Cycles, Paths, Connectivity, and Diameter in Distance Graphs

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Circulant Graph

For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the circulant graph C_n^D has vertex set

$$V\left(C_n^D\right) = [0, n-1] = \{0, 1, \ldots, n-1\}$$

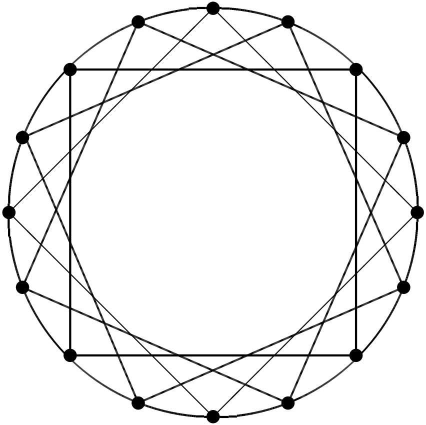
and edge set

$$E\left(C_n^D\right) = \{ij \mid i, j \in [0, n-1], \exists d \in D : j = (i \pm d) \bmod n\}.$$

We may always assume that $\max(D) \leq \frac{n}{2}$.

Circulant Graph

$n = 16,\ D = \{1, 4\}$



Circulant Graph

Because of their simplicity, extendability, regularity, reliability,...
the circulant graphs are interesting for many applications :

- Interconnection networks,
- local area computer networks,
- large area communication networks,
- parallel processing architectures,
- distributed computing,
- VLSI design,
- ...

Circulant Graph

Some surveys:

- Bermond et al., *Distributed Loop Computer Networks*
- Hwang, *A survey on multi-loop networks*
- Liu, *Distributed Loop Computer Networks*

The mathematical properties of circulant graphs are interesting and (very) well-studied.

- Cayley graphs of cyclic groups
- Cycles and paths
- Connectivity and diameter
- Isomorphism testing and recognition
- ...

Distance Graph

For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the distance graph P_n^D has vertex set

$$V\left(P_n^D\right) = [0, n-1]$$

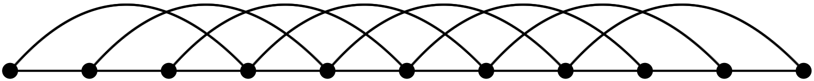
and edge set

$$E\left(P_n^D\right) = \{ij \mid i, j \in [0, n-1], j - i \in D\}.$$

We may always assume that $\max(D) \leq n - 1$.

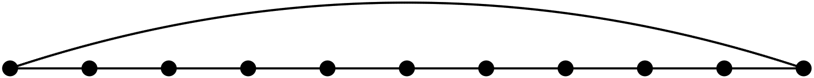
Distance Graph

$n = 11, \ D = \{1, 3\}$



Distance Graph

$n = 11,\ D = \{1, 10\}$



Distance Graph

$$P_n^D \subset C_{n+\max(D)}^D$$

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Proposition

The circulant graphs are exactly the regular distance graphs.

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Proof: "$\Rightarrow$"

$$C_n^D \cong P_n^{D \cup \{n-d \mid d \in D\}}.$$

$$P_n^D \subset C^D_{n+\max(D)}$$

Proposition

The circulant graphs are exactly the regular distance graphs.

(Skip proof?)

Proof: "$\Rightarrow$"

$$C_n^D \cong P_n^{D\cup\{n-d|d\in D\}}.$$

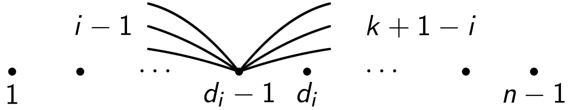
"$\Leftarrow$" Let P_n^D be a regular distance graph.
Let $D = \{d_1, d_2, \ldots, d_k\}$ be such that

$$1 \leq d_1 < d_2 < \ldots < d_k \leq n-1.$$

$\Rightarrow$ P_n^D is k-regular.

Distance Graph

Let $i \in [1, k]$.



$\Rightarrow (d_i - 1) + d_{k+1-i} \leq n - 1$
$\Rightarrow d_i + d_{k+1-i} > n - 1$
$\Rightarrow$

$$d_i + d_{k+1-i} = n \;\forall\; i \in [1, k]$$

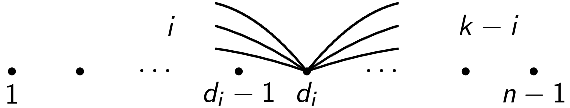
$\Rightarrow$

$$P_n^D \cong C_n^{\left\{d \in D | d \leq \frac{n}{2}\right\}}.$$

$\square$

Distance Graph

Let $i \in [1, k]$.



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$$d_i + d_{k+1-i} = n \;\forall\; i \in [1, k]$$

$\Rightarrow$

$$P_n^D \cong C_n^{\left\{d \in D \mid d \leq \frac{n}{2}\right\}}.$$

□

Distance Graph

We want to extend fundamental results concerning circulant graphs to distance graphs.

Until now mainly coloring problems were considered starting with Eggleton, Erdős, and Skilton who studied colorings of (infinite) distance graphs.

We will present results concerning:

- Long cycles and paths
- Connectivity and diameter
- Recognition

Long Cycles and Paths

Combining results due to Boesch and Tindell, Burkard and Sandholzer, and Garfinkel, we obtain the following.

Theorem (BT, BS, G)

For $n \in \mathbb{N}$ and a finite set $D \subseteq \mathbb{N}$, the following statements are equivalent.

(i) C_n^D *is connected.*

(ii) $\gcd(\{n\} \cup D) = 1$.

(iii) C_n^D *has a Hamiltonian cycle.*

Long Cycles and Paths

Theorem

For a finite set $D \subseteq \mathbb{N}$, the following statements are equivalent.

(i) *There is a constant $c_1(D)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a component of order at least $n - c_1(D)$.*

(ii) $\gcd(D) = 1$.

(iii) *There is a constant $c_2(D)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a cycle of order at least $n - c_2(D)$.*

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Proof Sketch:

Long Cycles and Paths

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Proof Sketch:
(i) $\Rightarrow$ (ii): $\exists\ i$: i and $i+1$ are in one component.
(iii) $\Rightarrow$ (i): trivial.

Long Cycles and Paths

(ii) $\Rightarrow$ (iii)

Long Cycles and Paths

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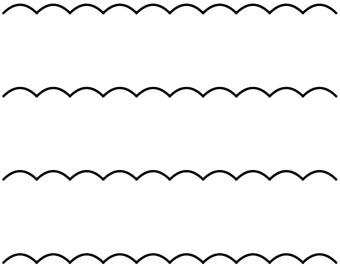

Long Cycles and Paths

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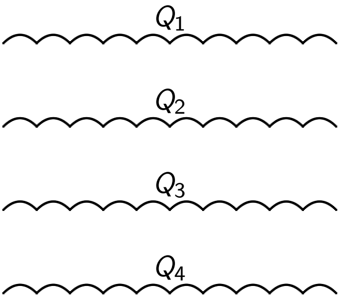

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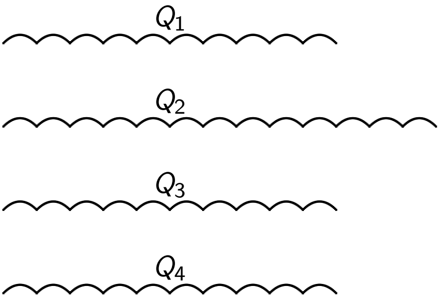

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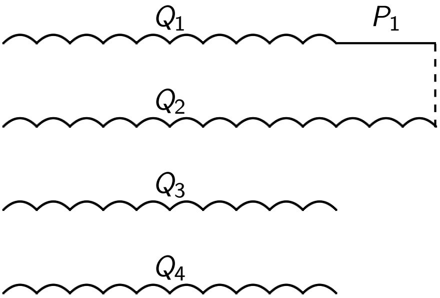

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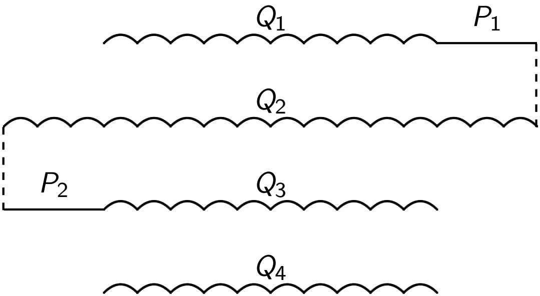

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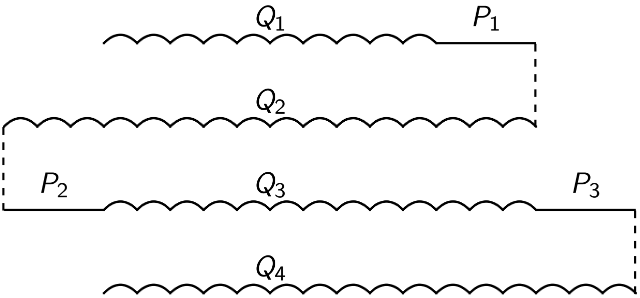

Long Cycles and Paths

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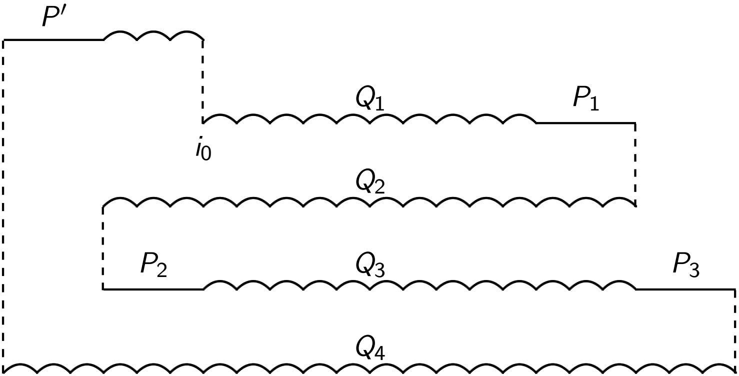

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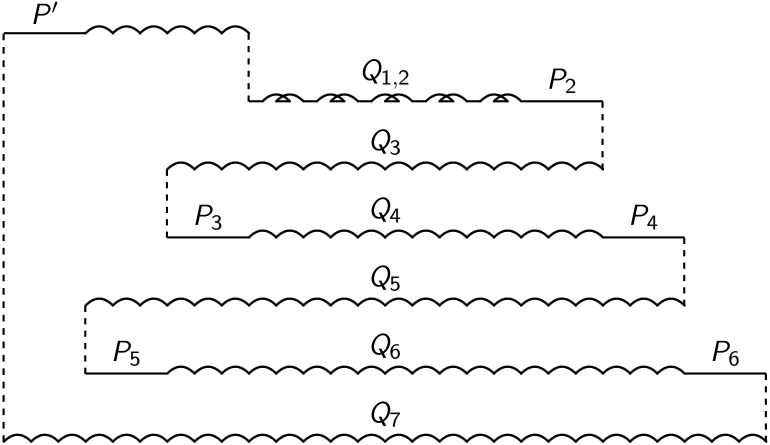

$\square$

Long Cycles and Paths

If all elements in $\{n\} \cup D$ are odd, then P_n^D is bipartite and every cycle misses at least one vertex.

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Conjecture

For a finite set $D \subseteq \mathbb{N}$, the following statements are equivalent.

(i) $\gcd(D) = 1$.

(ii) *There are two constants $c_3(D)$ and $c_4(D)$ such that for every $n \in \mathbb{N}$, the distance graph P_n^D has a path*

$$u_0 u_1 \dots u_l$$

of order at least $n - c_3(D)$ such that

$$u_j > u_i \text{ for all } 0 \leq i, j \leq l \text{ with } j - i \geq c_4(D).$$

- For every $\epsilon > 0$, there is such a path of order at least

$$(1-\epsilon)n - c_5(D,\epsilon).$$

- If $d_1, d_2 \in \mathbb{N}$ are such that $\gcd(\{d_1, d_2\}) = 1$, then

$$P_{d_1+d_2+1}^{\{d_1,d_2\}}$$

  has a Hamiltonian path which begins at 0 and ends at $d_1 + d_2$.
- If $D = \{6, 10, 15\} = \{2 \cdot 3, 2 \cdot 5, 3 \cdot 5\}$, then

$$0, 6, 12, 2, 8, 14, 4, 10, 16, 1, 7, 13, 3, 9, 15, 5, 11, 17$$

  is a Hamiltonian path in P_{18}^{D} which begins at 0 and ends at 17.

Connectivity and Diameter

Proposition (G)

C_n^D *is connected if and only if* $\gcd(\{n\} \cup D) = 1$.

Connectivity and Diameter

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Proof: Since C_n^D is vertex-transitive, it is connected if and only if it contains a path from 0 to 1. This happens if and only if there are integers l and l_d for $d \in D$ such that $1 = ln + \sum_{d \in D} l_d d$. $\square$

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Problem (Connectivity of P_n^D)

Instance:	*$n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$.*
Question:	*Is P_n^D connected?*

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Instance:	*$n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$.*
Question:	*Is P_n^D connected?*

Conjecture

CONNECTIVITY OF P_n^D *is NP-hard.*

Connectivity and Diameter

Clearly, P_n^D is connected if and only if for every $i \in [0, n-2]$, there are integers $x_1, x_2, \ldots, x_l$ such that

$$\begin{aligned} |x_i| &\in D \text{ for all } i \in [1, l], && (1) \\ 1 &= \sum_{j=1}^{l} x_j, \text{ and} && (2) \\ i + \sum_{j=1}^{k} x_j &\in [0, n-1] \text{ for all } k \in [0, l]. && (3) \end{aligned}$$

(1) and (2) are easy. What about (3)?

Problem (Bounded Partial Sums)

Instance: $x_0, x_1, x_2, \ldots, x_l \in \mathbb{Z}$ *and* $n \in \mathbb{N}$.

Question: *Is there a permutation* $\pi \in S_l$ *such that*

$$x_0 + \sum_{j=1}^{k} x_{\pi(j)} \in [0, n-1]$$

for all $k \in [0, l]$*?*

Connectivity and Diameter

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for all $k \in [0, l]$*?*

Proposition

Bounded Partial Sums *is NP-complete.*

Theorem

Let $n, d_1, d_2 \in \mathbb{N}$ *be such that* $d_1 < d_2$*. For* $i \in [0, d_1 - 1]$*, let*

$$r_i = (id_2) \ mod \ d_1 \ and \ s_i = (n - 1 - r_i) \ mod \ d_1.$$

Furthermore, for $i^* \in [1, d_1 - 1]$*, let*

$$\begin{aligned} d_{i^*}^+ &= \max\{r_i \mid i \in [0, i^* - 1]\} \ and \\ d_{i^*}^- &= \max\left\{s_{-i \ mod \ d_1} \mid i \in [0, d_1 - i^* - 1]\right\}. \end{aligned}$$

Finally, let

$$d^* = \max_{i^* \in [1, d_1 - 1]} \min\{d_{i^*}^+, d_{i^*}^-\}.$$

$P_n^{\{d_1, d_2\}}$ *is connected if and only if* $\gcd(\{d_1, d_2\}) = 1$ *and*

$$d^* + d_2 \le n - 1.$$

Connectivity and Diameter

Theorem

Let $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$ be such $\max(D) \leq n-1$.
If $\max(D) \leq \frac{n-1}{2}$, then P_n^D is connected, if and only if $gcd(D) = 1$.

Connectivity and Diameter

Theorem

Let $n \in \mathbb{N}$ *and* $D \subseteq \mathbb{N}$ *be such* $\max(D) \leq n - 1$.
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▸ (Skip Proof?)

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If $\max(D) \leq \frac{n-1}{2}$*, then* P_n^D *is connected, if and only if* $gcd(D) = 1$*.*

(Skip Proof?)

Proof: Let $i \in [0, n-2]$. Let $n_d \in \mathbb{Z}$ for $d \in D$ be such that

$$1 = \sum_{d \in D} n_d d.$$

We assume that $l = \sum_{d \in D} |n_d|$ is smallest possible.

The following algorithm yields a path in P_n^D from i to $i+1$.

$j \leftarrow 0$;
while $j < l$ **do**
  if $\left(\left(i \leq \frac{n-1}{2}\right) \wedge (\exists d \in D : n_d > 0)\right) \vee (\nexists d \in D : n_d < 0)$ **then**
    Choose $d \in D$ with $n_d > 0$;
    $i \leftarrow i + d$;
    $n_d \leftarrow n_d - 1$;
  else
    Choose $d \in D$ with $n_d < 0$;
    $i \leftarrow i - d$;
    $n_d \leftarrow n_d + 1$;
  end
  $j \leftarrow j + 1$;
end

$\square$

Connectivity and Diameter

The exact calculation/minimization of the diameter of C_n^D is a difficult and well-studied problem even for $|D| = 2$.

Using Wong and Coppersmith's arguments (J. ACM, 1974), we obtain

$$\begin{aligned} \operatorname{diam}\left(P_n^D\right) &\geq \frac{1}{2}\left(|D|!n\right)^{\frac{1}{|D|}} - |D| \text{ and} \\ \operatorname{diam}\left(P_{d^k}^{\{1,d,\ldots,d^{k-1}\}}\right) &\leq k(d-1). \end{aligned}$$

Problem (SHORT PATH IN P_n^D)

Instance: $n \in \mathbb{N}$, $D \subseteq \mathbb{N}$ *and* $l \in \mathbb{N}$.

Question: *Is there some* $u \in [0, n-2]$ *such that* P_n^D *contains a path of length at most* l *between* u *and* $u+1$*?*

Connectivity and Diameter

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Theorem

Short Path in P_n^D *is NP-hard.*

Problem (SHORT PATH IN P_n^D)

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Theorem

SHORT PATH IN P_n^D *is NP-hard.*

(Skip Proof?)

Proof: Let $x_1, x_2, \dots, x_k \in \mathbb{Z}$ be an instance of PARTITION.
Let

$$\begin{aligned} l &= 2(k+1) \\ d &= 2(k+1)\max\left(\{|x_1|, |x_2|, \dots, |x_k|\}\right) + 1 \\ n &= 2d^{3k+4} + 1 \\ D &= \{x_{i,j} \mid ...\} \end{aligned}$$

(n, D, l) is a "yes"-instance of SHORT PATH IN P_n^D if and only if $x_1, x_2, \dots, x_k$ is a "yes"-instance of PARTITION.

Connectivity and Diameter

	d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	$\dots$	d^{3k+2}	d^{3k+3}
$x_{1,1}$		x_1	1	1								
$x_{1,2}$				1		1						
$x_{1,3}$			1		1							
$x_{1,4}$		x_1			1	1						
$x_{2,1}$		x_2				1	1					
$x_{2,2}$							1		1			
$x_{2,3}$						1		1				
$x_{2,4}$		x_2						1	1			
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$x_{k+1,1}$	1										1	1
$x_{k+1,2}$			1									1

$\square$

Recognition

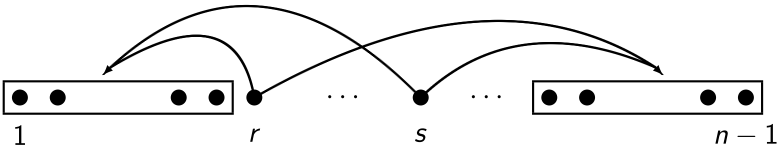
We only consider $D = \{d_1, d_2, \ldots, d_k\}$ with

$$1 = d_1 < d_2 < \ldots < d_k \leq \frac{n-1}{2}.$$

Proposition

If D is as above, then P_n^D has $2(d_{i+1} - d_i)$ vertices of degree $k + i$ for $0 \leq i \leq k-1$. Hence D is uniquely determined by the degree sequence of P_n^D.

An index of ambiguity of P_n^D.



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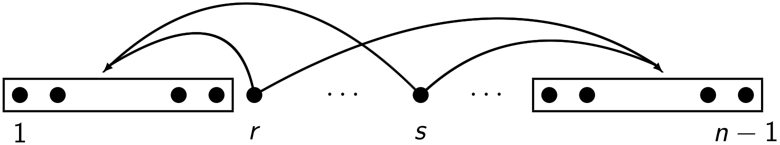

Theorem

Let D be as above. If P_n^D has l indices of ambiguity, then D and the ordering of its vertices can be obtained from P_n^D given up to isomorphism in time $O(4^l n^2)$.

Thank you for your attention!