Graph Partitioning and Traffic Grooming with Bounded Degree Request Graph

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Outline of the talk

1. Motivation: traffic grooming
2. Statement of the problem
3. The parameter $M(C, \Delta)$
4. Previous work (Muñoz and S., WG 2008)
5. Our results
   - Case $\Delta = 3$, $C = 4$
   - Case $\Delta \geq 4$ even
   - Case $\Delta \geq 5$ odd
   - Improved lower bound when $\Delta \equiv C \pmod{2C}$
6. Conclusions

Next section is...

Introduction

- WDM (Wavelength Division Multiplexing) networks
  - 1 wavelength (or frequency) = up to 40 Gb/s
  - 1 fiber = hundreds of wavelengths = Tb/s
- Idea:
  Traffic grooming consists in packing low-speed traffic flows into higher speed streams

  $\longrightarrow$ we allocate the same wavelength to several low-speed requests (TDM, Time Division Multiplexing)
- Objectives:
  - Better use of bandwidth
  - Reduce the equipment cost (mostly given by electronics)

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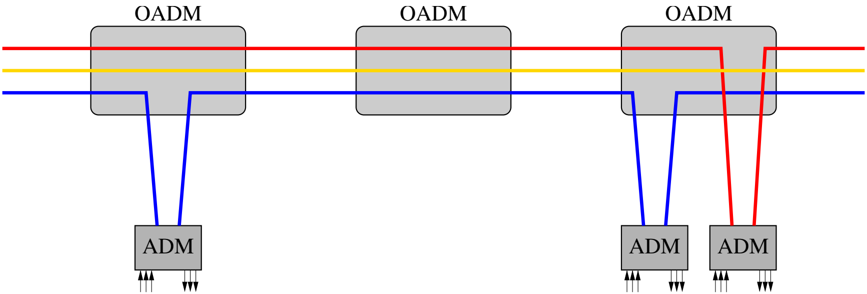
ADM and OADM

- **OADM** (Optical Add/Drop Multiplexer)= insert/extract a wavelength to/from an optical fiber
- **ADM** (Add/Drop Multiplexer)= insert/extract an OC/STM (electric low-speed signal) to/from a wavelength



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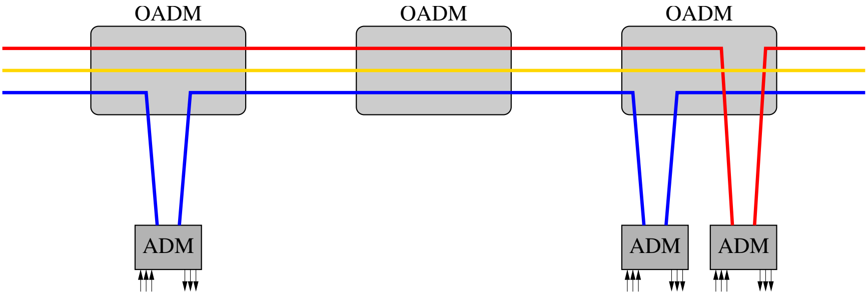

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Definitions

- **Request** (i, j): two vertices (i, j) that want to exchange (low-speed) traffic

- **Grooming factor C:**

$$C = \frac{\text{Capacity of a wavelength}}{\text{Capacity used by a request}}$$

Example:

Capacity of one wavelength $= 2400\ Mb/s$
Capacity used by a request $= 600\ Mb/s$ $\quad \Rightarrow \quad C = 4$

- **load** of an arc in a wavelength: number of requests using this arc in this wavelength ($\leq C$)

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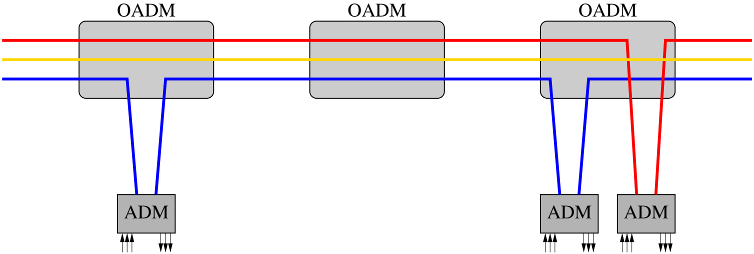

- **Idea:** Use an **ADM only at the endpoints of a request** (lightpaths) in order to save as many ADMs as possible

To fix ideas...

- Model:

Topology	$\longrightarrow$	graph G
Request set	$\longrightarrow$	graph R
Grooming factor	$\longrightarrow$	integer C
Requests in a **wavelength**	$\longrightarrow$	edges in a **subgraph** of R
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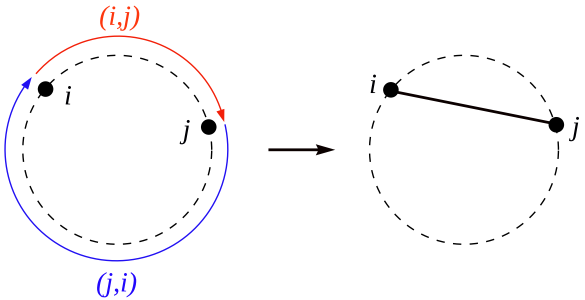
Unidirectional Ring with Symmetric Requests

- **Symmetric requests**: whenever there is the request (i, j), there is also the request (j, i).



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  $\rightarrow$ **grooming factor** $C \Leftrightarrow$ **each subgraph has** $\leq C$ **edges.**

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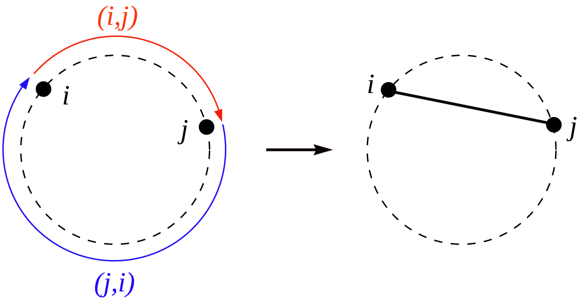

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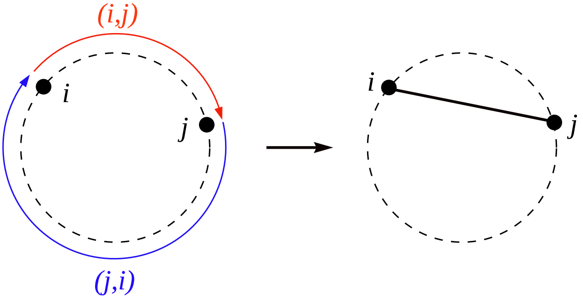

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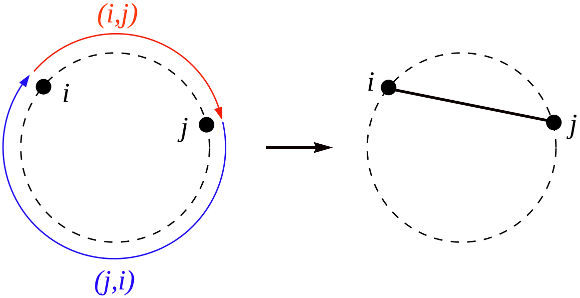

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Statement of the “old” problem

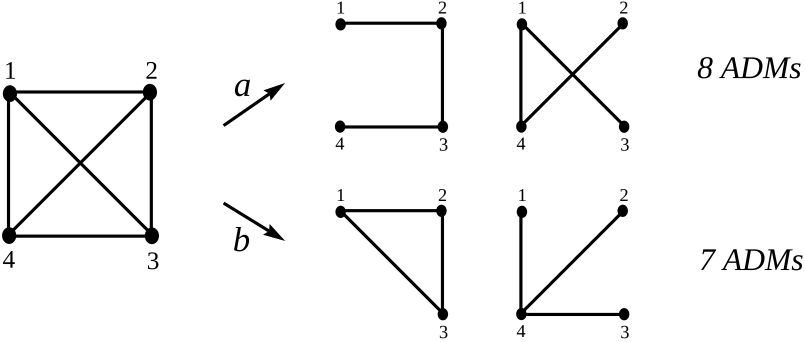
Traffic Grooming in Unidirectional Rings

Input A cycle C_n on n nodes (network);
An *undirected* graph R on n nodes (request set);
A grooming factor C.

Output A partition of $E(R)$ into subgraphs
$R_1, \ldots, R_W$ with $|E(R_i)| \leq C$, i=1...,W.

Objective Minimize $\sum_{\omega=1}^{W} |V(R_\omega)|$.

Example: $n = 4$, $R = K_4$, and $C = 3$



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New model [Muñoz and S., WG 2008]

- **Non-exhaustive** previous work (a lot!):
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- In all of them: place ADMs at nodes for a **fixed request graph**.
  → placement of ADMs **a posteriori**.

- **New model** [Muñoz and S., WG 2008]: place the ADMs at nodes such that the network can support **any request graph with maximum degree at most Δ.**
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Statement of the "new" problem

Traffic Grooming in Unidirectional Rings with Bounded-Degree Request Graph

Input An integer n (size of the ring);
An integer C (grooming factor);
An integer Δ (maximum degree).

Output An assignment of $A(v)$ ADMs to each $v \in V(C_n)$, in such a way that **for any graph** R on n nodes with **maximum degree at most** Δ, there exists a partition of $E(R)$ into subgraphs $R_1, \ldots, R_W$ s.t.:

(i) $|E(B_i)| \leq C$ for all $i = 1, \ldots, W$; and
(ii) each $v \in V(C_n)$ is in $\leq A(v)$ subgraphs.

Objective Minimize $\sum_{v \in V(C_n)} A(v)$,
and the optimum is denoted $A(n, C, \Delta)$.

Next section is...

$M(C, \Delta)$

Definition

Let $M(C, \Delta)$ be the smallest positive number M such that, for all $n \geq 1$, the inequality $A(n, C, \Delta) \leq Mn$ holds.

- Due to symmetry, it can be seen that $A(v)$ is the **same for all nodes** v, except for a subset whose size is **independent of** n.
- $M(C, \Delta)$ is always an **integer**.
- Equivalently:

  $M(C, \Delta)$ is the smallest integer M such that the edges of **any** graph with maximum degree at most Δ can be partitioned into subgraphs with at most C edges, in such a way that each vertex appears in at most M subgraphs.

- In the sequel we focus on determining $M(C, \Delta)$.

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More formally...

- Let $\mathcal{G}_\Delta$ be the class of (simple undirected) graphs with maximum degree at most Δ.
- For $G \in \mathcal{G}_\Delta$, let $\mathcal{P}_C(G)$ be the set of partitions of $E(G)$ into subgraphs with at most C edges.
- For $P \in \mathcal{P}_C(G)$, let

$$occ(P) = \max_{v \in V(G)} |\{B \in P : v \in B\}|$$

- And then,

$$M(C, \Delta) = \max_{G \in \mathcal{G}_\Delta} \left(\min_{P \in \mathcal{P}_C(G)} occ(P) \right)$$

- If the request graph is restricted to belong to a subclass of graphs $\mathcal{C} \subseteq \mathcal{G}_\Delta$, then the corresponding positive integer is denoted by $M(C, \Delta, \mathcal{C})$.

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Some properties of $M(C, \Delta)$ [Muñoz and S., WG 2008]

- W.l.o.g. we can assume that R has **regular degree** Δ.
- $C \geq C' \Rightarrow M(C, \Delta) \leq M(C', \Delta)$ for all $\Delta \geq 1$.
- $\Delta \geq \Delta' \Rightarrow M(C, \Delta) \geq M(C, \Delta')$ for all $C \geq 1$.
- **Upper bound:** $M(C, \Delta) \leq M(1, \Delta) = \Delta$.

Proposition (Lower Bound)

$M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

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- W.l.o.g. we can assume that R has **regular degree** Δ.
- $C \geq C' \Rightarrow M(C, \Delta) \leq M(C', \Delta)$ for all $\Delta \geq 1$.
- $\Delta \geq \Delta' \Rightarrow M(C, \Delta) \geq M(C, \Delta')$ for all $C \geq 1$.
- **Upper bound:** $M(C, \Delta) \leq M(1, \Delta) = \Delta$.

Proposition (Lower Bound)

$M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$ *for all* $C, \Delta \geq 1$.

Known values of $M(C, \Delta)$ [Muñoz and S., WG 2008]

- $\Delta = 1$: $M(C, 1) = 1$ for all C (trivial).
- $\Delta = 2$: $M(C, 2) = 2$ for all C (not difficult).
- $\Delta = 3$: Cubic graphs. First “interesting” case:
  - If $C \leq 3$, then $M(C, 3) = 3$.
  - If $C \geq 5$, then $M(C, 3) = 2$.
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Next section is...

Next subsection is...

Case $\Delta = 3$, $C = 4$

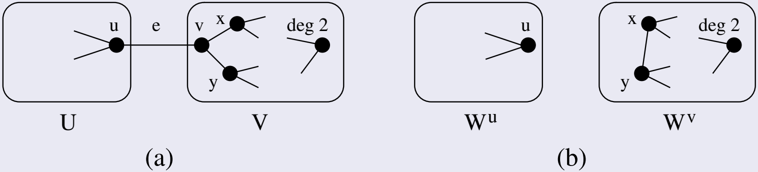
Proposition

$M(4,3) = 2$.

Idea of the proof.

(in fact, we prove a slightly stronger result)

- Let G be a minimal counterexample ($|V(G)|$ is minimal).
- If G has no bridges, then it can be "easily" proved.
- If G has a bridge e, then the property is true for U and V.



- Finally, we merge "carefully" the partitions of U and V to obtain a partition of G $\Rightarrow$ contradiction.

Next subsection is...

Case $\Delta \geq 4$ even

Theorem

Let $\Delta \geq 4$ be even. Then for any $C \geq 1$, $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Proof.

- The lower bound follows from [Muñoz and S., WG 2008].
- Construction:
  - Orient the edges of $G = (V, E)$ in an Eulerian tour.
  - Assign to each vertex $v \in V$ its $\Delta/2$ out-edges, and partition them into $\left\lceil \frac{\Delta}{2C} \right\rceil$ stars with (at most) C edges centered at v.
  - Each vertex v appears as a leaf in stars centered at other vertices exactly $\Delta - \Delta/2 = \Delta/2$ times.
  - The number of occurrences of each vertex in this partition is

$$\left\lceil \frac{\Delta}{2C} \right\rceil + \frac{\Delta}{2} = \left\lceil \frac{\Delta}{2} \left(1 + \frac{1}{C}\right) \right\rceil = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil.$$

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Let $\Delta \geq 4$ *be even. Then for any* $C \geq 1$, $M(C,\Delta) = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil$.

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Next subsection is...

Case $\Delta \geq 5$ odd

Proposition

Let $\Delta \geq 5$ be odd. Then for any $C \geq 1$, $M(C, \Delta) \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.

Sketch of proof.

- Since Δ is odd, $|V(G)|$ is even. Add a perfect matching M to G to obtain a $(\Delta + 1)$-regular multigraph G'. Orient the edges of G' in an Eulerian tour, and assign to each vertex $v \in V(G')$ its $(\Delta + 1)/2$ out-edges E_v^+.
- Remove M and partition E_v^+ into stars with C edges.
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  - Otherwise, if no edge of M is in E_v^+, then: $\left\lceil \frac{\Delta+1}{2C} \right\rceil + \Delta - \frac{\Delta+1}{2} = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{1-C}{2C} \right\rceil \leq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} + \frac{C-1}{2C} \right\rceil$.

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Case $\Delta \geq 5$ odd (II)

Corollary

Let $\Delta \geq 5$ be odd. If $\Delta \pmod{2C} = 1$ or $\Delta \pmod{2C} \geq C+1$, then $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Corollary (Case $C = 2$)

For any $\Delta \geq 5$ odd, $M(2, \Delta) = \left\lceil \frac{3\Delta}{4} \right\rceil$.

Proposition

Let $\Delta \geq 5$ be odd and let $\mathcal{C}$ be the class of Δ-regular graphs than contain a perfect matching. Then $M(C, \Delta, \mathcal{C}) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

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Next subsection is...

Improved lower bound when $\Delta \equiv C \pmod{2C}$

Theorem

Let $\Delta \geq 5$ be odd. If $\Delta \equiv C \pmod{2C}$, then $M(C,\Delta) = \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil + 1$.

Corollary (Case $C = 3$)

For any $\Delta \geq 5$ odd, $M(3,\Delta) = \left\lceil \frac{2\Delta+1}{3} \right\rceil$.

Idea of the proof of the Theorem.

- We prove that if $\Delta = kC$ with k odd, then $M(C,\Delta) \geq \left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil + 1$.
- Since both Δ and k are odd, so is C, and therefore $\left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil = k \cdot \frac{C+1}{2}$.
- We proceed to build a Δ-regular graph G with no C-edge-partition where each vertex is incident to at most $k \cdot \frac{C+1}{2} =: LB(C,\Delta)$ subgraphs.
- First, we construct a graph H where all vertices have degree Δ except one which has degree $\Delta - 1$. Furthermore, we build H so that it has girth strictly greater than C. Such a graph H exists by [Chandran, SIAM J. Dicr. Math., 2003].

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For any $\Delta \geq 5$ odd, $M(3, \Delta) = \left\lceil \frac{2\Delta+1}{3} \right\rceil$.

Idea of the proof of the Theorem.

- We prove that if $\Delta = kC$ with k odd, then $M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil + 1$.
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Improved lower bound when $\Delta \equiv C \pmod{2C}$

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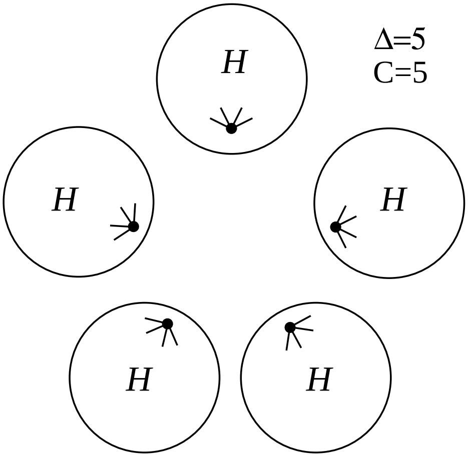

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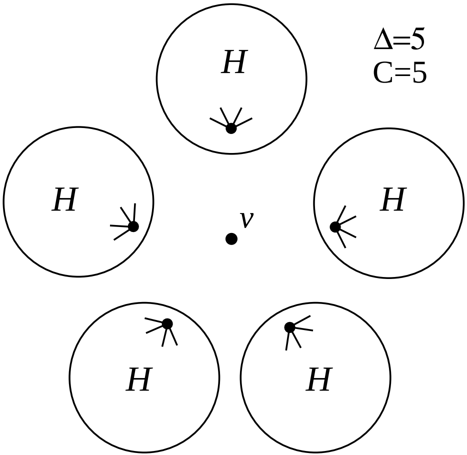

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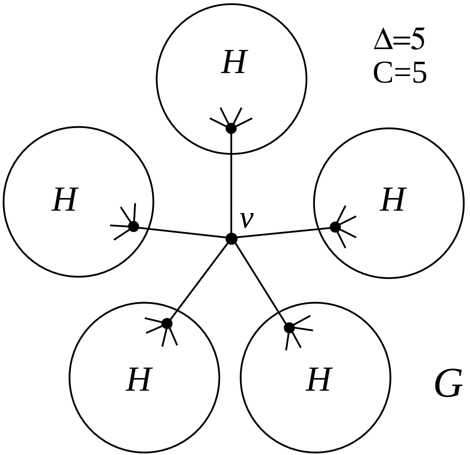

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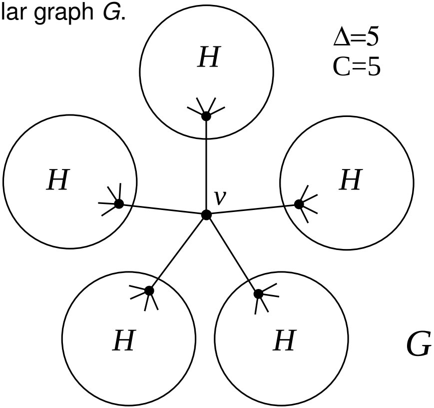

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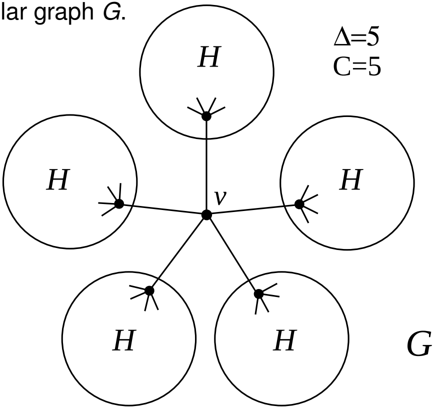

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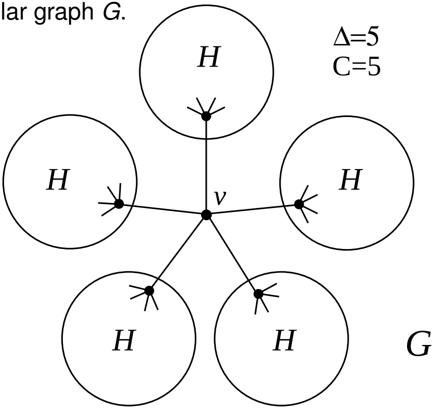

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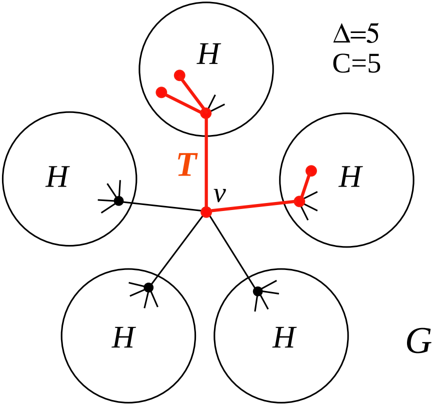
- Since $LB(C, \Delta) < \Delta$, v must have degree at least 2 in some subgraph $T \in \mathcal{B}$.



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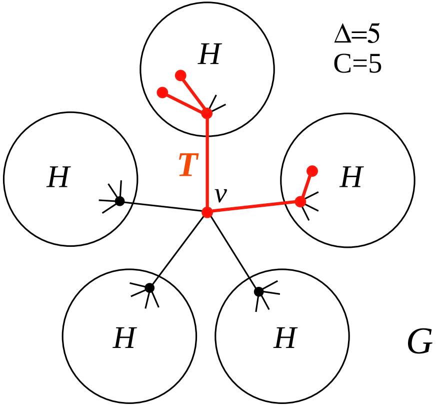

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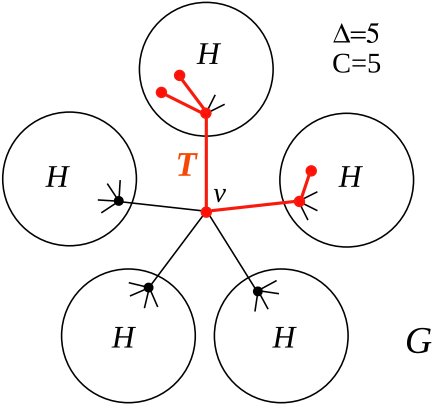

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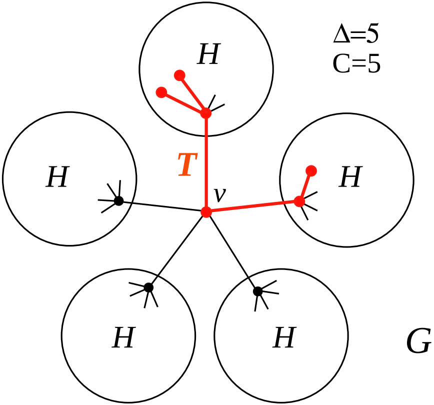

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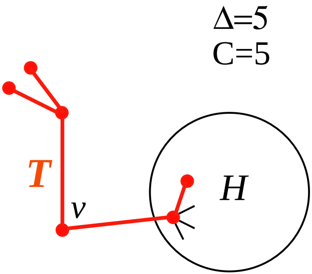
- Let $\mathcal{B}' = \{B \cap H\}_{B \in (\mathcal{B} - \{T\})}$, with the empty subgraphs removed. That is, $\mathcal{B}'$ contains the subgraphs in $\mathcal{B}$ that partition the edges in H that are not in T.



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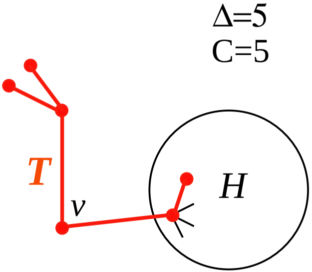

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- Therefore, the total number of edges of the trees in $\mathcal{B}'$ is

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- As $\alpha \leq \frac{C-3}{2}$, from (1) we get

$$\sum_{T \in \mathcal{B}'} |E(T)| \;\geq\; \frac{nkC - 1}{2} - \frac{C-3}{2} \;=\; \left(\frac{nk-1}{2}\right) \cdot C + 1. \qquad (2)$$

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- Therefore, using (1) and (3), we get that the total number of occurrences of the vertices of H in some tree of $\mathcal{B}$ is

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- which implies that at least one vertex of H appears in at least $\mathrm{LB}(C,\Delta)+1$ subgraphs, which is a contradiction to $\mathcal{B}$ being a C-edge-partition of G in which each vertex appears in at most $\mathrm{LB}(C,\Delta)$ subgraphs.

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Next section is...

Summary of results: values of $M(C,\Delta)$

$C\vert\Delta$	1	2	3	4	5	6	7	...	Δ even	Δ odd
1	1	2	3	4	5	6	7	...	Δ	Δ
2	1	2	3	3	4	5	6	...	$\frac{3\Delta}{4}$	$\frac{3\Delta}{4}$
3	1	2	3 (2)	3	4	5 (4)	5	...	$\frac{2\Delta}{3}$	$\frac{2\Delta+1}{3}$ $\left(\frac{2\Delta}{3}\right)$
4	1	2	2	3	4	4	5	...	$\frac{5\Delta}{8}$	$\geq \frac{5\Delta}{8}$ $(=)$
5	1	2	2	3	4 (3)	4	5	...	$\frac{3\Delta}{5}$	$\geq \frac{3\Delta}{5}$ $(=)$
6	1	2	2	3	≥ 3 $(=)$	4	5	...	$\frac{7\Delta}{12}$	$\geq \frac{7\Delta}{12}$ $(=)$
7	1	2	2	3	≥ 3 $(=)$	4	5 (4)	...	$\frac{4\Delta}{7}$	$\geq \frac{4\Delta}{7}$ $(=)$
8	1	2	2	3	≥ 3 $(=)$	4	≥ 4 $(=)$	...	$\frac{9\Delta}{16}$	$\geq \frac{9\Delta}{16}$ $(=)$
9	1	2	2	3	≥ 3 $(=)$	4	≥ 4 $(=)$	...	$\frac{5\Delta}{9}$	$\geq \frac{5\Delta}{9}$ $(=)$
...	...	...	...	...	...	...	...	...	...	...
C	1	2	2	3	≥ 3 $(=)$	4	≥ 4 $(=)$	...	$\frac{C+1}{C}\frac{\Delta}{2}$	$\geq \frac{C+1}{C}\frac{\Delta}{2}$ $(=)$

Table: Known values of $M(C,\Delta)$. The red cases remain open.
The (blue) cases in brackets only hold if the graph has a perfect matching.
The symbol "$(=)$" means that the corresponding lower bound is attained.

Conclusions and further research

- We have studied a new model of **traffic grooming** that allows the network to support **dynamic** traffic without reconfiguring the electronic equipment at the nodes.
- We established the value of $M(C, \Delta)$ for "almost all" values of C and Δ, leaving **open** only the case where:
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- For these open cases, we provided upper bounds that differ from the optimal value by at most one.
- **Further Research**:
  - Determine $M(C, \Delta)$ for the remaining cases:

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Gràcies!