vector τ is to lie within an interval $[\tau_{i_{min}}, \tau_{i_{max}}]$ where $\tau_{i_{min}}$ and $\tau_{i_{max}}$ are the minimum and maximum values of actuator i force/torque, respectively. Note that usually $\tau_{i_{min}} = -\tau_{i_{max}}$ but a more general case is considered here in order to include parallel cable-driven manipulators for which $0 \leq \tau_{i_{min}} < \tau_{i_{max}}$ (a nonnegativity constraint due to the fact that cables can only pull and not push [1]). Let us also define the box $[\tau]$ (hypercube) of admissible actuator forces/torques as

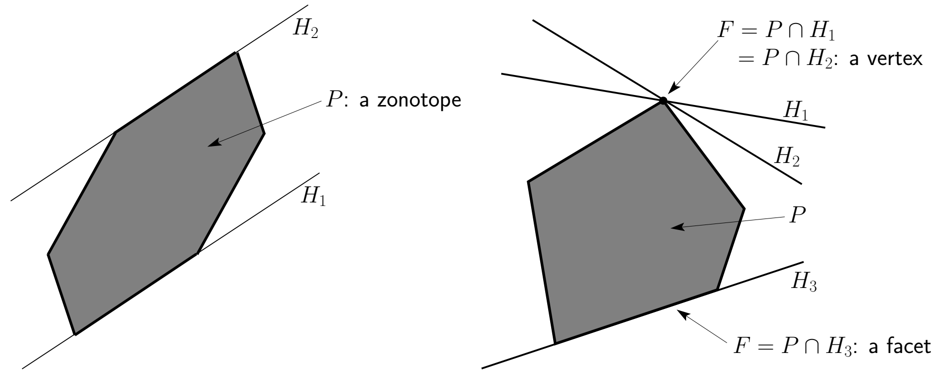
$$[\tau] = \{\tau \mid \tau_i \in [\tau_{i_{min}}, \tau_{i_{max}}], \ \forall\, i,\ 1 \leq i \leq m\}\,. \tag{2}$$

The present work deals with the set of wrenches A defined as

$$A = \{\mathbf{f} \mid \mathbf{f} = \mathbf{W}\tau,\ \tau \in [\tau]\} \tag{3}$$

which is, for a given configuration of a parallel manipulator, the set of platform wrenches that can be generated by the actuators with each τ_i in its admissible range $[\tau_{i_{min}}, \tau_{i_{max}}]$. Following [1] and [5], A is called the *available wrench set*.

This set is known to be a *convex polytope*. In fact, as pointed out in [5], since A is the image of the box $[\tau]$ under the linear map given by matrix $\mathbf{W}$, A is affinely isomorphic to a particular type of polytope called a *zonotope* [7]. In Figure 1(a), a two-dimensional zonotope is shown. It is the image of a three-dimensional box.



(a) A two-dimensional zonotope P and two parallel facet-defining hyperplanes H_1 and H_1 of P.

(b) A convex polygon P and three supporting hyperplanes H_i (lines). Both H_1 and H_2 define a vertex of P whereas H_3 is a facet-defining hyperplane (a facet of a convex polygon being an edge).

Fig. 1 A two-dimensional zonotope and a two-dimensional convex polytope (polygon).

3 Faces and Representation of a Convex Polytope

Let P be an n-dimensional convex polytope. An inequality $\mathbf{c}^T\mathbf{x} \leq d$, where $\mathbf{c}$ is an n-dimensional column vector and d a scalar, is said to be *valid* for P if it is satisfied for all $\mathbf{x} \in P$. Equivalently, $\mathbf{c}^T\mathbf{x} \leq d$ is valid for P if P is fully included in the halfspace $H^- = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} \leq d\}$. An hyperplane $H = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} = d\}$ is said to be a *supporting hyperplane* of P if $\mathbf{c}^T\mathbf{x} \leq d$ is a valid inequality for P and $P \cap H$ is not empty.

A *face* F of a convex polytope P is a subset of P which can be written as $F = P \cap H$ for some supporting hyperplane $H = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} = d\}$ of P. The dimension $\dim(F)$ of a face F is defined as the dimension of its affine hull aff(F), the affine hull of F being the smallest affine set containing F or, equivalently, the intersection of all the affine sets that contain F. Faces of dimension 0, 1 and $n-1$ are called *vertices*, *edges* and *facets*, respectively. A *facet-defining hyperplane* is a supporting hyperplane H of P such that $F = P \cap H$ is a facet of P. Figure 1(b) illustrates these definitions by means of a two-dimensional example (a convex polygon). Let us note that if F is a face of a polytope P, $F = P \cap \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} = d\}$, then

$$d = \max_{\mathbf{x} \in P} \mathbf{c}^T\mathbf{x}. \tag{4}$$

It is a well-known fact [7], [8] that a convex polytope can be represented either as the convex hull of a finite set of points or as the intersection of a finite set of closed halfspaces. In fact, the facet-defining hyperplanes provide the latter representation since *a full dimensional polytope P is the intersection of the halfspaces bounded by its facet-defining hyperplanes* [7], [8]. Precisely, if $\{F_i,\ 1 \leq i \leq f\}$ is the set of facets of P, $H_i = \{\mathbf{x} \mid \mathbf{c}_i^T\mathbf{x} = d_i\}$ the facet-defining hyperplane supporting P along

F_i ($F_i = P \cap H_i$) and $H_i^- = \{\mathbf{x} \mid \mathbf{c}_i^T \mathbf{x} \leq d_i\}$ the closed halfspace bounded by H_i that contains P, then

$$P = \bigcap_{i=1}^{f} H_i^- . \tag{5}$$

Defining the $f \times n$ matrix $\mathbf{C}$ by $\mathbf{C} = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_f)^T$ and the f-dimensional column vector $\mathbf{d}$ by $\mathbf{d} = (d_1, d_2, \ldots, d_f)^T$, Eq. (5) gives P as *the set of solutions to a finite system of linear inequalities* since it implies that $P = \{\mathbf{x} \mid \mathbf{C}\mathbf{x} \leq \mathbf{d}\}$. The representation given in Eq. (5) is minimal in the sense that none of the facet-defining hyperplanes H_i^-, $1 \leq i \leq f$, can be removed.

4 The Available Wrench Set as a System of Linear Inequalities

This section sketches a proof of the characterization stated below which enables the representation of the available wrench set A as a system of finitely many linear inequalities.

The wrench set A has been defined in Section 2 as the image of the box of admissible actuator forces/torques $[\tau]$ under $\mathbf{W}$. In the remainder of this paper, the wrench matrix $\mathbf{W}$ is assumed to have full rank n so that A is full dimensional, *i.e.*, A is of dimension n.

Characterization of the available wrench set facet-defining hyperplanes:
First assertion—A facet-defining hyperplane $H = \{\mathbf{x} \mid \mathbf{c}^T \mathbf{x} = d\}$ of the available wrench set A is such that $\mathbf{c}$ is orthogonal to $n-1$ linearly independent column vectors $\mathbf{w}_i$ of $\mathbf{W}$ and

$$d = \sum_{I^+} \tau_{i_{max}} \mathbf{c}^T \mathbf{w}_i + \sum_{I^-} \tau_{i_{min}} \mathbf{c}^T \mathbf{w}_i \tag{6}$$

where I^- and I^+ are index sets defined as

$$I^+ = \{i,\, 1 \leq i \leq m \mid \mathbf{c}^T \mathbf{w}_i > 0\}, \quad I^- = \{i,\, 1 \leq i \leq m \mid \mathbf{c}^T \mathbf{w}_i < 0\}. \tag{7}$$

Second assertion—Conversely, to any set of $n-1$ linearly independent column vectors $\mathbf{w}_i$ of $\mathbf{W}$, there correspond two facet-defining hyperplanes $H_1 = \{\mathbf{x} \mid \mathbf{c}_1^T \mathbf{x} = d_1\}$ and $H_2 = \{\mathbf{x} \mid \mathbf{c}_2^T \mathbf{x} = d_2\}$ of A. These two hyperplanes are *parallel* (*e.g.* H_1 and H_2 in Figure 1(a)) and such that $\mathbf{c}_1 \neq \mathbf{0}$ is orthogonal to the $n-1$ linearly independent $\mathbf{w}_i$, $\mathbf{c}_2 = -\mathbf{c}_1$ and

$$d_1 = \sum_{I_1^+} \tau_{i_{max}} \mathbf{c}_1^T \mathbf{w}_i + \sum_{I_1^-} \tau_{i_{min}} \mathbf{c}_1^T \mathbf{w}_i, \quad d_2 = \sum_{I_2^+} \tau_{i_{max}} \mathbf{c}_2^T \mathbf{w}_i + \sum_{I_2^-} \tau_{i_{min}} \mathbf{c}_2^T \mathbf{w}_i \tag{8}$$

where $I_1^+ = \{i \mid \mathbf{c}_1^T\mathbf{w}_i > 0\}$, $I_1^- = \{i \mid \mathbf{c}_1^T\mathbf{w}_i < 0\}$, $I_2^+ = \{i \mid \mathbf{c}_2^T\mathbf{w}_i > 0\}$ and $I_2^- = \{i \mid \mathbf{c}_2^T\mathbf{w}_i < 0\}$. Note that $I_2^+ = I_1^-$, $I_2^- = I_1^+$ and $H_2 = \{\mathbf{x} \mid \mathbf{c}_1^T\mathbf{x} = -d_2 = \sum_{I_1^-} \tau_{i_{max}}\mathbf{c}_1^T\mathbf{w}_i + \sum_{I_1^+} \tau_{i_{min}}\mathbf{c}_1^T\mathbf{w}_i\}$.

4.1 Proofs of the First and Second Assertions

Proof of the first assertion—Let $H = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} = d\}$ be a facet-defining hyperplane of A and $F = A \cap H$ the corresponding facet. Consider an arbitrary point (a wrench) $\mathbf{x}_F$ of F. Since $\mathbf{x}_F \in A$, we have

$$\mathbf{x}_F = \sum_{i=1}^{m} \tau_i \mathbf{w}_i, \quad \tau_i \in [\tau_{i_{min}}, \tau_{i_{max}}] \tag{9}$$

and since $\mathbf{x}_F \in H$, according to Eq. (4), we have $\mathbf{c}^T\mathbf{x}_F = d = \max_{\mathbf{x}\in A} \mathbf{c}^T\mathbf{x}$. Hence, the τ_i which define $\mathbf{x}_F$ in Eq. (9) are such that

$$\sum_{i=1}^{m} \tau_i \mathbf{c}^T\mathbf{w}_i = \max_{\mathbf{x}\in A} \mathbf{c}^T\mathbf{x}. \tag{10}$$

Let us decompose the sum in Eq. (10) as follows

$$\sum_{i=1}^{m} \tau_i \mathbf{c}^T\mathbf{w}_i = \sum_{I^+} \tau_i \mathbf{c}^T\mathbf{w}_i + \sum_{I^-} \tau_i \mathbf{c}^T\mathbf{w}_i + \sum_{I^0} \tau_i \mathbf{c}^T\mathbf{w}_i = \sum_{I^+} \tau_i \mathbf{c}^T\mathbf{w}_i + \sum_{I^-} \tau_i \mathbf{c}^T\mathbf{w}_i \tag{11}$$

where $I^0 = \{i, 1 \le i \le m \mid \mathbf{c}^T\mathbf{w}_i = 0\}$ and I^+ and I^- are defined in Eq. (7). According to Eq. (10), since the τ_i maximize the sum in Eq. (11), necessarily, $\tau_i = \tau_{i_{max}}$ for all $i \in I^+$ and $\tau_i = \tau_{i_{min}}$ for all $i \in I^-$. In other words, $\mathbf{x}_F$ is given by

$$\mathbf{x}_F = \sum_{I^+} \tau_{i_{max}} \mathbf{w}_i + \sum_{I^-} \tau_{i_{min}} \mathbf{w}_i + \sum_{I^0} \tau_i \mathbf{w}_i, \quad \tau_i \in [\tau_{i_{min}}, \tau_{i_{max}}] \text{ for all } i \in I^0. \tag{12}$$

Consequently, by definition of the index set I^0, $d = \mathbf{c}^T\mathbf{x}_F$ can be written as

$$d = \sum_{I^+} \tau_{i_{max}} \mathbf{c}^T\mathbf{w}_i + \sum_{I^-} \tau_{i_{min}} \mathbf{c}^T\mathbf{w}_i \tag{13}$$

so that Eq. (6) is proved.

Note that any point $\mathbf{x}$ of A which can be written as $\mathbf{x}_F$ in Eq. (12) belongs to the facet-defining hyperplane H since $\mathbf{c}^T\mathbf{x} = d$. Thus, such a point $\mathbf{x}$ belongs to the facet F (since $F = A \cap H$) and we have

$$F = \{\mathbf{x} \mid \mathbf{x} = \sum_{I^+} \tau_{i_{max}} \mathbf{w}_i + \sum_{I^-} \tau_{i_{min}} \mathbf{w}_i + \sum_{I^0} \tau_i \mathbf{w}_i, \ \tau_i \in [\tau_{i_{min}}, \tau_{i_{max}}] \text{ for all } i \in I^0\}. \tag{14}$$

Moreover, basic properties of affine sets imply that the affine hull aff(F) of F is

$$\text{aff}(F) = \{\mathbf{x} \mid \mathbf{x} = \sum_{I^+} \tau_{i_{max}} \mathbf{w}_i + \sum_{I^-} \tau_{i_{min}} \mathbf{w}_i + \sum_{I^0} \tau_i \mathbf{w}_i,\ \tau_i \in \mathbb{R} \text{ for all } i \in I^0\} \quad (15)$$

and that its dimension is

$$\dim(\text{aff}(F)) = \text{rank}(\{\mathbf{w}_i \mid i \in I^0\}) = \text{rank}(\{\mathbf{w}_i \mid \mathbf{c}^T \mathbf{w}_i = 0\}). \quad (16)$$

Finally, F being a facet of A, $\dim(\text{aff}(F)) = n-1$ and hence Eq. (16) implies that there exists $n-1$ linearly independent $\mathbf{w}_i$ among those of the set $\{\mathbf{w}_i \mid \mathbf{c}^T \mathbf{w}_i = 0\}$. In other words, $\mathbf{c}$ is orthogonal to $n-1$ linearly independent $\mathbf{w}_i$ completing the proof of the first assertion.

Proof of the second assertion—Let us consider a set of $n-1$ linearly independent column vectors $\mathbf{w}_i$ of $\mathbf{W}$ which, without loss of generality, can be assumed to be $\mathbf{w}_1$, $\mathbf{w}_2$, ..., $\mathbf{w}_{n-1}$. Let $\mathbf{c}_1$ be any nonzero vector orthogonal to $\mathbf{w}_1$, $\mathbf{w}_2$, ..., $\mathbf{w}_{n-1}$, d_1 be given by Eq. (8) and H_1 be the hyperplane $H_1 = \{\mathbf{x} \mid \mathbf{c}_1^T \mathbf{x} = d_1\}$.

With arguments similar to those used above in the proof of the first assertion, it can be shown that H_1 is a supporting hyperplane of A and that the corresponding face $F_1 = A \cap H_1$ is given by Eq. (14) with $I_1^0 = \{i \mid \mathbf{c}_1^T \mathbf{w}_i = 0\}$ instead of I^0 and I_1^+ and I_1^- instead of I^+ and I^-, respectively. Then, according to Eq. (16) (with F_1, I_1^0 and $\mathbf{c}_1$ in place of F, I^0 and $\mathbf{c}$, respectively) and since the $n-1$ linearly independent $\mathbf{w}_1$, $\mathbf{w}_2$, ..., $\mathbf{w}_{n-1}$ belong to $\{\mathbf{w}_i \mid i \in I_1^0\}$, we have $\dim(\text{aff}(F_1)) = n-1$ so that F_1 is a facet of A. H_1 is thus a facet-defining hyperplane of A.

Defining the vector $\mathbf{c}_2$ as $\mathbf{c}_2 = -\mathbf{c}_1$, the hyperplane $H_2 = \{\mathbf{x} \mid \mathbf{c}_2^T \mathbf{x} = d_2\}$ with d_2 defined in Eq. (8) is parallel to H_1 and the same type of arguments as those used above show that H_2 is a facet-defining hyperplane of A which completes the proof.

4.2 A Finite System of Linear Inequalities

Referring to Section 3, the available wrench set A can be written as the intersection of the halfspaces bounded by its facet-defining hyperplanes which provides a representation of A as the set of solutions to a system of linear inequalities.

Let C be a finite set of nonzero vectors $\mathbf{c}_j$, $C = \{\mathbf{c}_j,\ 1 \leq j \leq p\}$, such that, on the one hand, each $\mathbf{c}_j \in C$ is orthogonal to $n-1$ linearly independent columns $\mathbf{w}_i$ of $\mathbf{W}$ and, on the other hand, for any set of $n-1$ linearly independent columns $\mathbf{w}_i$, there exists one and only one $\mathbf{c}_j \in C$ orthogonal to these $n-1$ columns $\mathbf{w}_i$. Then, according to the characterization of the facet-defining hyperplanes of A stated at the beginning of Section 4, we have

$$A = \bigcap_{j=1}^{p} (H_{j1}^- \cap H_{j2}^-) \quad (17)$$

where $H_{j1}^- = \{\mathbf{x} \mid \mathbf{c}_j^T \mathbf{x} \le d_{j1}\}$ and $H_{j2}^- = \{\mathbf{x} \mid (-\mathbf{c}_j)^T \mathbf{x} \le d_{j2}\}$ are two closed halfspaces such that $\mathbf{c}_j \in C$ and d_{j1} and d_{j2} are defined similarly to d_1 and d_2 in Eq. (8).

Moreover, in order to obtain a *minimal representation*, *i.e.*, a representation in which, to each facet of A, there corresponds one and only one halfspace H_{jk}^-, $k = 1$ or 2, C must be such that no two of its vectors are collinear (*i.e.*, $\forall(\mathbf{c}_j, \mathbf{c}_l) \in C \times C$, $\mathbf{c}_j \ne \mathbf{c}_l$, there does not exist α such that $\mathbf{c}_l = \alpha \mathbf{c}_j$). The number of facets f of A is equal to $2p$ if and only if C satisfies this property. When C does not satisfy this property, $f < 2p$. Finally, note that if no set of n columns $\mathbf{w}_i$ of $\mathbf{W}$ is a linearly dependent set then C necessarily satisfies the aforementioned property, *i.e*, $f = 2p$. Note however that a minimal representation is not mandatory in order to test if a required wrench set T is included in A.

5 Hyperplane Shifting Method

The characterization of the available wrench set A presented in the previous section leads naturally to the following method that provides a representation of A as the solution set of a system of linear inequalities. In [5], this method is referred to as the hyperplane shifting method. Compared to the version introduced in [5], step 2 of the one presented below avoids many useless computations. Moreover, it is pointed out how to obtain a minimal representation of A.

This method consists in considering all the possible combinations of $n-1$ columns of the wrench matrix $\mathbf{W}$ in turn. At the beginning of the method, $j = 0$. For the current combination $\{\mathbf{w}_i \mid i \in I^0\}$, where I^0 is the current subset of $n-1$ elements of $\{1, 2, \ldots, m\}$, do

Step 1: Test if the $n-1$ column vectors $\mathbf{w}_i$, $i \in I^0$, are linearly independent. If it is the case, $j = j+1$, determine a nonzero vector $\mathbf{c}_j$ orthogonal to these $n-1$ columns $\mathbf{w}_i$ and go to step 2.

Step 2: Let I^+ and I^- be the subsets of $\{1, 2, \ldots, m\}$ defined as $I^+ = \{i \mid \mathbf{c}_j^T \mathbf{w}_i > 0\}$ and $I^- = \{i \mid \mathbf{c}_j^T \mathbf{w}_i < 0\}$. Compute d_{j1} and d_{j2} as follows

$$d_{j1} = \sum_{I^+} \tau_{i_{max}} \mathbf{c}_j^T \mathbf{w}_i + \sum_{I^-} \tau_{i_{min}} \mathbf{c}_j^T \mathbf{w}_i, \quad d_{j2} = -\sum_{I^-} \tau_{i_{max}} \mathbf{c}_j^T \mathbf{w}_i - \sum_{I^+} \tau_{i_{min}} \mathbf{c}_j^T \mathbf{w}_i. \tag{18}$$

At the end of the method, $j = p$ and the $2p$ halfspaces $H_{j1}^- = \{\mathbf{x} \mid \mathbf{c}_j^T \mathbf{x} \le d_{j1}\}$ and $H_{j2}^- = \{\mathbf{x} \mid (-\mathbf{c}_j)^T \mathbf{x} \le d_{j2}\}$, $j = 1, \ldots, p$, provide a representation of A as stated in Eq. (17). Gathering the vectors $\mathbf{c}_j$ and $-\mathbf{c}_j$ in a matrix $\mathbf{C}$ and the d_{j1} and d_{j2} in a vector $\mathbf{d}$ in an appropriate way, Eq. (17) gives A as the solution set of the system of linear inequalities $\mathbf{Cx} \le \mathbf{d}$. For this representation of A to be minimal, in Step 1, it must be ensured that $\mathbf{c}_j$ is not collinear to any of the previously computed ones, *i.e.*, not collinear to any of the $\mathbf{c}_k$, $k = 1, \ldots, j-1$. Indeed, if such a collinearity exists, $\mathbf{c}_j$ can be left out from consideration since it yields a redundant inequality.

Step 1 can be implemented as follows. Let $\mathbf{W}_{I^0}$ be the $n \times n-1$ matrix whose columns are the current $n-1$ column vectors $\mathbf{w}_i$, $i \in I^0$. Let d be the dimension of the nullspace of $\mathbf{W}_{I^0}^T$ where $d \neq 0$ since $\mathbf{W}_{I^0}^T$ has more columns than rows. When $d > 2$, the current $n-1$ column vectors $\mathbf{w}_i$, $i \in I^0$, are linearly dependent. Otherwise, $d = 1$ and $\mathbf{c}_j$ can be any nonzero vector in the nullspace of $\mathbf{W}_{I^0}^T$ since then $\mathbf{w}_i^T \mathbf{c}_j = \mathbf{0}$ for all $i \in I^0$. Hence, with the help of a routine that determines the nullspace of a matrix, the hyperplane shifting method is quite straightforward to implement.

6 Conclusion

This paper has dealt with the characterization of the facet-defining hyperplanes of the available wrench set of a parallel manipulator which enables to represent this set as the set of solutions to a system of linear inequalities. With such a representation, being given a wrench set T that the parallel manipulator is required to generate, it is generally straightforward to test whether or not T is fully included in the available wrench set, *i.e.*, whether or not T is feasible.

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