Edge-Paths and Shortcuts for Detecting an Odd Hole

Shimon Shrem
Joint work with: Martin C. Golumbic & Michal Stern
Computer Science, University of Haifa
Caesarea Rothschild Institute

Outline

- Motivation
- Γ-BFS
- Edge-Paths
- Shortcuts of a Hole
- Main Theorem
- Future Work

Motivation

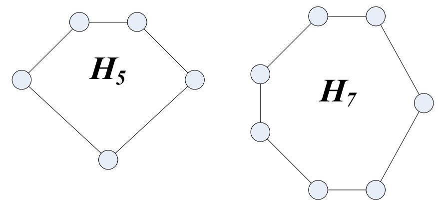
- Strong Perfect Graph Conjecture [Berge, 61]

  a graph is perfect if and only if it does not contain an odd hole nor an odd antihole.

- Strong Perfect Graph Theorem [M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas, 2002].
- The general problem is still open.
- Holes and antiholes have been extensively studied in many different contexts in algorithmic graph theory.

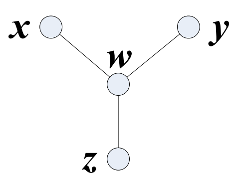
A Hole

- Hole



A ***hole*** is a chordless cycle on five or more vertices.

- Claw Graph



A ***claw*** is the graph consisting of vertices w, x, y, z and only the edges $(w,x),(w,y),(w,z)$.

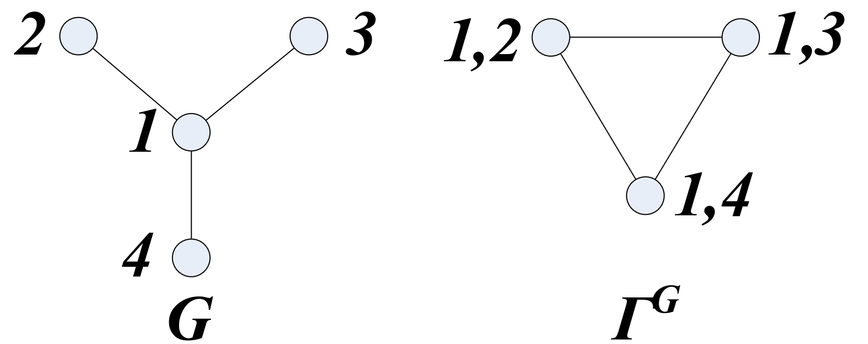
- Claw-free Graph

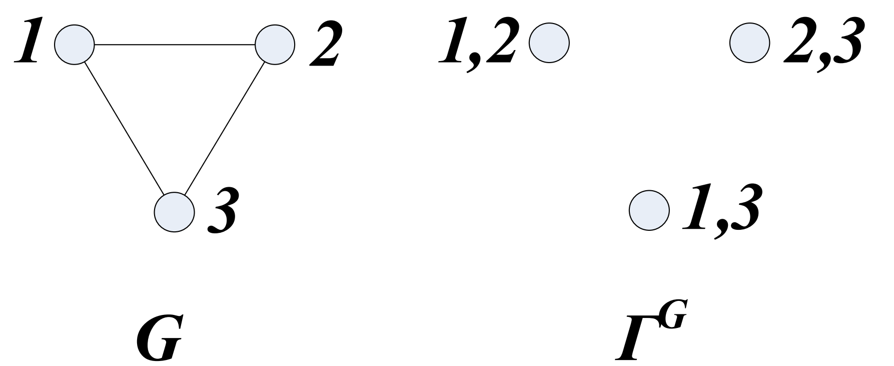
A graph is called ***claw-free*** if none of its induced subgraphs is a claw.

- In this talk, we consider only claw-free graphs.

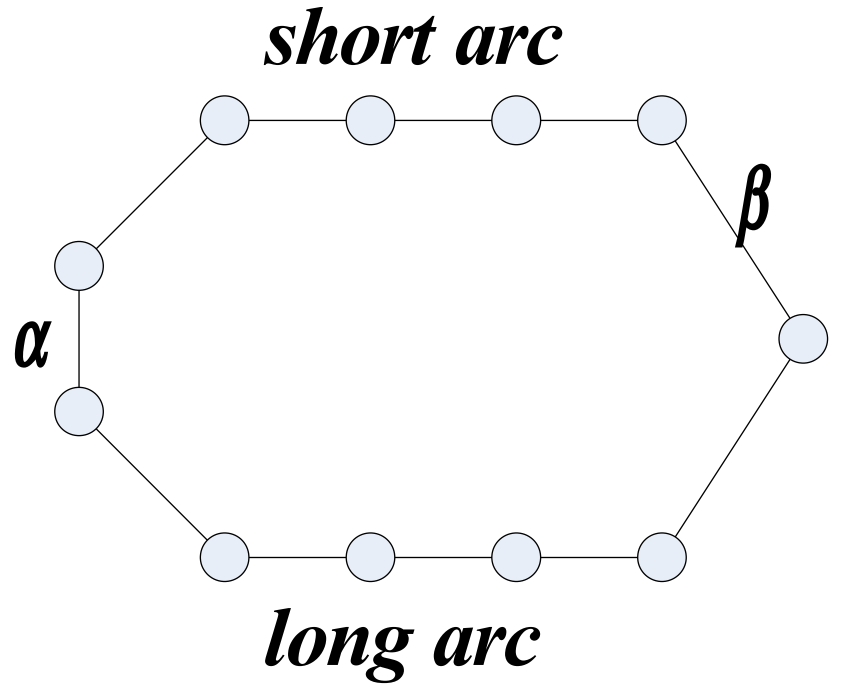
The Graph Γ^G

- The Gallai forcing graph Γ^G of a graph $G=(V,E)$ is the graph with:
  - Vertex set $U= E(G)$
  - Two vertices (a,b), $(a',b') \in U(\Gamma^G)$ are connected iff either:
    - $a = a'$ and $(b,b') \notin E(G)$ or
    - $b = b'$ and $(a,a') \notin E(G)$

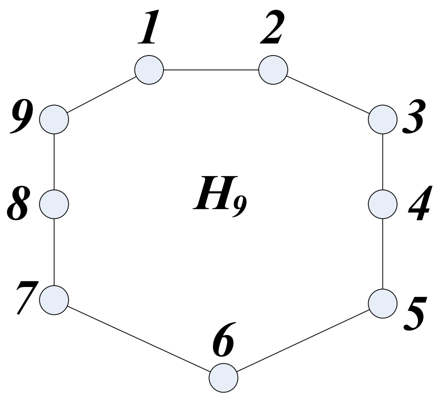




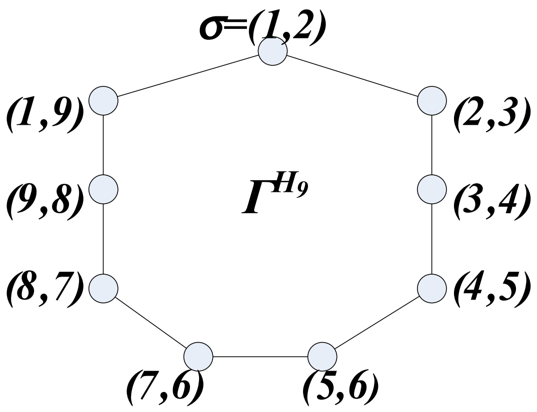
Short & Long Arcs in an Odd Hole



Γ-BFS – Simple Example

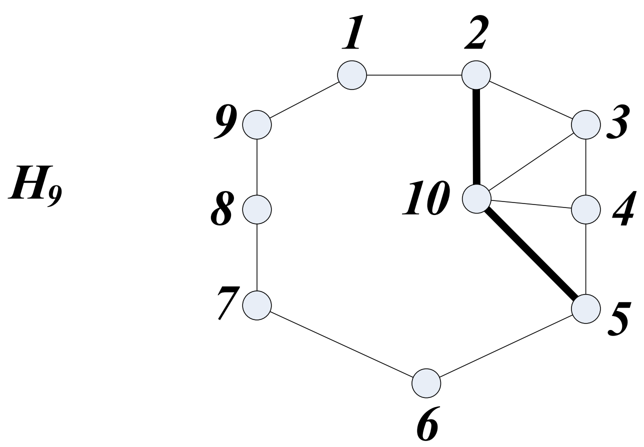


Vertices of G	Discovery Time
1,2	0
3,9	1
4,8	2
5,7	3
6	4

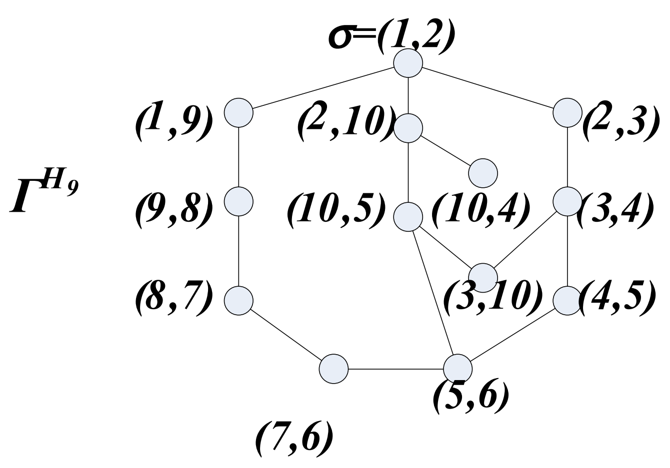


Vertices of Γ^G	Discovery Time
σ=(1,2)	0
(2,3), (1,9)	1
(3,4), (9,8)	2
(4,5), (8,7)	3
(5,6), (7,6)	4

Γ-BFS – Shortcut Example



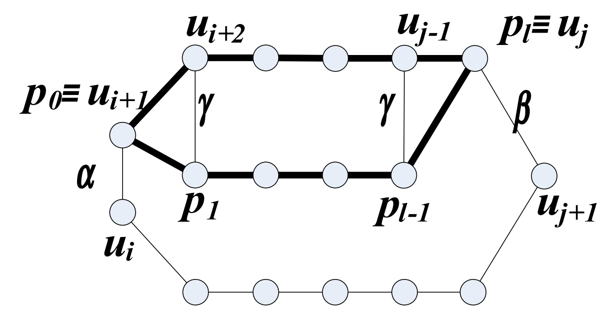
Vertices of G	Discovery Time
1, 2	0
3, 9, 10	1
4, 8, 11	2
5 , 7, 6	3



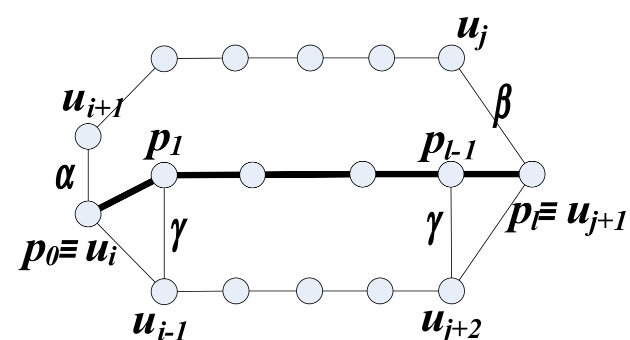
Vertices of Γ^G	Discovery Time
σ=(1,2)	0
(2,3), (2,10), (1,9)	1
(3,4), (9,8), (10,5), (10,4)	2
(4,5), (8,7), (5,6)	3
(7,6)	4

Edge-Paths

- *Edge-path* from $\alpha = (u_i, u_{i+1})$ to $\beta = (u_j, u_{j+1})$:
- P is a simple chordless path.
- $|P| \leq |A_{i+1,j}| < |A_{j+1,i}|$.
- Let $P' \subseteq P$ be the set of the inner vertices of P that have neighbors on H, then P' has neighbors only on the long arc between and or P' has neighbors only on the short arc between and , but not on both.
- From all the paths that satisfy 1-3 , we chose P to be with minimal length.



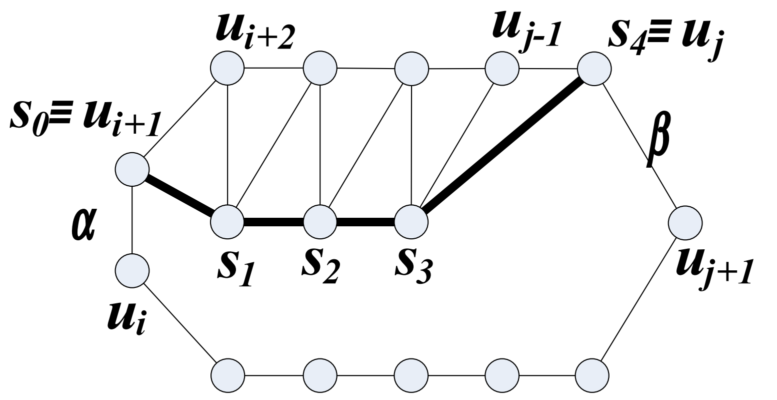
Type 1



Type 2

Shortcuts of a Hole

- Let S be an edge-path with respect to H from a source $\alpha = (u_i, u_{i+1})$ to a target $\beta = (u_j, u_{j+1})$. We say that S is a shortcut of H if $|S| < |A_S|$.



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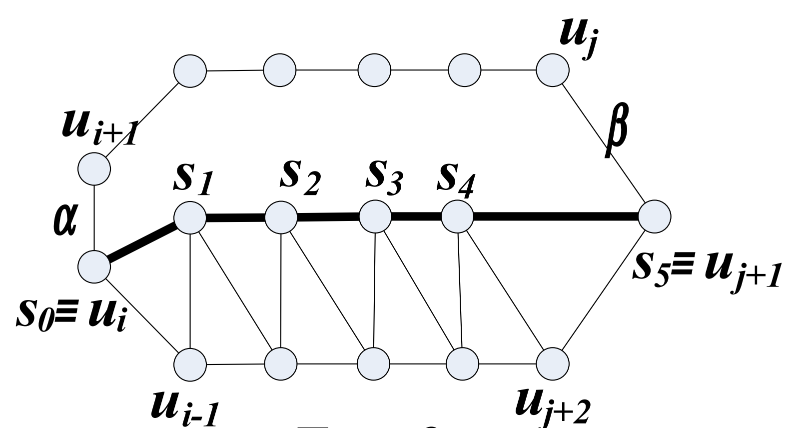
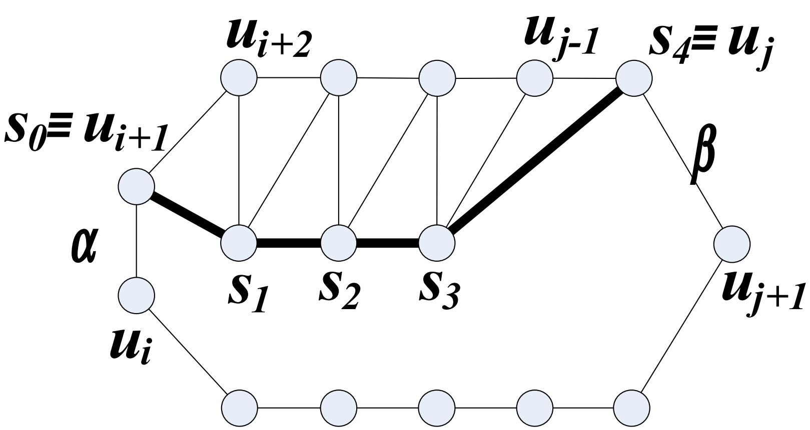


Type 2

Shortcuts of a Hole

- Theorem 1:

  For any shortcut $S= s_0s_1\ldots s_l$ of H, $|S| =|A_S|-1$.



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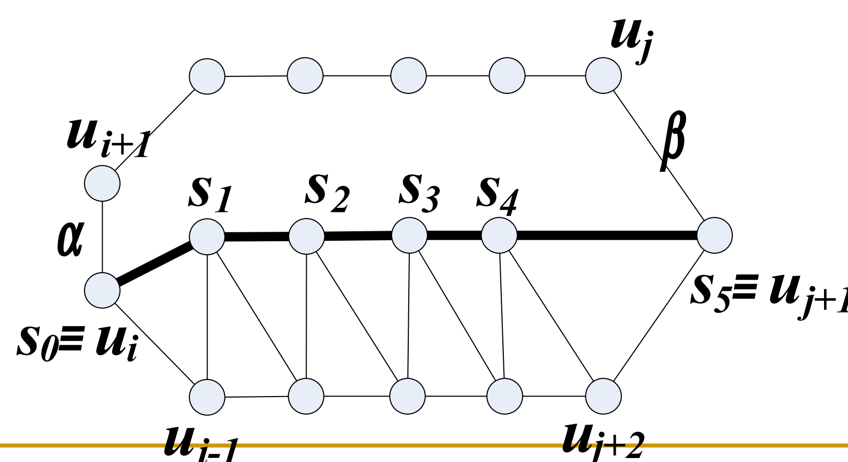
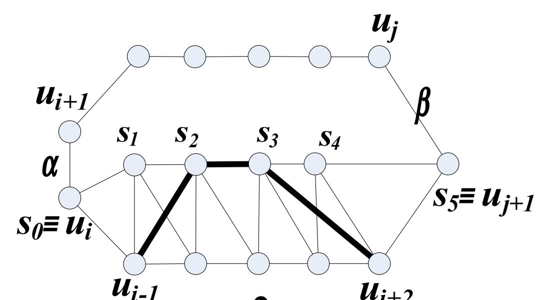


Type 2

Shortcuts of a Hole

- *Minimal shortcut* does not **"contain"** any other shortcut.
- Theorem 2:

  The Structure of Minimal Shortcuts.



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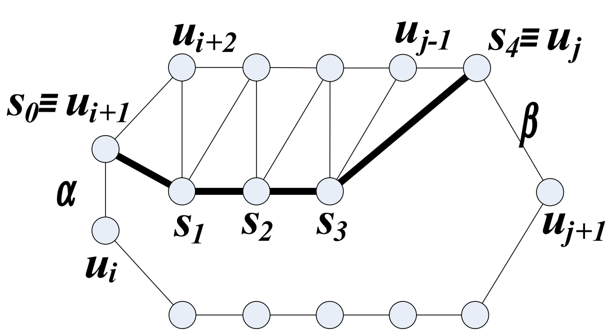

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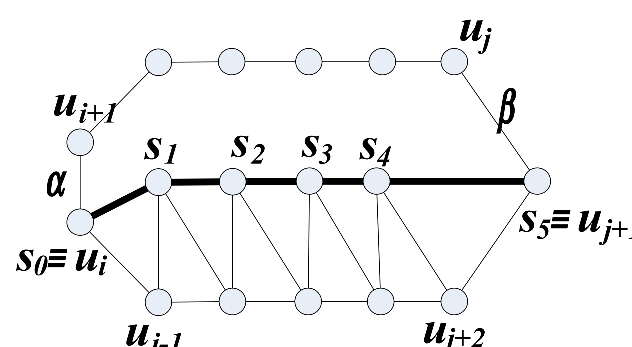
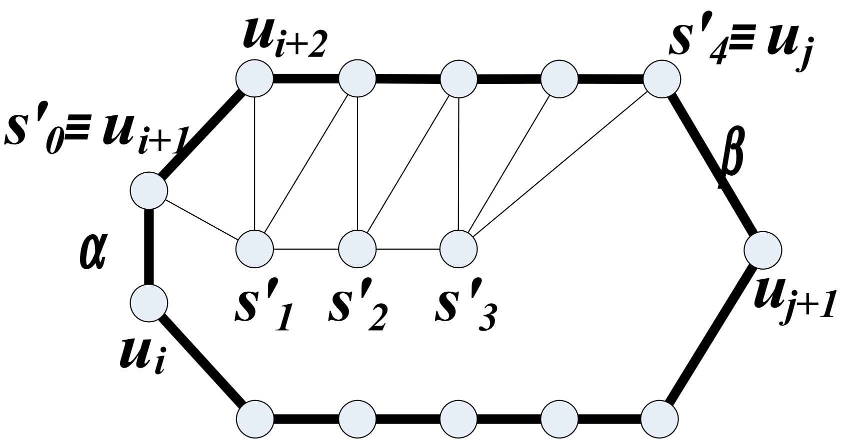
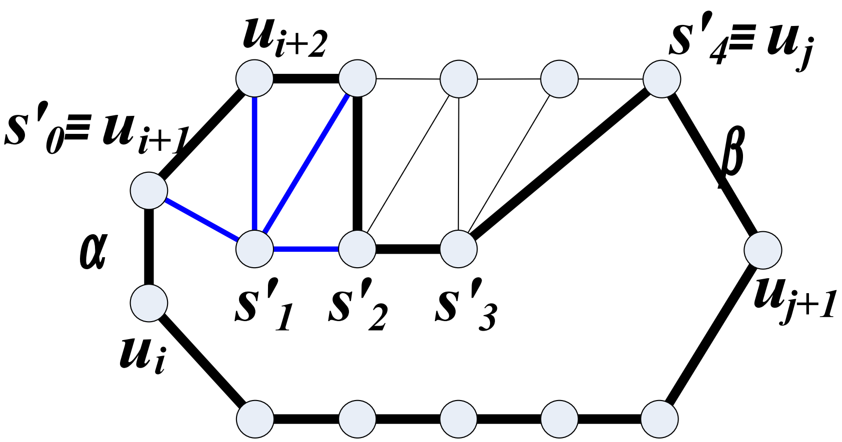


Type 2

Tents on Minimal Shortcuts

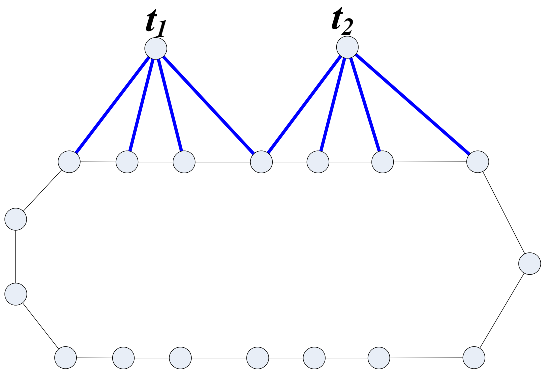


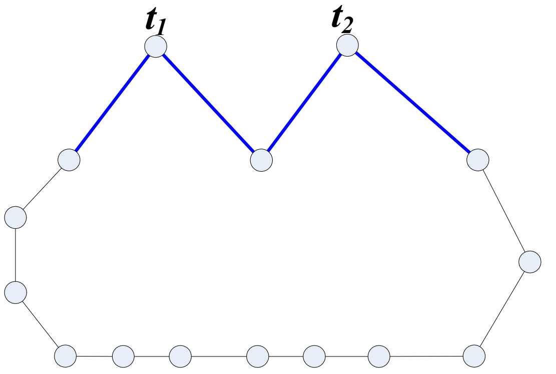


Tents in a Claw-Free Graph

- Lemma

  Let *H* be a smallest odd hole in a *claw*-free graph *G*, and let t_1 and t_2 be tents of *H*. Then all the neighbors of t_1 and t_2 on *H* lie consecutively on at most a P_6 .

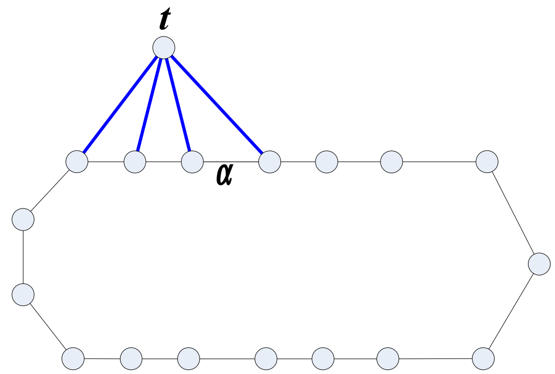


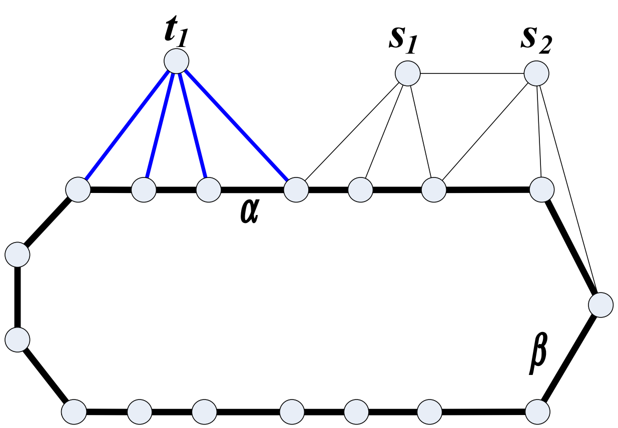


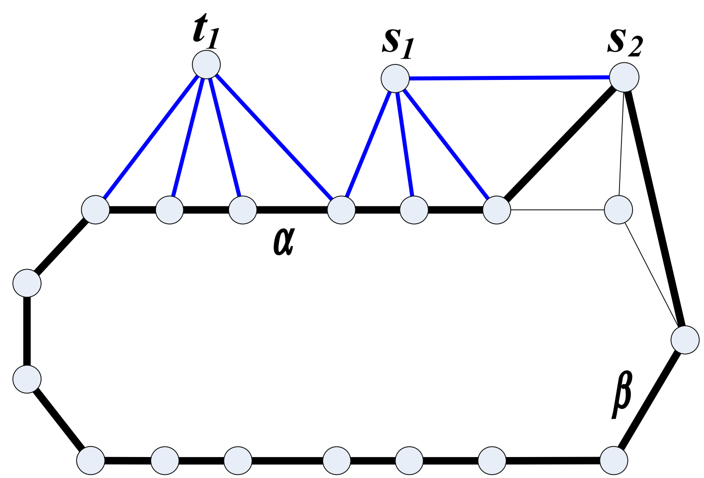
Edge without Shortcuts

- Theorem:

  Let t_1 be a tent of H, then there is a smallest odd hole H^* and an edge $\alpha \in E(H^*)$, such that has no shortcut of H^*.



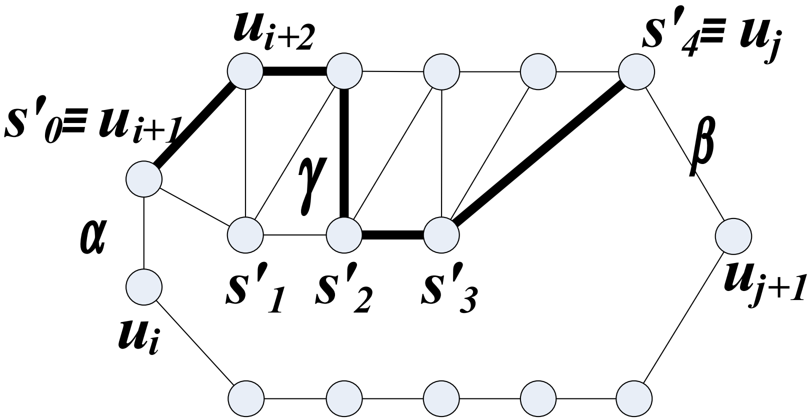




Edge without Shortcuts

- Main Theorem:

  Let H be a smallest odd hole in a claw-free graph G. Then there is a smallest odd hole H^* and an edge $\gamma \in E(H^*)$, such that γ has no shortcut of H^*.



- We use the main theorem to get a polynomial algorithm for detecting an odd hole in a claw-free graph.

Future Work

- Anti-Hole detection in a claw-free graph (preliminary work).
- General Problem (preliminary work).