# QUAD workshop

## QUantifiers And Determiners

A tentative introduction


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ANR franco autrichienne FISP FIne Structure of Proofs.

## An heteroclite domain

Logic, philosophy of language, linguistics, psycholinguistics, cognition, didactics,...

There are quantifiers in natural language, some quantifiers have been studied logically.

# A Quantified logics

## A.1. Quantified statements in ancient/medieval logic

- All students passed.

$$A_{P,Q} = \forall x.(P(x) \Rightarrow Q(x))$$

- No student passed.

$$E_{P,Q} = \forall x.(P(x) \Rightarrow \neg Q(x))$$

- Some student passed.

$$I_{P,Q} = \exists x.(P(x) \wedge Q(x))$$

- Not all student passed. *(original phrasing)*

$$O_{P,Q} = \exists x.(P(x) \wedge \neg Q(x))$$

  Some students did not pass.
  *(different focus, but less ambiguous)*
  not all *is not lexicalised in all (?) languages.*

## A.2. Rules rather than models

Syllogisms (vowel= type of statement among $A, E, I, O$)

Barbara:

All M are P,
All S are M,
Hence all S are P

Baroco :

All P are M,
Some S are not M,
Hence Some S are not P

## A.3. Square of opposition

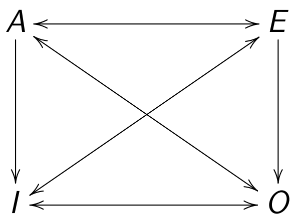
Under some conditions (e.g. $\exists x.P(x)$ holds) the formulae A E I O constitute a square of opposition:

i) $A \dashv\vdash \neg O$ and $E \dashv\vdash \neg I$

ii) It is never the case that $\top \vdash A$ and $\top \vdash E$

iii) It is never the case that $I \vdash \bot$ and $E \vdash \bot$

iv) $A \vdash I$ and $E \vdash O$



# A.4. Algebraic Logic, Boole, De Morgan, Peirce,...

Regarding quantification Boole tried some rules for developping, factorising, verifying formulas:

$$\forall x(I(x) \Rightarrow (F(x) \vee M(x)))$$
$$\equiv$$
$$(\forall x(I(x) \Rightarrow F(x))) \vee (\forall x(I(x) \Rightarrow M(x)))$$

$I$ : *individual* $F$ : *female* $M$ : *male*

????

# A.5. Frege: modern predicate calculus / quantified logic

The two mathematical quantifiers: $\forall x$ or $(x)$ and $\exists x$ or E$x$

Usual first order logic

Single sorted: contradicts ”A property is asserted of some individual as part of a class.”

# A.6. Interpretation of a sentence in a model $M$

$n$-ary relation: part of $M^n$.

Free variables are assigned ($\mathscr{A}$) a value in $M$.

Connectives are interpreted naturally: & means ”and” etc.

$\forall x.P(x)$ for holds for $\mathscr{A}$ whenever $P(x)$ holds for $\mathscr{A} \cup x \mapsto m$ for every assignment $m$ in $M$ to the free variable $x$ in $P(x)$

$\exists x.P(x)$ for holds for $\mathscr{A} \cup x \mapsto m$ when $P(x)$ holds for some assignment $m$ in $M$ to the free variable $x$ in $P(x)$

# A.7. Proof systems

Frege's system is complicated.

Hilbert axioms (requires substitution).

Gentzen LK (sequent calculus), NK (natural deduction).

## A.8. Results, properties

Observe that numbers (which are quantifiers) can be defined: at least $n$, at most $n$, exactly $n$,

Soundness: a provable formula is true in every model.

Completeness of pure logic: if a formula $F$ is true in every model then it is provable. *Kurt Gödel Die Vollstândigkeit der Axiome des logischen Funktionenkalküls Monatshefte für Mathematik und Physik (1930)*

Compactness if every subset of a possibly infinite collection of formulas has a model so does the whole collection.

Löwenheim Skolem a set of formulas that admits an infinite models admits models of any infinite cardinality.

## A.9. Open question for this classical/Fregean view of FOL

The only quantifiers are $\forall$ and $\exists$ on individuals, and one a whole single domain.

Restricted quantification on a subdomain can be defined:

(1) a. $\forall x \in M\ P(x) \quad \equiv \quad \forall x\ (M(x) \Rightarrow P(x))$
b. $\exists x \in M\ P(x) \quad \equiv \quad \exists x\ (M(x) \& P(x))$

This treatment does not apply to other quantifiers:

(2) a. for 1/3 of the $x \in M\ P(x) \not\equiv$ for 1/3 of the $x\ (M(x) \Rightarrow P(x))$
b. for few $x \in M\ P(x) \not\equiv$ for few $x\ (M(x) \& P(x))$

## A.10. Higher order?

(3) He believes everything he is said.

Probably what he is said are propositions and not individuals.

Beware that completeness fails from second order onwards because completeness entails compacity.

> $D$: every two place predicate that defines an injective function is surjective. The universe is finite.
>
> $F_n$ = there are more than $n$ individuals.

There is a notion of model, for which one has completeness, but than the above set of formulas has a model.

# B Linguistics

## B.1. Quantifiers in natural language

*Exists, for all* Universal quantifiers are rare.

and many more:

*Exactly, at most, at least* $n$

*2/3, 45%*

*few, many,*

Subtle differences between model theoreticallly equivalent formulations: tout/chaque.

## B.2. Meaning and scope

In order to understand each other properly we need to determine quantifier scope.

In a pizzeria:

The children will have a pizza.

Depending on the size of the the pizza, of the children and of their hungriness.

(4) A circle is connected to each square.

(5) A guard stands in front of each museum door.

## B.3. Generative/transformational grammar

Scope and such considered as a syntactic phenomenon.

Quantifier raising: quantifier undergo covert movement to encompass a large part of the sentence.

Preferred quantifier scope is left right and other scopes ought to be triggered by some feature.

## B.4. Categorial grammar / Montague semantics

Interesting solution because it encompasses syntax.

Categorial grammar : a syntax that relies on the logical structure of the sentence.

The syntactic structure, the parse tree is non canonical...

Meaning is computed for the meaning of its parts (Fregean compositionality) and the syntactic structure (Montague).

Meaning: logical formula interpreted thereafter or situations in which the sentence is true?

## B.5. Categorial grammar

| (Syntactic category)$^*$ | $=$ | Semantic type | |
|---|---|---|---|
| $S^*$ | $=$ | $t$ | a sentence is a proposition |
| $np^*$ | $=$ | $e$ | a noun phrase is an entity/individual |
| $n^*$ | $=$ | $e \rightarrow t$ | a common noun is a property of individuals |
| $(A\backslash B)^* = (B/A)^*$ | $=$ | $A \rightarrow B$ | extends the translation to complex categories |

## B.6. Constants for logical operators

| Constant | Type |
|---|---|
| $\exists$ | $(e \to t) \to t$ |
| $\forall$ | $(e \to t) \to t$ |
| $\wedge$ | $t \to (t \to t)$ |
| $\vee$ | $t \to (t \to t)$ |
| $\supset$ | $t \to (t \to t)$ |

# B.7. From natural langue to a logical language

| | | |
|---|---|---|
| *love* | $\lambda x \lambda y\ (\text{aime}\ y)\ x$ | $x : e,\ y : e, \text{love} : e \to (e \to t)$ |
| << love >> is a binary predicate | | |

| | | |
|---|---|---|
| *Garance* | $\lambda P\ (P\ \text{Garance})$ | $P : e \to t,\ \text{Garance} : e$ |
| << Garance >> is viewed as the properties << Garance >> enjoys | | |

## B.8. Montague semantics — algorithm

1. syntactic analysis $S$
2. mapped to a lambda term of type $t$ (bases types: $e$ et $t$)
3. insertion of semantic lambda terms
4. beta reduction

   $\rightarrow$ term of type $t$

   = logical formula

   = sentence meaning (roughly speaking ;-)

# B.9. Example of a semantic analysis: les enfants prendront une pizza

| **word** | ***csyntactici category*** $u$<br>***type sémantique*** $u^*$<br>***semantics:*** $\lambda$***-term of type*** $u^*$<br>$x^v$ ***means*** $x$ ***(variable, constante) of type*** $v$ |
|---|---|
| les | $(S/(np\backslash S))/n$ (subject)<br>$((S/np)\backslash S)/n$ (object)<br>$(e \to t) \to ((e \to t) \to t)$<br>$\lambda P^{e\to t}\, \lambda Q^{e\to t}\, (\forall^{(e\to t)\to t}\, (\lambda x^e (\Rightarrow^{t\to(t\to t)}\, (P\ x)(Q\ x))))$ |
| une | $((S/np)\backslash S)/n$ (object)<br>$(S/(np\backslash S))/n$ (subject)<br>$(e \to t) \to ((e \to t) \to t)$<br>$\lambda P^{e\to t}\, \lambda Q^{e\to t}\, (\exists^{(e\to t)\to t}\, (\lambda x^e (\wedge^{t\to(t\to t)} (P\ x)(Q\ x))))$ |
| enfant(s) | $n$<br>$e \to t$<br>$\lambda x^e (\mathtt{enfant}^{e\to t}\ x)$ |
| pizza | $n$<br>$e \to t$<br>$\lambda x^e (\mathtt{pizza}^{e\to t}\ x)$ |
| prendront | $(np\backslash S)/np$<br>$e \to (e \to t)$<br>$\lambda y^e\, \lambda x^e\, ((\mathtt{prendront}^{e\to(e\to t)}\ x)y)$ |

## B.10. Syntactic analysis ∃∀

There are two possible syntactic analysis. Here is one.

∃∀

$$\dfrac{\dfrac{\dfrac{\dfrac{(S/(np\backslash S))/n \quad n}{(S/(np\backslash S))}/_e \quad \dfrac{(np\backslash S)/np \quad [np]^1}{(np\backslash S)}/_e}{S}/_e}{S/np}/_i(1) \qquad \dfrac{((S/np)\backslash S)/n \quad n}{(S/np)\backslash S}\backslash_e}{S}\backslash_e$$

# B.11. Syntax $\rightarrow$ semantic $\lambda$-term of the sentence

$\exists\forall$

$$
\dfrac{
 \dfrac{
  \dfrac{
   \dfrac{\overset{\textit{les}}{(\mathbf{e}\to\mathbf{t})\to(\mathbf{e}\to\mathbf{t})\to\mathbf{t}} \quad \overset{\textit{enfants}}{(\mathbf{e}\to\mathbf{t})}}{(\mathbf{e}\to\mathbf{t})\to\mathbf{t}}\to_e
   \quad
   \dfrac{\overset{\textit{prendront}}{\mathbf{e}\to\mathbf{e}\to\mathbf{t}} \quad \overset{\textit{o}}{[\mathbf{e}]^1}}{\mathbf{e}\to\mathbf{t}}\to_e
  }{\mathbf{t}}\to_e
 }{\mathbf{e}\to\mathbf{t}}\to_i(1)
 \quad
 \dfrac{\overset{\textit{une}}{(\mathbf{e}\to\mathbf{t})\to(\mathbf{e}\to\mathbf{t})\to\mathbf{t}} \quad \overset{\textit{pizza}}{(\mathbf{e}\to\mathbf{t})}}{(\mathbf{e}\to\mathbf{t})\to\mathbf{t}}\to_e
}{\mathbf{t}}\to_e
$$

The obtained $\lambda$-term is

$\exists\forall = (\textit{une pizza})(\lambda o^{\mathbf{e}} (\textit{les enfants})(\textit{prendront } o))$

One has to

1. insert lexical lambda terms
2. reduce/compute

# B.12. Calculus, step by step 1/2

$(\textit{une pizza})$
$= (\lambda P^{e\to t}\, \lambda Q^{e\to t}\, (\exists^{(e\to t)\to t}\, (\lambda x^e (\wedge^{t\to(t\to t)}(P\ x)(Q\ x)))))(\lambda z^e (\mathtt{pizza}^{e\to t}\ z))$
$= (\lambda Q^{e\to t}\, (\exists^{(e\to t)\to t}\, (\lambda x^e (\wedge^{t\to(t\to t)}((\lambda z^e (\mathtt{pizza}^{e\to t}\ z))\ x)(Q\ x)))))$
$= (\lambda Q^{e\to t}\, (\exists^{(e\to t)\to t}\, (\lambda x^e (\wedge^{t\to(t\to t)}((\mathtt{pizza}^{e\to t}\ x)))(Q\ x))))$

$(\textit{les enfants})$
$= (\lambda P^{e\to t}\, \lambda Q^{e\to t}\, (\forall^{(e\to t)\to t}\, (\lambda x^e (\Rightarrow^{t\to(t\to t)}\, (P\ x)(Q\ x)))))(\lambda u^e (\mathtt{enfant}^{e\to t}\ u))$
$= (\lambda Q^{e\to t}\, (\forall^{(e\to t)\to t}\, (\lambda x^e (\Rightarrow^{t\to(t\to t)}\, ((\lambda u^e (\mathtt{enfant}^{e\to t}\ u))\ x)(Q\ x)))))$
$= (\lambda Q^{e\to t}\, (\forall^{(e\to t)\to t}\, (\lambda x^e (\Rightarrow^{t\to(t\to t)}\, (\mathtt{enfant}^{e\to t}\ x)(Q\ x)))))$

$(\textit{les enfants})(\textit{prendront } o) =$
$(\lambda Q^{e\to t}\, (\forall^{(e\to t)\to t}\, (\lambda w^e (\Rightarrow^{t\to(t\to t)}\, (\mathtt{enfant}^{e\to t}\ w)(Q\ w)))))((\lambda y^e\, \lambda x^e\, ((\mathtt{prendront}^{e\to(e\to t)}\ x)$
$= (\lambda Q^{e\to t}\, (\forall^{(e\to t)\to t}\, (\lambda w^e (\Rightarrow^{t\to(t\to t)}\, (\mathtt{enfant}^{e\to t}\ w)(Q\ w)))))(\lambda x^e\, ((\mathtt{prendront}^{e\to(e\to t)}\ x)\ o))$
$= \forall^{(e\to t)\to t}\, (\lambda w^e (\Rightarrow^{t\to(t\to t)}\, (\mathtt{enfant}^{e\to t}\ w)((\lambda x^e\, ((\mathtt{prendront}^{e\to(e\to t)}\ x)\ o))\ w)))$
$= \forall^{(e\to t)\to t}\, (\lambda w^e (\Rightarrow^{t\to(t\to t)}\, (\mathtt{enfant}^{e\to t}\ w)(((\mathtt{prendront}^{e\to(e\to t)}\ w)\ o))))$

# B.13. Calculus, step by step 2/2

$$
\begin{aligned}
&(\textit{une pizza})(\lambda o\ (\textit{les enfants})(\textit{prendront}\ o))\\
&= (\lambda Q^{e\to t}\ (\exists^{(e\to t)\to t}\ (\lambda x^{e}(\wedge^{t\to(t\to t)}((\texttt{pizza}^{e\to t}\ x)))(Q\ x))))\\
&\qquad (\lambda o \forall^{(e\to t)\to t}\ (\lambda w^{e}(\Rightarrow^{t\to(t\to t)}\ (\texttt{enfant}^{e\to t}\ w)(((\texttt{prendront}^{e\to(e\to t)}\ w)\ o)))))\\
&= (\exists^{(e\to t)\to t}\ (\lambda x^{e}(\wedge^{t\to(t\to t)}((\texttt{pizza}^{e\to t}\ x)))\\
&\qquad ((\lambda o \forall^{(e\to t)\to t}\ (\lambda w^{e}(\Rightarrow^{t\to(t\to t)}\ (\texttt{enfant}^{e\to t}\ w)(((\texttt{prendront}^{e\to(e\to t)}\ w)\ o)))))\ x)))\\
&= (\exists^{(e\to t)\to t}\ (\lambda x^{e}(\wedge^{t\to(t\to t)}((\texttt{pizza}^{e\to t}\ x)))\\
&\qquad (\forall^{(e\to t)\to t}\ (\lambda w^{e}(\Rightarrow^{t\to(t\to t)}\ (\texttt{enfant}^{e\to t}\ w)((\texttt{prendront}^{e\to(e\to t)}\ w)\ x))))))
\end{aligned}
$$

One usually write the above term like this:

$$\exists x.\ \textit{pizza}(x) \wedge \forall w.\ (\textit{enfant}(w) \Rightarrow \textit{prendront}(w,x))$$

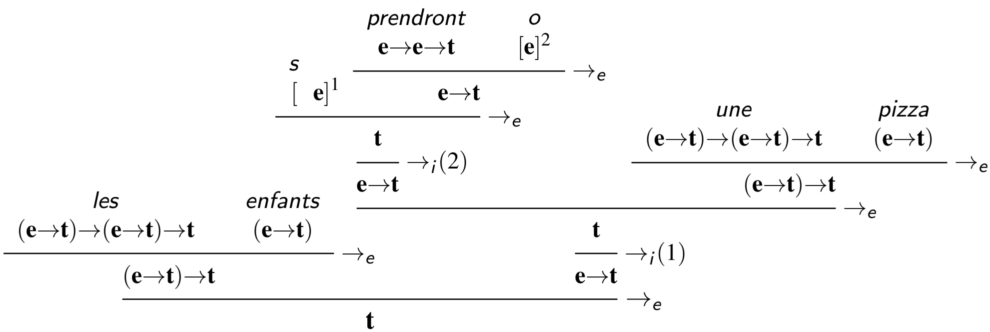
# B.14. There is another syntactic analysis ...

$\forall\exists$

$$
\dfrac{\dfrac{(S/(np\backslash S))/n \quad n}{(S/(np\backslash S))}/_e \qquad \dfrac{\dfrac{\dfrac{\dfrac{[np]^1 \qquad \dfrac{(np\backslash S)/np \quad [np]^2}{(np\backslash S)}/_e}{S}\backslash_e}{S/np}/_i(2) \qquad \dfrac{((S/np)\backslash S)/n \quad n}{(S/np)\backslash S}\backslash_e}{S}\backslash_e}{np\backslash S}\backslash_i(1)}{S}/_e
$$

Which yields the following semantic structure:

$\forall\exists$




$\lambda$-term of the sentence:

$$\forall\exists = (\textit{les enfants})(\lambda s.\ (\textit{une pizza})(\lambda o\ ((\textit{prendront}\ o)\ s)))$$

we intert the lexical $\lambda$-terms and compute:

((*une pizza*) et (*les enfants*) déjà faits)

## B.15. Calculus (bis repetita placent)

$$(\mathit{une\ pizza})(\lambda o\ ((\mathit{prendront}\ o)\ s))$$
$$= (\lambda Q^{e\to t}\ (\exists^{(e\to t)\to t}\ (\lambda x^e(\wedge^{t\to(t\to t)}((\mathtt{pizza}^{e\to t}\ x)))(Q\ x))))$$
$$(\lambda o\ (((\lambda y^e\ \lambda x^e\ ((\mathtt{prendront}^{e\to(e\to t)}\ x)\ y))\ o)\ s)))$$
$$= (\lambda Q^{e\to t}\ (\exists^{(e\to t)\to t}\ (\lambda x^e(\wedge^{t\to(t\to t)}((\mathtt{pizza}^{e\to t}\ x)))(Q\ x))))$$
$$(\lambda o\ ((\mathtt{prendront}^{e\to(e\to t)}\ s)\ o))$$
$$= (\exists^{(e\to t)\to t}\ (\lambda x^e(\wedge^{t\to(t\to t)}((\mathtt{pizza}^{e\to t}\ x)))((\lambda o\ ((\mathtt{prendront}^{e\to(e\to t)}\ s)\ o))\ x)))$$
$$= (\exists^{(e\to t)\to t}\ (\lambda x^e(\wedge^{t\to(t\to t)}((\mathtt{pizza}^{e\to t}\ x)))((\mathtt{prendront}^{e\to(e\to t)}\ s)\ x)))$$

$$\forall\exists = (\mathit{les\ enfants})(\lambda s.\ (\mathit{une\ pizza})(\lambda o\ ((\mathit{prendront}\ o)\ s)))$$
$$= (\lambda Q^{e\to t}\ (\forall^{(e\to t)\to t}\ (\lambda u^e(\Rightarrow^{t\to(t\to t)}\ (\mathtt{enfants}^{e\to t}\ u)(Q\ u)))))$$
$$(\lambda s.\ (\exists^{(e\to t)\to t}\ (\lambda x^e(\wedge^{t\to(t\to t)}((\mathtt{pizza}^{e\to t}\ x)))((\mathtt{prendront}^{e\to(e\to t)}\ s)\ x))))$$
$$= (\forall^{(e\to t)\to t}\ (\lambda u^e(\Rightarrow^{t\to(t\to t)}\ (\mathtt{enfants}^{e\to t}\ u)$$
$$((\lambda s.\ (\exists^{(e\to t)\to t}\ (\lambda x^e(\wedge^{t\to(t\to t)}((\mathtt{pizza}^{e\to t}\ x)))((\mathtt{prendront}^{e\to(e\to t)}\ s)\ x))))\ u)))))$$

$$= (\forall^{(e\to t)\to t}\ (\lambda u^e(\Rightarrow^{t\to(t\to t)}\ (\mathtt{enfants}^{e\to t}\ u)$$
$$(\exists^{(e\to t)\to t}\ (\lambda x^e.\ (\wedge^{t\to(t\to t)}((\mathtt{pizza}^{e\to t}\ x)))((\mathtt{prendront}^{e\to(e\to t)}\ u)\ x))))))$$

This is usually written as:

$$\forall u.\ \mathit{enfants}(u) \Rightarrow \exists .x\ \mathit{pizza}(x) \wedge \mathit{prendront}(u,x)$$

# B.16. Other quantifiers

Also have scope issues:

Put eight drops in three spoon of water.

## B.17. Preferences?

We clearly have preferences for readings sentences with multiple quantifiers, some being syntactical (left right order) other being triggered by world knowledge and common sense reasoning.

Regarding syntactic preferences Davide Catta and Mehdi Mirzapour (PhDs) proposed a combination of Morrill syntactic measure and penalties for swapping quantifiers in the preferred reading.

## B.18. Quantifier: critics of the standard solution 1/3

Syntactical structure of the sentence $\neq$ logical form $\rightarrow$ lack of compositionality.

Associated problem: several syntactic categories depending on the place of the determine/quantifier in the syntactic tree.

(6) Orlando di Lasso composed some motets.

(7) syntax (Orlando di Lasso (composed (some (motets))))

(8) semantics: (some (motets)) $\underline{(\lambda x.\ \text{OdL composed } x)}$

The underlined predicate is not a proper phrase.

# B.19. Quantifier: critics of the standard solution 2/3

Asymmetry class / predicate

(9) a. Some politicians are crooks.
b. ?? Some crooks are politicians.

(10) a. Some students are employees.
b. Some employees are students.

The different focus makes a big difference.

# B.20. Quantifier: critics of the standard solution 3/3

There can be a reference before the utterance of the main predicate (if any):

(11) Cars, cars, cars,... (Blog)

(12) Premier voyage, New-York. (B. Cendrars)

(13) What a thrill — My thumb instead of an onion. (S. Plath)

(14) Lundi, mercredi et vendredi, une machine de couleurs, mardi et jeudi, une machine de blanc, le samedi, les draps, le dimanche, les serviettes. (Blog)

Even when there is a main predicate, I do think that we interpret the quantified NP as soon as we hear it.

# C CCG, Skolem terms, and scope issues (cf. other slide series)

# D Generics, universals and quantification

## D.1. Ancient and medieval philosophical ideas

A long debated question in logic and metaphysics (from Plato, Aristotle, **Porphyre**, scholastics...)

Universal ”dog” vs. the set of individuals ”dogs”

What is a concept of a dog?

a substance, that exists independently of the individuals falling under this concept

a name without reality, i.e. an abbreviation a word for the class of all individuals falling under the concept

a concept that is a mental construction related to the empirical relation to the set of individuals

A good question (Abélard/Roscelin debate) :

If an illness causes the extinction of all tall dogs, would your concept of dog be altered.

## D.2. Ancient and new mathematics

Now let speak about proofs and reasoning on a collection of individuals:

$$\forall x \in \mathscr{C} F(x) \equiv \&_{x \in \mathscr{C}} F(x) \equiv P(c_1) \& P(c_2) \& P(c_3) \& P(c_4) \& \cdots$$

How can we prove and refute such a formula? TWO ways:

- We can prove $P(c_1)$ then $P(c_2)$ then $P(c_3)$ then $P(c_4)$... Once we did so for all elements in $\mathscr{C}$ we can perform the conjunction of all these formulae.
- Let $x$ be any element of $\mathscr{C}$...(reasoning) ... $P(x)$ holds. As $x$ does not possess anything special apart from being in $\mathscr{C}$, the property holds for any element in $\mathscr{C}$.

As far as the collection under consideration is finite no difference.

## D.3. The dual nature of universal quantification

(15) a. Un chien a quatre pattes.
b. Tout chien a quatre pattes.
c. Le chien a quatre pattes.
d. Les chiens ont quatre pattes.
e. Tous les chiens ont quatre pattes.
f. Chaque chien a quatre pattes.

(16) a. Each dog has four legs.
b. Dogs have four legs.
c. A dog has four legs.

## D.4. Distributive readings and individuals

Collection of individuals: cannot accept exceptions, coincidence of properties that can be conjuncted.

- Domain may be complicated:

  *"Every one sitting at the table with the uncle of the bride had white shirts."*

- Proof by reasoning.

- Refutation: a sentence involving a distributive quantifier on domain *D* can only be refuted by an individual and not by a class unless it is clear that this class intersects the domain *D*.

(17) a. Each bird with both black and white feathers flies.
b. Not this wound bird.(perfect)
c. Not autruches. (not good refutation, since the intersection is not obvious)

## D.5. Generics NPs and sentences

Generic element: ideal, properties derived by reasoning, can accept exceptions.

- Domain cannot be complicated.

  A person sitting at the table with the uncle of the bride had a white shirt. (cannot mean all of them).

- Refutation:

  The refutation of a sentence involving a generic can only be refuted by another rule: Proof: a proof for each individual.

- (18)
  a. Birds fly.
  b. Not this wound bird. (not a refutation, generic readings admit exceptions)
  c. Not autruches. (perfect)

# D.6. Proof rules with generics — introduction

Usual rule when a property has been established for an $x$ which does not enjoy any particular property (i.e. is not free in any hypothesis), one can conclude that the property holds for all individuals:

$$\frac{\begin{array}{c}\text{no free occurrence of } x \\ \text{in any } H_i \\ H_1, \ldots, H_n \vdash P(x)\end{array}}{H_1, \ldots, H_n \vdash \forall x.\ P(x)}\ \forall_i$$

Can be formulated with a generic element:

$$\frac{\begin{array}{c}\text{no free occurrence of } x \\ \text{in any } H_i \\ H_1, \ldots, H_n \vdash P(x)\end{array}}{H_1, \ldots, H_n \vdash P(\tau_x P(x))}\ \forall_i$$

$\tau_x P(x)$ enjoys the property $P(\_)$ when every individual does.

# D.7. Proof rules with generics — elimination

The $\forall$ elimination rule says that when a property has been established for all individuals it can be inferred for any particular terms or individual:

$$\frac{H_1, \ldots, H_n \vdash \forall x.\ P(x)}{H_1, \ldots, H_n \vdash P(a)} \forall_e$$

This can be formulated with a generic individual using $\tau$ that is a **subnector** i.e. an operator that builds a term (of type individual) from a formula (Curry's terminology).

$$\frac{H_1, \ldots, H_n \vdash P(\tau_x P(x))}{H_1, \ldots, H_n \vdash P(a)} \forall_e$$

If $\tau_x P(x)$ enjoys the property $P(\_)$ then any individual does.

$\tau_x Drink(x)$ is sober: he drinks iff every one does.

## D.8. Distinction chaque/tout

Christian Retoré / Alda Mari:

CHAQUE: known domain, no exception, contingent, descriptive/synthetic statements: $\&_{x \in D} P(x)$ (easily takes wide scope)

TOUT: vague domain, admits exceptions, général, prescriptive/analytic statements: $P(\tau_x D(x))$

QUELQUES/CERTAINS Audrey BEDEL Existential quantification in French: what difference between quelque and un certain? Cog-Master internship, 2017.

**This raises a general question in semantics that goes beyond quantification: how to interpret logical formulas differently than in set theoretic models?**

## D.9. Russell's iota

As opposed to the generic dog $\tau_x dog(x)$ there is "This dog", "The dog that is sleeping on the sofa,..." the unique individual satisfying $P$: a term $\iota_x P(x)$.
Russell introduced $\iota$ for definite descriptions. It is the ancestor of Hilbert's $\varepsilon$.
A technical problem with $\iota$ is that the negation of there exists a unique individual such that P is that there are no such individual or at least two.
As observed by von Heusinger, it should be observed that there is little difference between the logical form of definite descriptions and indefinite noun phrase...
The uniqueness is not always observed,

(19) Recueilli très jeune par les moines de l'abbaye de Reichenau, sur **l'ile du lac de Constance**, en Allemagne, qui le prennent en charge totalement; Hermann étudie et devient l'un des savants les plus érudits du XIème siècle.

## D.10. Hilbert's epsilon

$$F(\varepsilon_x F(x)) \equiv \exists x.\ F(x)$$

A term (of type individual) $\varepsilon_x F(x)$ associated with $F(x)$: as soon as an entity enjoys $F(\_)$ the term $\varepsilon_x F(x)$ enjoys $F(\_)$.

The operator $\varepsilon$ binds the free occurrences of $x$ in $F$.

$\varepsilon_x Drink(x)$ is a soak: he drinks iff someone drinks.

## D.11. Syntax of epsilon in first order logic

Terms and formulae are defined by mutual recursion:

- Any constant in $\mathscr{L}$ is a term.
- Any variable in $\mathscr{L}$ is a term.
- $f(t_1, \dots, t_p)$ is a term provided each $t_i$ is a term and $f$ is a function symbol of $\mathscr{L}$ of arity $p$
- $\varepsilon_x A$ is a term if $A$ is a formula and $x$ a variable — any free occurrence of $x$ in $A$ is bound by $\varepsilon_x$
- $\tau_x A$ is a term if $A$ is a formula and $x$ a variable — any free occurrence of $x$ in $A$ is bound by $\tau_x$
- $s = t$ is a formula whenever $s$ and $t$ are terms.
- $R(t_1, \dots, t_n)$ is a formula provided each $t_i$ is a term and $R$ is a relation symbol of $\mathscr{L}$ of arity $n$
- $A \& B$, $A \vee B$, $A \Rightarrow B$, $\neg A$ when $A$ and $B$ are formulae.

## D.12. Rules for $\varepsilon$

Hilbert's work: fine! (Grundlagen der Mathematik, with P. Bernays)

Introduction of the universal quantifier

Rule 1: From $P(x)$ with $x$ generic infer:
$P(\tau_x.P(x))$

Introduction of the existential quantifier:

Rule 2: From $P(t)$ infer $P(\varepsilon_x P(x)) \equiv \exists x\ P(x)$

A classical (as opposed to intuitionistic) observation:

$$P(\varepsilon_x P(x)) \equiv \exists x P(x) \equiv \neg\forall x \neg P(x) \equiv \neg\neg P(\tau_x \neg P(x))$$

$$P(\tau_x P(x)) \equiv \forall x P(x) \equiv \neg\exists x \neg P(x) \equiv \neg\neg P(\varepsilon_x \neg P(x))$$

Hence: $\tau_x P(x) = \varepsilon\neg P(x)$ and $\varepsilon_x P(x) = \tau\neg P(x)$

One is enough, usually people chose $\varepsilon$ (e.g. Bourbaki in their set theory book).

## D.13. Relation to first order logic

The quantifier free epsilon calculus is a strict conservative extension of first order logic.

- Strict: there are formulae not equivalent to any formula of first order logic, e.g. $P(\varepsilon_x Q(x))$ with $P, Q$ unary predicate symbols.
- Conservative: regarding first order formula the epsilon calculus derive the same formulae.

## D.14. A false (but nevertheless useful) intuition

Although $\exists x.(S(x) \wedge P(x))$ and $S(\varepsilon_x P(x))$ are related, they are not equivalent for any $P$ and $S$.

$$S(\varepsilon_x P(x)) \nvdash \qquad \not\dashv \exists x.(S(x) \wedge P(x))$$

Indeed, let $S(x)$ be $(x = x)$ and let $P(x)$ be $(x \neq x)$ i.e. $\neg S(x)$.

Then

$$S(\varepsilon_x P(x)) \;\equiv\; S(\varepsilon_x.\neg S(x)) \;\equiv\; S(\tau_x.S(x)) \;\equiv\; \forall x.S(x) \;\equiv\; \forall x.(x = x)$$

which is clearly true.

But $\exists x.(S(x) \& P(x)) \;\equiv\; \exists x.(S(x) \& \neg S(x)) \;\equiv\; \exists x.(x = x \;\&\; x \neq x)$ which is clearly false.

The argument works with any formula of one variable that is universally true like here $S(x) \equiv (x = x)$.

## D.15. Main results

$\varepsilon$-elimination (1st & 2nd $\varepsilon$-theorems), yielding the first correct proof of Herbrand theorem.

**First epsilon theorem** When inferring a formula $C$ without the $\varepsilon$ symbol nor quantifiers from formulae $\Gamma$ not involving the $\varepsilon$ symbol nor quantifiers the derivation can be done within quantifier free predicate calculus.

**Second epsilon theorem** When inferring a formula $C$ without the $\varepsilon$ symbol from formulae $\Gamma$ not involving the $\varepsilon$ symbol, the derivation can be done within usual predicate calculus.

Epsilon was introduced by Hilbert firstly for arithmetic, in order to establish arithmetic consistency by elementary means (this was before Gödel's incompleteness theorem), using **epsilon substitution method**.

Little else is known (non standard formulae, full cut-elimination, models), erroneous results cf. Zentralblatt.

## D.16. Admittedly slightly unpleasant

Heavy notation:

$\forall x \exists y P(x,y)$ is
$\exists y P(\tau_x P(x,y), y)$ is
$P(\tau_x P(x, \varepsilon_y P(\tau_x P(x,y), y)), \varepsilon_y P(\tau_x P(x,y), y))$

## D.17. "Loose" use of $\varepsilon$

Some $A$ are $B$. (E sentences of Aristotle)

$B(\varepsilon x.\ A(x))$

Not equivalent to an ordinary formula, in particular not equivalent to the standard: $\exists x.\ A\&B(x)$ but

$$B(\varepsilon x.\ A(x)) \wedge A(\varepsilon x.\ A(x)) \vdash \exists x.\ B\&A(x)$$

Indeed:

$$B(\varepsilon x.\ A(x)) \wedge A(\varepsilon x.\ A(x))$$
$$\vdash B(\varepsilon x.\ B\&A(x)) \wedge A(\varepsilon x.\ B\&A(x))$$
$$\vdash B\&A(\varepsilon x.\ (B\&A(x))$$

On the other hand, one has:

$$\exists x.A(x)\&\forall y(A(y) \Rightarrow B(y)) \vdash B(\varepsilon x.A(x))$$

because $\varepsilon$-terms are usual terms.

## D.18. Intuitive interpretation

Kind of Henkin witnesses but actually there is no good interpretation that would entail completeness.

Here is a pleasant intituitive interpretation rule due to von Heusinger: both “*a*” and “*the*” are interpreted by the an epsilon term, but the “*a*” always refers to a **new** individual in the class, while “*the*” refers to the most **salient** one.

(20) A student entered the lecture hall. He sat down. A student left the lecture hall.

(21) A student arrived lately. The professor looked upset. The student left.

## D.19. Categorical model

[Sorry for giving little details, this construction is ”heavy” category theory]

Very recently Fabio Pasquali proposed a categorical model: an epsilon logic/language can be interpreted in a Boolean hyper doctrine with a specific property (corresponding to the Axiom of Choice).

Formulae, terms and proofs are all interpreted by arrows. The $\exists x F$ arrow correspond to the $\varepsilon$-term arrow.

Such an hyper doctrine can be constructed from any elementary Topos enjoying the Axiom of choice.

It is important to have a many sorted logic defined with types for his construction.

## D.20. Epsilon/tau A E I O formulae

- All students passed.

$$A_{P,Q} = \forall x.(P(x) \Rightarrow Q(x))$$

- No student passed.

$$E_{P,Q} = \forall x.(P(x) \Rightarrow \neg Q(x))$$

- Some student passed.

$$I_{P,Q} = \exists x.(P(x) \wedge Q(x))$$

- Not all student passed. (original phrasing)
  Some students did not pass. (different focus, but less ambiguous)

$$O_{P,Q} = \exists x.(P(x) \wedge \neg Q(x))$$

# D.21. Epsilon E I A O formulae

Consider the epsilon versions of I and A:

$$I_{S,P} := S(\varepsilon_x P(x))$$

$$A_{S,P} := S(\tau_x P(x))$$

Hence we have no choice for the E and O:

$$E_{S,P} := \neg I_{S,P} = \neg S(\varepsilon_x P(x))$$

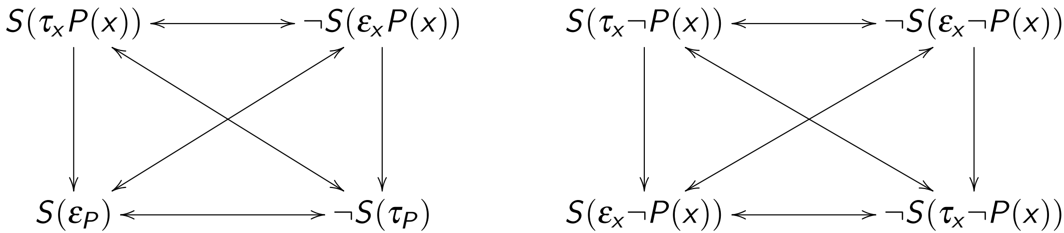
$$O_{S,P} := \neg A_{S,P} = \neg S(\tau_x P(x))$$

As we have seen earlier $I_{S,P} \equiv S(\varepsilon_x P(x))$ is not always equivalent to $\exists x.P(x) \& S(x)$

## D.22. Hilbertian square of opposition

Let $\mathscr{S}(S, P)$ be the square obtained with the following figures: $A_{S,P}$, $I_{S,P}$, $E_{S,P}$ and $O_{S,P}$.

In the Hilbert's $\varepsilon$-calculus, for every formulas $S(x)$ and $P(x)$ (one free variable), either $\mathscr{S}(S, P)$ or $\mathscr{S}(S, \neg P)$ is a square of opposition provided $S(\tau_x P(x)) \vdash S(\varepsilon_x P(x))$ or $S(\tau_x \neg P(x)) \vdash S(\varepsilon_x \neg P(x))$.



## D.23. Typed Hilbert operators

If single sorted logic, Frege / Montague style: $\varepsilon : (\mathbf{e} \to \mathbf{t}) \to \mathbf{e}$ if many sorted: $e_1, ..., e_n$ as base types, possibly subtyping inclusions between base types. With types variables $(\alpha, \beta, ....)$, quantification over type variables: $\Pi\alpha. T[\alpha]$.

$\varepsilon^* : \Pi\alpha.\ \alpha$ or $\varepsilon : \Pi\alpha.\ (\alpha \to \mathbf{t}) \to \alpha$ ???

Either type/formula entails the other: $\varepsilon^* = \varepsilon\{\Lambda\alpha.\alpha\}(\lambda x^{\Pi\alpha.\alpha}.\ x\{\mathbf{t}\}) : \Lambda\alpha.\ \alpha$ and $\varepsilon = \varepsilon^*\{\Lambda\alpha.\ (\alpha \to \mathbf{t}) \to \alpha\}$

$\varepsilon$ is more general because types can be mirrored as predicates, but not the converse.

There is no problem of consistency with such constants whose type in unprovable (like fix point $Y$).

## D.24. Intuitive interpretation and logic: some perspectives

**Cohabitation of types and formulae of first/higher order logic:**

Typing ($\sim$ presupposition) is irrefutable $\mathit{sleeps}(x : \mathit{cat})$

Type to Formula:

type *cat* mirrored as a predicate $\widehat{\mathit{cat}} : \mathbf{e} \to \mathbf{t}$

Formula to Type?

Formula with a single free variable $\sim$ type?

$\mathit{cat}(x) \wedge \mathit{belong}(x, \mathit{john}) \wedge \mathit{sleeps}(x) \sim$ type?

At least it is not a natural class.

# D.25. Computing the proper semantics reading

A cat. cat$^{animal \to \mathbf{t}}$ $(\varepsilon\{animal\} cat^{animal \to \mathbf{t}}) : animal$

Presupposition $F(\varepsilon_x F(x))$ is added: $cat(\varepsilon\{animal\} cat^{animal \to \mathbf{t}})$

For applying $\varepsilon$ to a type say $cat$,
any type has a predicative counterpart $cat$ (type) $\widehat{}[cat] : \mathbf{e} \to \mathbf{t}$.
(domains can be restrained / extended)

# D.26. Avoiding the infelicities of standard Montague semantics

$\varepsilon_x F(x)$ : individual.

1. Can be interpreted as an individual without the main predicate:
   it is a term.
2. Follows syntactical structure:
   it is a term, the semantics of an NP.
3. Asymmetry subject/predicate:
   $P(\varepsilon Q) \not\equiv Q(\varepsilon P)$.

## D.27. E-type pronouns

$\varepsilon$ solves the so-called E-type pronouns interpretation (Gareth Evans) where the semantic of the pronoun is the copy of the semantic of its antecedent:

(22) A man came in. He sat dow.

(23) $"He" = "A\ man" = (\varepsilon_x\ M(x))$.

## D.28. Difference with choice functions

Choice functions, Skolem symbols:

- One per formula: given one formula one enrich the formal language with a new function symbol and usually, there are no function symbols, when interpreting natural language: as a dictionary, the logical lexicon should be finite.
- No specific deduction system.
- The symmetry problem is still there: it does not go beyond classical logic and the E sentences are still improperly symmetric.
- choice function are not syntactically defined they have to be added one by one in the FOL language.

## D.29. Universal quantification

Observe that our setting allow two ways to do so (as for the epsilon):

if the noun is a type, the operator should apply to a type and yields an object of this type: $\Pi\alpha.\ \alpha$

when it is a property the type is $\Pi\alpha.\ (\alpha \to \mathbf{t}) \to \alpha$

## D.30. Scope ambiguity with $\varepsilon, \tau$

Ongoing work by William Babonnaud MPRI CS master internship, Paris.

(24) Every student has a number (exam sit / social category).

(25) Every Parisian/French enjoys leaving in a beautiful town.

(26) A corridor lead to the bedrooms.

(27) A guard stands in front of the museum doors.

With esplion/tau:

(28) a. The children ate a pizza

b. $ate(\tau x.\ children(x), \varepsilon x.\ pizza(x))$

Underspecification: ambiguous relation to $\forall child \exists pizza\ ate(child, pizza)$ and $\exists pizza \forall child\ \ ate(child, pizza)$

## D.31. Subnectors and generalised quantifiers

Formally (in the logical syntax) nothing prevent to introduce sub- nectors for complex quantifiers like 2/3 or few or many etc.

# E Conclusion

## E.1. Some interesting topics that were left out

Interaction of scope phenomena:

- Negation (Certains enfants n’ont pas peur des chiens.)
- Pronouns (as in donkey sentences).
- Belief verbs and de re / de dicto readings (James Bond believes that some department member is a spy.)

## E.2. Some logical questions

We focused on the computational analysis of sentences with quantifiers, what about the proper wording of quantified statements e.g. in maths (which makes an extensive use of universal quantifiers).

Quantification over individual concepts / quantifying over individuals. (in case you do not believe that Tullius is Cicero)

Quantifying in type theory.

Models and proof systems for quantified logic.

Argumentation (proofs/refutations) and quantification (ludics, dialogical logic)