Algorithms for classes of graphs with bounded expansion

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Overview of the talk

- Classes of sparse graphs
  tree-width, tree-depth, graphs with bounded expansion
- Structural properties of graphs with bounded expansion
  orientations, coloring resuls, short-path queries
- Local parameters
  locally bounded tree-width, locally bounded expansion
- Deciding graph properties
  deciding MSOL and FOL properties for classes of sparse graphs
- Extensions to relational structures

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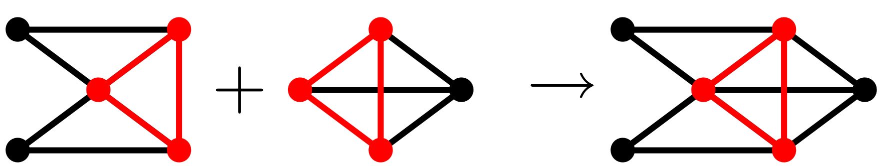
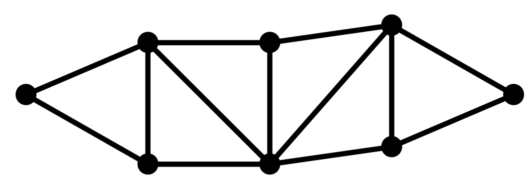
Tree-width

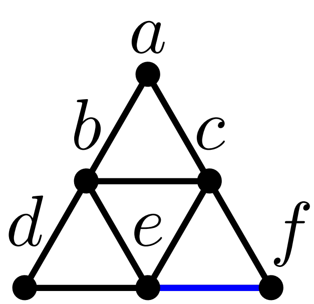
- a *k-tree* is obtained from K_{k+1} by gluing along cliques of order k
- a *partial k-tree* is a subgraph of k-tree
- the *tree-width $\mathrm{tw}(G)$* of a graph G is the smallest k such that G is a partial k-tree
  observe that $\omega(G) \leq \mathrm{tw}(G) + 1$
- every forest has tree-width one
  every outer-planar graph has tree-width two

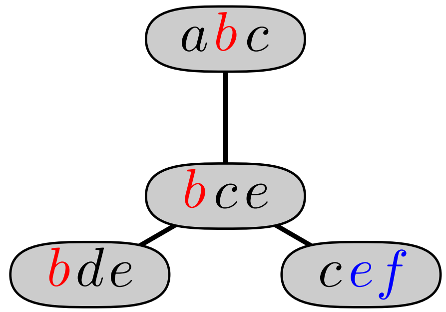




Tree-decomposition

- decomposition tree with nodes, each has a bag of vertices
- bags containing a single vertex form a subtree
- for each edge, there is a bag containing its both end-vertices

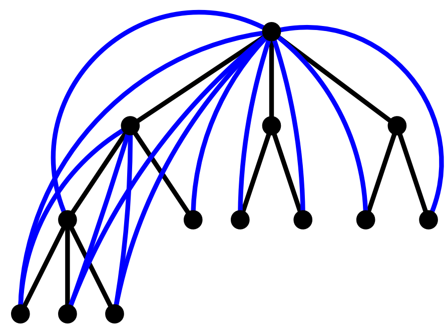




- the width of a decomposition is its maximum bag size minus one
- the tree-width of a graph is the smallest width of its decomposition
- computable in linear time [Bodlaender, 1996]

Tree-depth

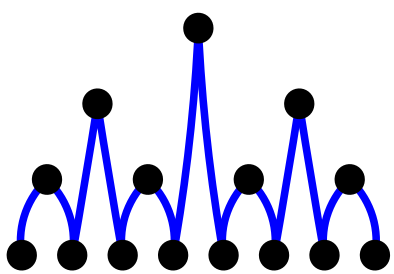
- transitive closure $\overline{T}$ of a tree T
- the tree-depth $\text{td}(G)$ of G is the smallest depth of a tree T such that G is a subgraph of $\overline{T}$



- observe that $\text{tw}(G) \leq \text{td}(G) - 1$
- the ranking number of a graph
  the minimal number of colors such that every path joining two vertices of color i contains a vertex of color bigger than i

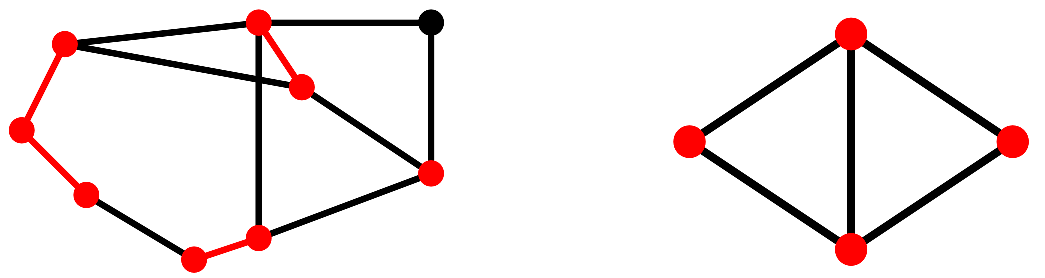
Tree-width vs. tree-depth

- tree-depth is not bounded by a function of tree-width
  $\mathrm{td}(P_n) = \lceil \log_2(n+1) \rceil$
- bounded by a function of the tree-width and the order
  $\mathrm{td}(G) \leq (\mathrm{tw}(G)+1)\log_2 n$
- related to the length of the longest path
  $\mathrm{td}(G) \leq \ell(G) \leq 2^{\mathrm{td}(G)}$
- other characterizations of tree-depth by separations



Graph minors, shallow minors

- a minor of a graph is obtained by contracting edges and removing vertices and/or edges
- alternatively, remove some vertices and edges and then contract connected subgraph to vertices



- d-shallow minor if the radii of contracted subgraphs are at most d

Minor-closed classes of graphs

- a graph class is minor-closed if it contains minors of all its members
  classes of graphs embeddable on a fixed surface
  classes of graphs with bounded tree-width/tree-depth
- graph minor series by Robertson and Seymour
- every minor-closed class $\mathcal{G}$ has a finite list of obstructions,
  i.e., a graph $G \in \mathcal{G}$ iff none of $H_1, \ldots, H_k$ is a minor of G
- testing the existence of a minor is polynomial time solvable
- structural characterization of minor-closed classes of graphs
  through clique-sums of graphs almost embedded on surfaces

Classes of graphs with bounded expansion

- introduced by Nešetřil and Ossona de Mendéz in 2006
- maximum average degree

$$\mathrm{mad}(G) = \max_{H \subseteq G} \frac{||H||}{|H|}$$

- let $\mathcal{G}\nabla d$ be the set of all d-shallow minors of $G \in \mathcal{G}$

$$\nabla_d(\mathcal{G}) = \max_{G \in \mathcal{G}\nabla d} \mathrm{mad}(G)$$

grad (greatest reduced average density) with rank d of $\mathcal{G}$

- a class $\mathcal{G}$ has bounded expansion if $\nabla_d(\mathcal{G})$ is finite for every d

Examples

- proper minor-closed classes of graphs
  every minor-closed class of graphs is degenerate e.g., graphs on surfaces, graphs with bounded tree-width
- classes of graphs with bounded maximum degree
  the maximum degree of a d-shallow minor is at most Δ^d
- proper topologically closed classes of graphs
  graph classes excluding certain graphs as subdivisions

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Fraternal orientations

- consider a class $\mathcal{G}$ of graphs with bounded expansion
- every graph $G \in \mathcal{G}$ has an orientation with bounded in-degrees
- fraternal augmentation

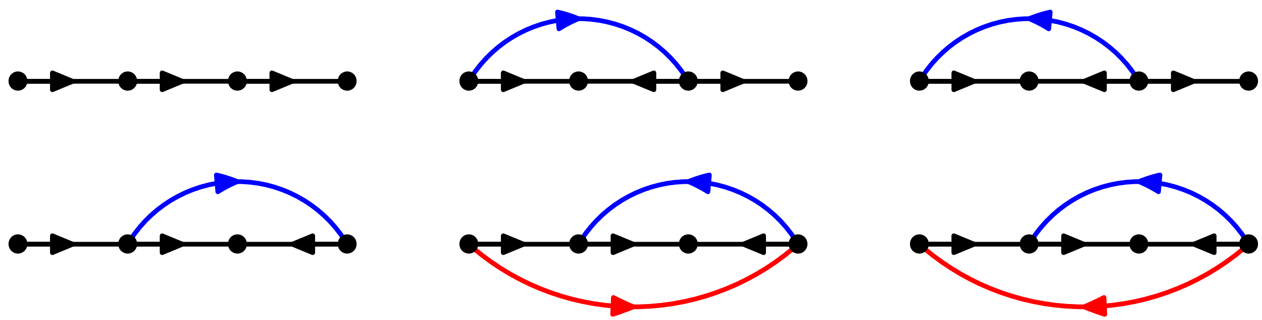
  transitive closure: $ab \wedge bc \Rightarrow ac$

  fraternal closure: $xz \wedge yz \Rightarrow xy$

- the class of augmented graphs has again bounded expansion
- the process can be iterated

Short path queries

- based on previous ideas, cf. Kowalik and Kurowski (2004)
- linear time precomputation, constant time query



Dynamic short path queries

- the algorithm of Brodal and Fagerberg (1999) as black-box orientations with maximum in-degree at most $D(d)$ for graphs G with $\mathrm{mad}(G) \leq d$ can be mainted in time $O(\log n)$ for edge insertions and $O(1)$ for edge deletions

- dynamic ℓ-path query data structure for classes of graphs with bounded expansion
$O(n)$ build time, $O(1)$ query time
$O(\log^{\ell} n)$ insertion time and $O(1)$ deletion time

Bounded tree-depth coloring

- since $\nabla_0(G) = \mathrm{mad}(G)$ for $G \in \mathcal{G}$ is bounded, the chromatic number of all graphs in $\mathcal{G}$ is bounded
- the chromatic number remains bounded for augmentations!
- bounded tree-depth coloring [Nešetřil and Ossona de Mendéz, 2006] For every class $\mathcal{G}$ of graphs with bounded expansion and every k , there exists K such that every $G \in \mathcal{G}$ has a vertex coloring with K colors such that every union of $i \leq k$ color classes induces a subgraph of tree-depth at most i.
- generalization of tree-width coloring for minor-closed classes of graphs of DeVos, Ding, Oporowski, Sanders, Reed, Seymour and Vertigan (2004)

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Local parameters

- a class $\mathcal{G}$ of graphs has locally bounded tree-width, if there exists f such that the d-neighborhood of every vertex of $G \in \mathcal{G}$ has tree-width at most $f(d)$
  In other words: the classes $\mathcal{G}_d$ formed by d-neighborhoods of vertices of graphs in $\mathcal{G}$ have bounded tree-width
- $\mathcal{G}$ locally excludes a minor, if each $\mathcal{G}_d$ has a forbidden minor
- $\mathcal{G}$ has locally bounded expansion, if each $\mathcal{G}$ has bounded expansion
- introduced by Eppstein (2000), Dawar, Grohe and Kreutzer (2007), Nešetřil and Ossona de Mendéz (2008), respectively

Trichotomy of graph classes

- Trichotomy theorem [Nešetřil, Ossona de Mendéz, 2008]

$$\lim_{r\to\infty} \limsup_{G\in\mathcal{G}\nabla d} \frac{\log ||G||}{\log |G|} \in \{0,1,2\}$$

  for every infinite class $\mathcal{G}$ of graphs.

- for graph classes with bounded expansion, the limit is always one
- if the limit is zero, then graphs in $\mathcal{G}$ have bounded number of edges
- if the limit is two, then $\mathcal{G}\nabla D$ contains all graphs for some D
- classes $\mathcal{G}$ with the limit equal to one are said to be nowhere dense

Nowhere dense graph classes

$$\lim_{r\to\infty} \limsup_{G\in\mathcal{G}\nabla d} \frac{\log ||G||}{\log |G|} = 1$$

- a lot properties similar to graph classes with bounded expansion
- their grads are bounded by $O(n^{\varepsilon})$for every $\varepsilon > 0$
- they have augmentations with maximum in-degrees $O(n^{\varepsilon})$
- they have low tree-depth colorings with $O(n^{\varepsilon})$ colors
- classes of locally nowhere dense graphs are again nowhere dense

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MSOL PROPERTIES

- monadic second order logic (MSOL) formula
  quantification over elements and their sets
  3-coloring:

$$\exists V_1, V_2, V_3, V_1 \cup V_2 \cup V_3 = V \text{ such that } \forall uv \in E \wedge_{i=1}^{3} \{u, v\} \not\subseteq V_i$$

- Theorem of Courcelle (1990)
  For every class of graphs with bounded tree-width,
  every MSOL property can be decided in linear time.

- extensions to counting, etc.

FOL properties

- first order logic (FOL) formula
  quantification only over elements $\Rightarrow$ polynomial time decidable
  example: a dominating set of at most 7 vertices, subgraph testing
- almost-linear time algorithm for classes of graphs with locally bounded tree-width [Frick and Grohe, 2001]
  for every $\varepsilon > 0$, there exists an algorithm running in time $O(n^{1+\varepsilon})$
- fixed parameter algorithms for minor-closed classes of graphs and graphs locally excluding a minor [Dawar, Grohe, Kreutzer, 2007]

Our algorithms

- For every FOL formula φ and every class $\mathcal{G}$ of graphs with bounded expansion, there is a linear time algorithm deciding φ.

- For every FOL formula φ and every nowhere dense class $\mathcal{G}$ of graphs, there is an almost-linear time algorithm deciding φ.

joint work with Zdeněk Dvořák and Robin Thomas

Main tools for the algorithm

- Gaifman's theorem [1982]
  Every FOL formula is equivalent to a boolean combination of formulas of the following type:

  $$\exists x_1, \ldots, x_k \, (\wedge_{i,j} d(x_i, x_j) > 2r \ \ \wedge \ \ \wedge_i \varphi_i [N_r(x_i)] \,)$$

  locally decidable formulas

- evaluating formulas with free variables
  dynamic programming based on low tree-depth colorings
  with predicates relating the mutual position of free variables and their colors

Extension to relational structures

- for a relational structure (X,R), the Gaifman graph is a graph with vertex set X where $x, y \in X$ are adjacent if there are in a common relation in $\mathcal{R}$
- class of relational structure has bounded expansion / is nowhere dense if the class of their Gaifman graphs has bounded expansion / is nowhere dense
- structural results presented in the talk extend to this setting our FOL algorithms work for classes of relational structures with bounded expansion as well as those nowhere dense

Thank you for your attention!