The Parameterized Complexity of Some Minimum Label Problems

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Problem Definition

Input: an undirected graph G whose edges are labeled/colored;
a property Π

Problem: compute a set of edges in G satisfying property Π that uses the minimum number of colors

Examples of Π:

- Spanning Tree
- Path between two designated vertices s and t
- Cut between two designated vertices s and t
- Perfect Matching
- Maximum Matching
- Edge Dominating Set

Motivation

- Applications:
  - Telecommunication networks
  - Multi-model transportation

- Example: Telecommunication Networks
  - Nodes can communicate via different types of media
  - Compute a connected topology that uses the fewest number of communication types/media

$\Leftrightarrow$ Minimum Label Spanning Tree (MLST)

Previous Work

- Mainly dealt with classical complexity and approximation: all aforementioned problems are NP-hard

- Minimum Label Spanning Tree:
  - NP-hard Broersma & Li [1997]; Chang & Leu [1997]
  - APX-hard Krumke & Wirth [1998]
  - Can be approximated to a ratio ln(n-1) + 1

Previous Work

- Exact Algorithms:
  - Minimum Label Path: $O(n \times \min\{|C|^{d(s,t)}, 2^{|C|}\})$ Broersma et al. [2005]
  - Minimum Label Cut: $O(n^2|C|!)$ Broersma et al. [2005]

Our Work

- We look at the minimum label graph problems from the parameterized complexity angle
  - Parameter: number of labels
  - Parameter: size of the solution (cardinality of the set of edges forming the solution)

- We show that most of the problems under consideration are W[2]-hard when parameterized by the number of labels/colors, even on graphs whose pathwidth is at most a constant

Our Work

- We show that most of these problems are FPT when parameterized by the solution size, with the exception of Minimum Label Path and Minimum Label Cut, which are W[1]-hard, even on graphs of pathwidth at most 2 and 4, respectively

- We present nontrivial FPT algorithms for Minimum Label Maximum Matching and Minimum Label Edge Dominating Set parameterized by the solution size

Parameterized Complexity: A Quick Review

- A parameterized problem Q is a set of instances of the form (x, k)
- Q is *FPT* if it can be solved in time $f(k)n^{O(1)}$
- Q *is fpt-reducible* to Q' if there exists a reduction T that:

$$(x, k) \rightarrow (x', g(k)),$$

and T runs in time $f(k)|x|^{O(1)}$

Parameterized Complexity: A Quick Review

- W-hierarchy:

  At the bottom: FPT

  W[i]: $i \geq 1$

- W[1]-complete: Clique, IS

  W[2]-complete: DS, Set Cover, Hitting Set

W-hardness Results

- Theorem. Parameterized by the number of used labels:
  - MLEDS and MLMM are W[2]-hard on trees of pw ≤ 2
  - MLST and MLP are W[2]-hard on graphs with pw ≤ 2
  - MLC and MLPM are W[2]-hard on graphs with pw ≤ 2
  - MLHC is W[2]-hard on graphs with pw ≤ 5

Proof. FPT-reduction from the W[2]-hard problem Hitting Set

W-hardness Results

- Theorem. Parameterized by the solution size:
  - MLC is W[1]-hard on graphs of pw ≤ 4
  - MLP is W[2]-hard on graphs with pw ≤ 2

Proof. FPT-reduction from the W[1]-hard problem
Multicolored Clique

Tractability Results

Minimum Label Maximum Matching (MLMM)

Input: an undirected graph G whose edges are colored;
a parameter k: size of a maximum matching in G

Problem: compute a matching in G of size k whose edges are colored with the minimum number of colors

- Theorem. MLMM is FPT

The Algorithm

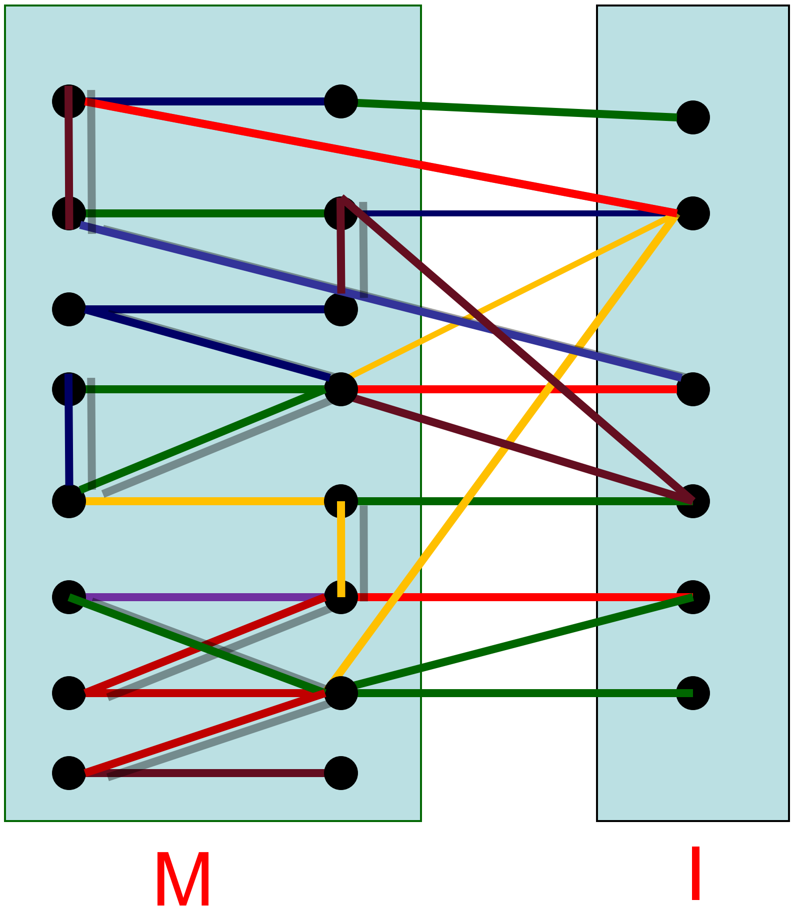
- The algorithm is a search-tree algorithm
- It grows a partial solution into an optimal solution OPT
- Let M be a maximal matching in G and $I = V(G) \setminus V(M)$; note that I is an independent set and that $|M| \leq k$
- The algorithm consists of 3 stages

Stage 1

Branch on every e ∈ G[M]:

e in OPT:
- remove e and its endpoints
- decrement k
- record color(e)

e ∉ OPT: remove e

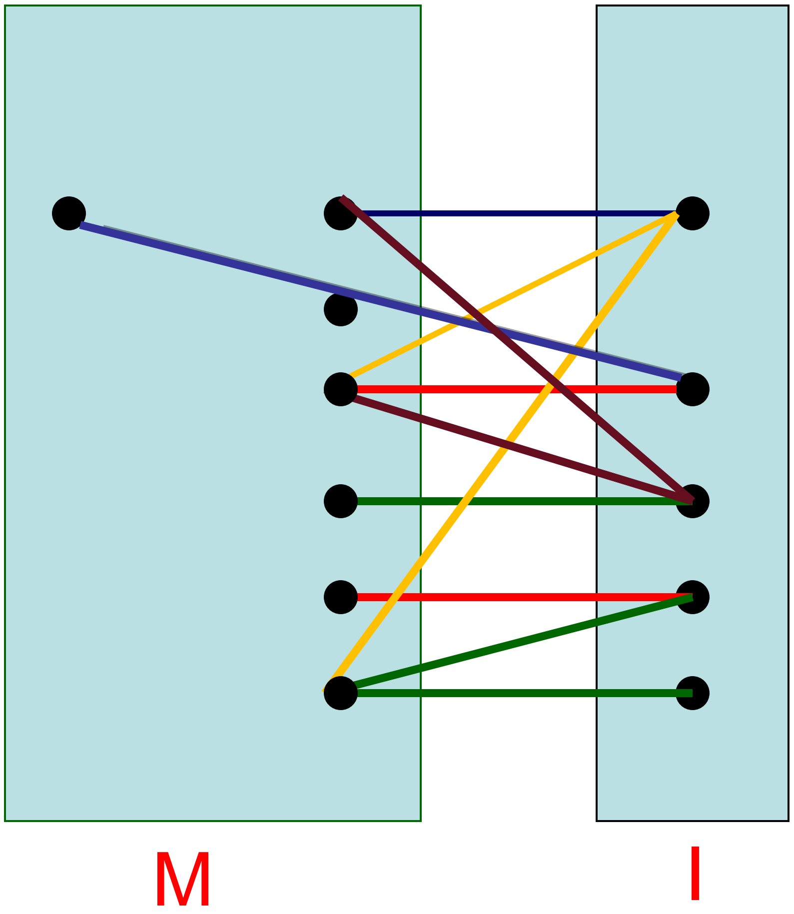


Stage 1

Branch on every remaining vertex $v \in V(M)$:

$v \in OPT$: keep v

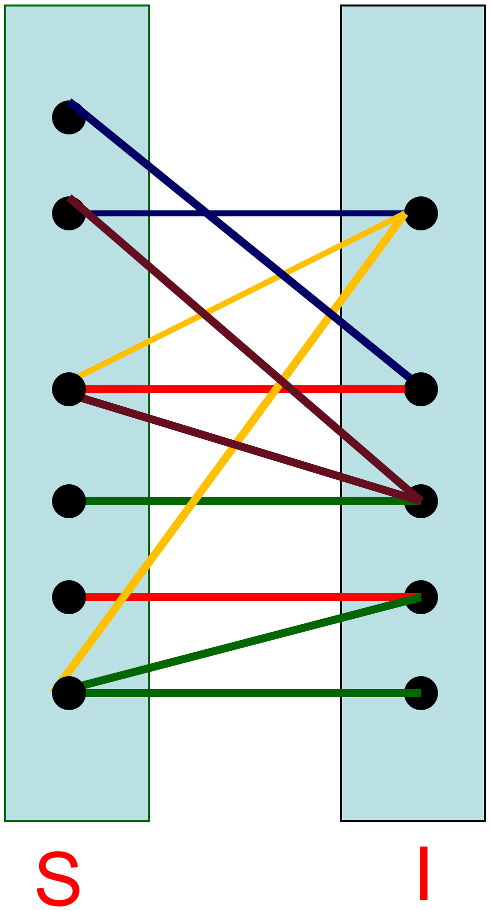
$v \notin OPT$: remove v



Stage 1

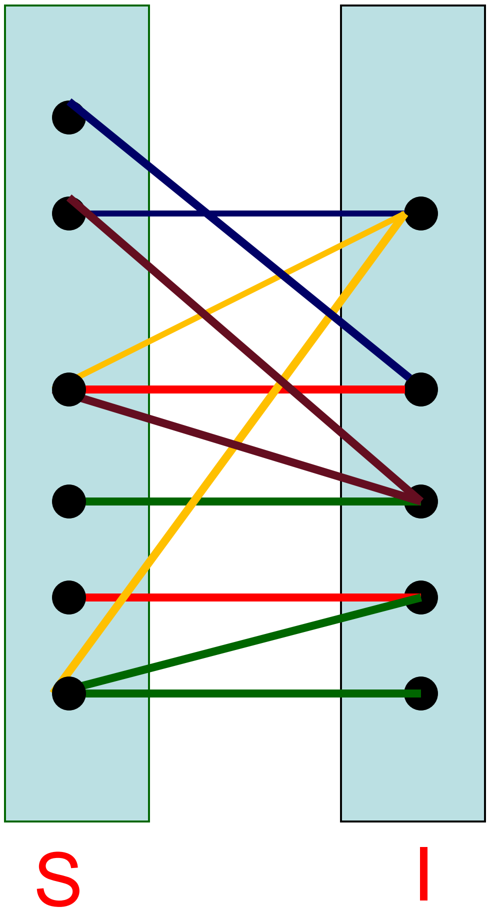
B=(S, I) is bipartite
S has at most k vertices

Number of search-tree paths
is $O((8ek)^k)$



Stage 2

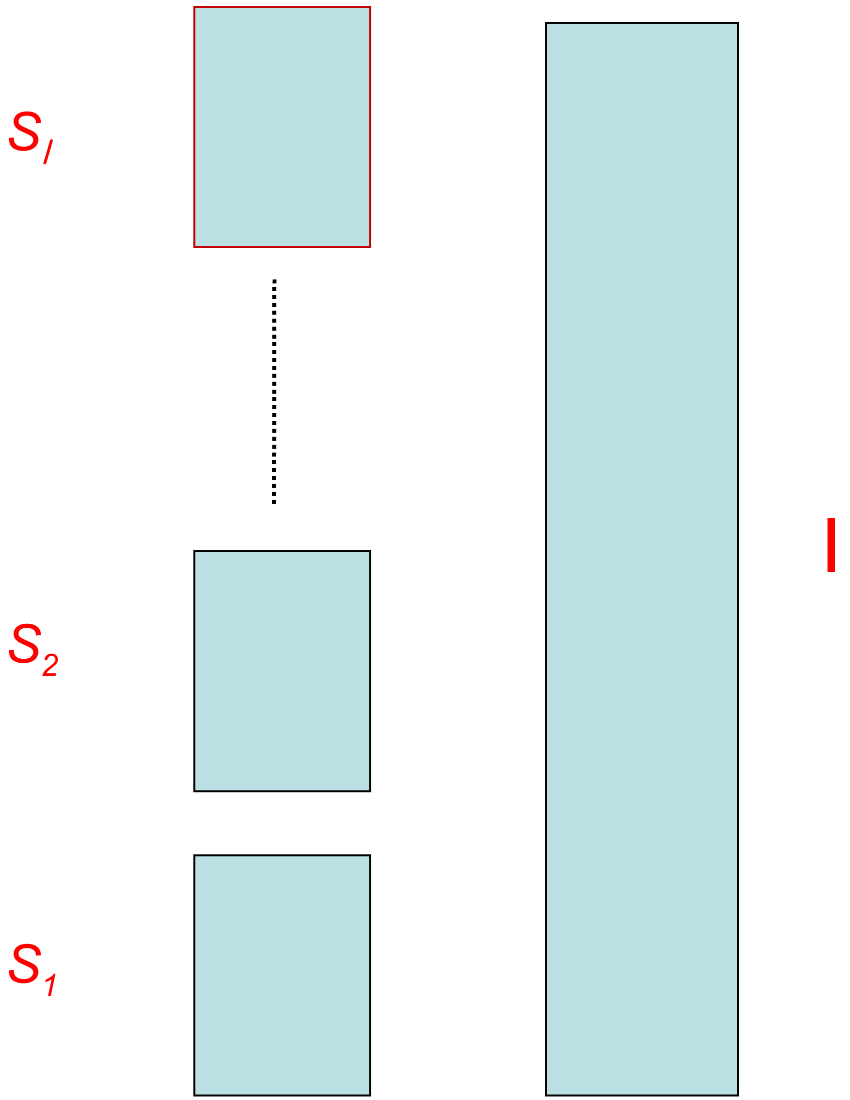
- B=(S, I) is bipartite
- S has at most k vertices
  (assume w.l.o.g. |S|= k)
- Every vertex in S is an endpoint of an edge in OPT
- Problem: Compute a matching from S into I that uses the minimum number of colors
- Constraint: some of the colors have already been used



Stage 2

- A matching is *monochromatic* if all its edges have the same color
- We try every possible partition of S into groups $S_1, \ldots, S_l$, such that all vertices in S_i are matched in OPT by a monochromatic matching of distinct color
- Fix such a partition $S_1, \ldots, S_l$

Stage 2

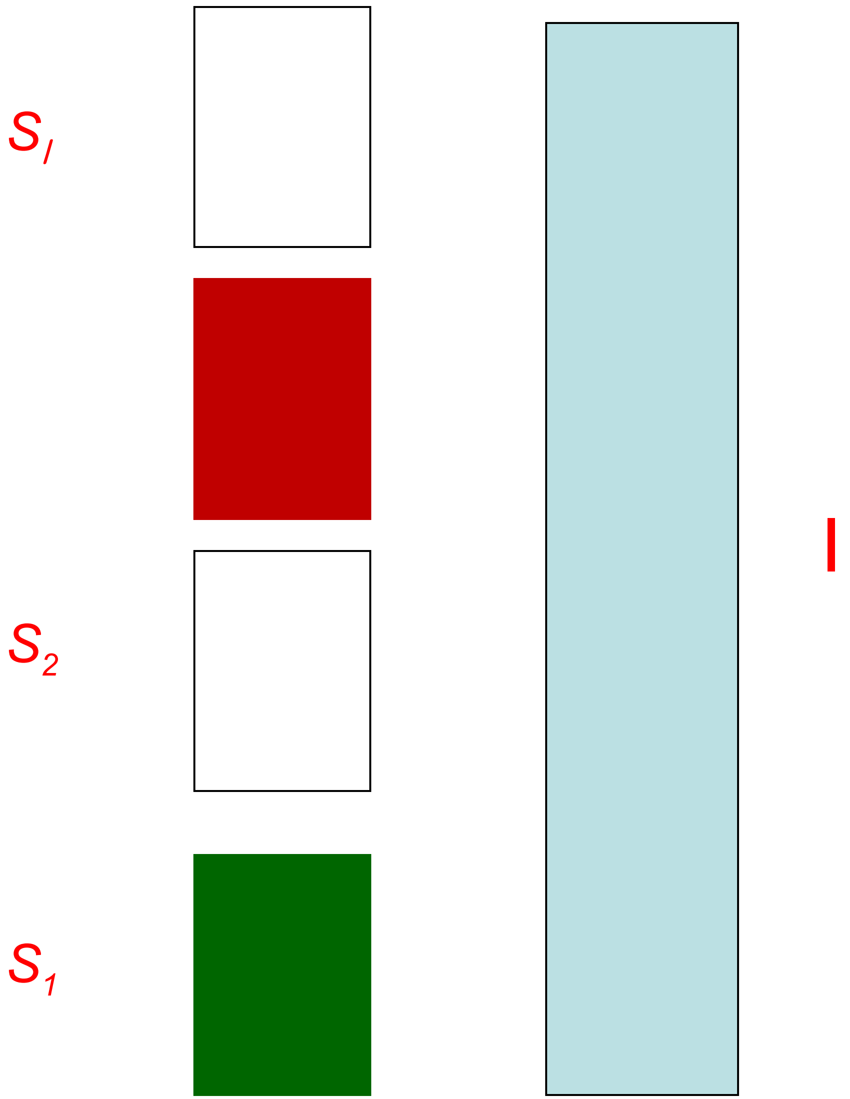


Stage 2

- It is possible that a group S_i uses the color of an edge that was determined to be in OPT in Stage 1

- Therefore, for each subset of the colors used in Stage 1, we try every possible one-to-one mapping from this subset to $S_1, \ldots, S_l$

Stage 2



Stage 2

- For each S_i, if S_i has a pre-assigned color c_i let

  $\mathcal{M}_i = \{M_i : M_i$ is a monochromatic matching of color c_i that matches S_i into $I\}$

Otherwise, let

  $\mathcal{M}_i = \{M_i : M_i$ is a monochromatic matching that matches S_i into $I\}$

- Let $h(k)$ be a function of k to be determined later

Stage 2

We branch further to make $B=(S, I)$ satisfy the following:

Assumption. For each $i=1,\ldots, l$

- $|\mathcal{M}_i| > h(k)$
- Either the number of colors in $\mathcal{M}_i$ is more than $h(k)$ or it is exactly 1
- If $\mathcal{M}_i$ has exactly one color in it, the every vertex in S_i has more than $h(k)$ edges incident on it in $\mathcal{M}_i$

The number of branches in Stage 2 is $O(k^{k+1}\, k! + h(k)^{3k})$, and the running time along each branch is $O(m\sqrt{n} + h(k)n)$

Stage 3

- Let $h(k) = k^2 + k$, and consider a partition of S into S_1, …, S_l where each $\mathcal{M}_i$ satisfies the previous assumption

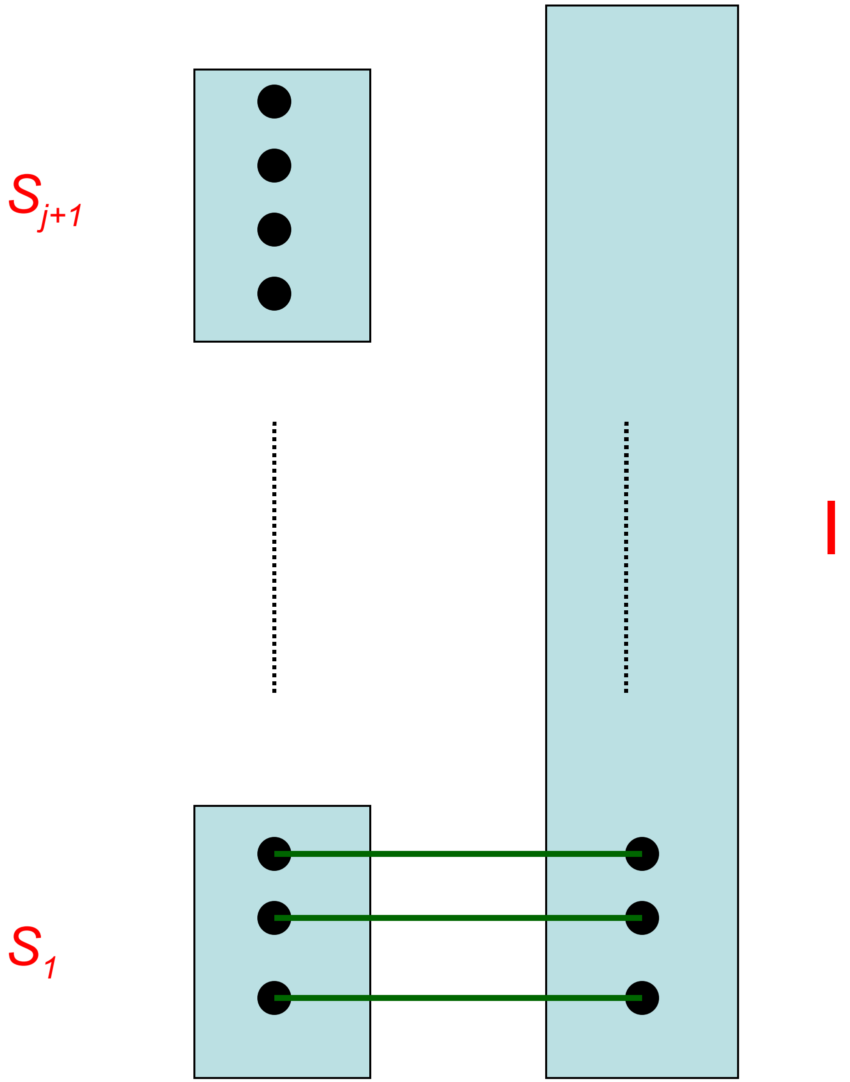
Then:

- Theorem. In $O(k^3)$ we can compute a matching M' matching S into I such that the set of edges in M' incident on S_i is a monochromatic matching in $\mathcal{M}_i$

Proof

Case 1: M_{j+1} contains more than h(k) colors

Case 2: M_{j+1} contains a single color



Running Time

Choosing $h(k) = k^2 + k$:

- The number of search-tree paths in Stage 1 is $O((8ek)^k)$
- The number of search-tree paths in Stage 2 is $O((8ek)^k k^{7k})$
- Stage 3 has no branching
- The running time along each path in the search-tree is $O(km\sqrt{n} + k^3n)$
- Therefore, the running time of the algorithm is $O((8e)^k k^{7k+3}m\sqrt{n})$

Minimum Label Edge Dominating Set

- Theorem. Parameterized by the solution size, Minimum Label Edge Dominating Set (MLEDS) is FPT

  Proof. Similar in flavor to that of MLMM

Concluding Remarks

- We showed that most minimum label problems are (parameterized) intractable w.r.t. the number of labels, even on graphs of bounded pathwidth, whereas most of these problems are FPT w.r.t. solution size

- Structured graph problems have received a lot of attention recently:
  - Applications in Computational Biology and Networking
  - They are hard
  - Try to parameterize by different parameters: treewidth, pathwidth, VC, etc…
- Study other minimum label problems: Minimum Label Feedback Arc Set on directed graphs