Refinement of universel quantification in Proof Theory

Christophe Fouqueré
Université Paris 13, Sorbonne Paris Cité,
LIPN, F–93430, Villetaneuse, France.
CNRS, UMR 7030, F–93430, Villetaneuse, France.
cf@lipn.fr

Myriam Quatrini
IMM –UMR 7373, Université Aix-Marseille
CNRS, F–13284 Marseille Cedex 07, France
Myriam.Quatrini@univ-amu.fr

Abstract

In Logic, first order quantifiers have a quite standard treatment. Nevertheless, some refinements already exist: the existential quantification is not the same in classical and in intuitionnistic Logic. More recently, a new logical theory, called Ludics, coming from proof theory and its connections with computation theory seems to open new possibilities in order to refine universal quantification. In this framework we can distinghish between quantification due to a uniform behaviour of the elements of a given domain, and quantification as a generalized conjunction indexed on the domain. Such a refinement of the first order universal quantifier seems to be useful for refining the linguistic treatment of quantifiers in natural languages.

1 Introduction

Proof theory may be seen at the interface between, on one side, game theory: formal proofs are strategies, on the other side, functional programming: formal proofs are programs. Therefore, the notion of interaction is at the core of Proof Theory, either as the communication between players in Game Theory, or as the application of a program on an argument in functionnal programming. Ludics, as defined by Girard (2001), is a theory of interaction: its primitive notion is interaction itself both as communication and as rewriting process. This is made possible by the definition of new objects called *designs*. Roughly speaking, designs are formal proofs viewed according to a bottom/up reading (from conclusion to hypothesis). But designs are not really proofs since, 1. inside designs formulae disappear and are replaced by addresses, 2. designs may be infinite. It is only once designs are defined that we may recover logical formulae as types, i.e. closed sets of designs, and formal proofs are retrieved as designs in a type having additional properties. We show in this paper that first order formulae may be interpreted in two directions. We finally consider that this may shed light on the interpretation of quantification in natural language.

2 Quantifiers in Ludics

Ludics is a fully complete model for second-order multiplicative additive Linear Logic[1]. First-order was not considered. However, as we shall see later, first-order may be recovered in two ways. To understand why this can be possible, a few technical details of Ludics should be given. First of all, designs are forests of *actions*, actions being localized: an action has an address. Designs may be considered isomorphic with respect to delocalization,i.e. being given a function from addresses to addresses. Furthermore, such a function can be viewed itself as a design. This is a way to interpret the identity axiom: a *fax* delocalizes a design into another one, letting unchanged the frame of the design. Second, as mentioned in the introduction, a formula is interpreted as a closed set of designs, hence the universal quantifier may be interpreted as a closed set of designs pairwise isomorphic. Another feature of Ludics is that it models additivity: the closed union of closed sets of designs is in fact the direct sum (on a domain) of formulas.

With these properties of Ludics, we are able to define two kinds of quantifiers. Fleury and Quatrini (2004) proposed an interpretation of first order universal quantified formulae in Ludics. The crucial point was to characterize properties of the designs which would be relevant for being interpretation of formal proofs of universally quantified formula. This is done in the following way: being given a family of designs, indexed on a given domain, this family models a universally quantified formula, still denoted by $\forall x F(x)$, if this family is *uniform*: all the designs in such a family are, in some way, the same, that is equal up to delocalisation. In that way, a proof of the universally quantified formula is modeled by a (daimon-free) design in the family. Such an interpretation of first-order universality means that in proving $F(x)$, x is only a parameter, that does not change the shape of the proof.

Besides such an interpretation of a universally quantified formula, another interpretation is available, denoted by $\&_x F(x)$. In this case, the universally quantified formula $\&_x F(x)$ is interpreted as a generalised additive conjunction on a given domain. This is possible because Ludics accepts infinite (additive) conjunctions. The difference of this latter formula, with respect to the formula $\forall x F(x)$, is that it is obtained by a family of proofs of $F(x)$ which do not have to have the same shape: the shape of a proof of $F(x)$ crucially depends on the parameter x.

3 Applications to Natural Language Studies

The refinement of the first order universal quantifier mentioned above may be useful for refining the linguistic treatment of quantifiers in natural languages. Possible applications are here only sketched and remain to be precisely studied. Therefore, examples given below are just devoted to illustrate such open tracks.

The determiner *all the* seems requiring both ludical quantifiers $\forall x$ and $\&_x$, according to the utterances. For example,the logical meaning of the determiner is not the same in the following utterances in which it occurs:

(1) *All man who is authorized to enter in the academy is a geometrician.*

(2) *All man who is present in the cafeteria today is a logician.*

In (1), the authorization is given *under the same circumstances* for each man. In (2), the fact that each man in the cafeteria is a logician depends specifically on each man considered: the

[1] Extensions of Ludics have been developed to consider also exponentials. However we will not take care of this in this paper as it is not relevant for our purposes.

fact that a man is a logician is true or false, however the proof that a specific man is a logician depends on this man and his characteristics.

Several tools may be used for grasping such a difference. For example, we may observe that the second utterance needs a context while the first one does not (as it is also said in Mari and Rétoré (2016)). We may also observe that both have distinct inferential effects. The addition of an utterance like *Alex is entered* enables to derive *Alex is a geometrician* but does not enable to derive *Alex is a logician*.

The ludical interpretation of *All* is then quite immediate: in (1), *All* requires $\forall$, whereas in (2), *All* requires &. In the two cases the domain is the family of all men.

The refinement of universal quantification in Ludics enables not only to distinguish logical forms but also to distinguish inferential behaviours of the two universal quantifiers. This arises as a result of interaction being at the core of Ludics. By interaction, computations for formulas typed with & are fundamentally different from computations for formulas typed with $\forall$. In the first case, the computation is dependent on the parameter, whereas this is not the case with $\forall$.

Being able to distinguish the two interpretations of *All* has always been a challenge in classical logic: there is only one classical quantifier! Ludics offers means for differentiating the two universal quantifiers in Natural Language, without limiting inferentiality.

References

Fleury, M.-R. and M. Quatrini (2004). First order in ludics. *Mathematical Structures in Computer Science 14*(2), 189–213.

Girard, J.-Y. (2001). Locus solum: From the rules of logic to the logic of rules. *MSCS 11*(3), 301–506.

Mari, A. and C. Rétoré (2016). Conditions d'assertion de «chaque» et de «tout» et règles de déduction du quantificateur universel. *Revue internationale de linguistique française 72*, 89–106.