Chordal digraphs

Daniel Meister   Jan Arne Telle

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Daniel Meister *Jan Arne Telle*

Department of Informatics, University of Bergen, Norway

Chordal graphs

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important and famous graph class

with many graph-theoretic and algorithmic applications

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closely related to treewidth and
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What is a/the

Chordal graphs

closely related to treewidth and
efficient solvability of difficult problems

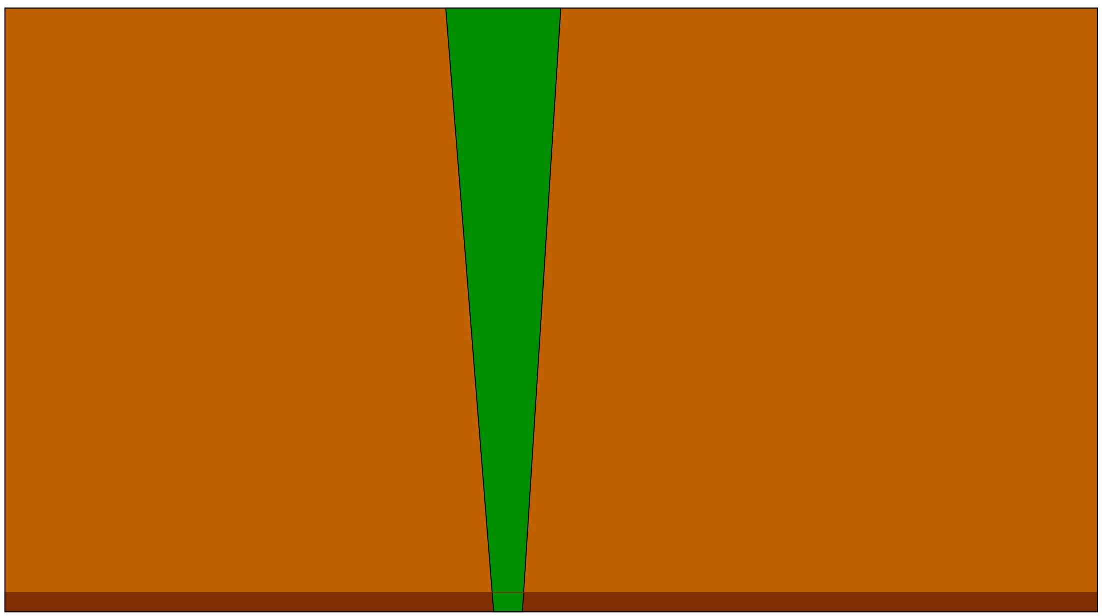
directed analogue?

Chordal graphs

appear in many applications directly or as important subproblem

What is a/the directed analogue?

class should be big enough to contain enough structure,
small enough to admit efficient solutions



Overview

Outline of the talk

Chordal (undirected) graphs

Presentation part

1 Everybody's darlings: chordal (undirected) graphs

2 Introducing: chordal digraphs

3 Characterisations of chordal digraphs

4 Forbidden subgraphs of semicomplete chordal digraphs

Chordal (undirected) graphs

Inductive construction

start from an empty graph,

- select clique
- add new vertex
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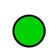

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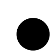

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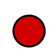

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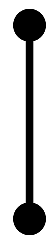

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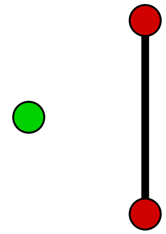

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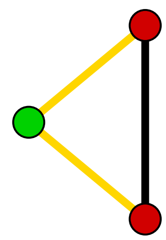

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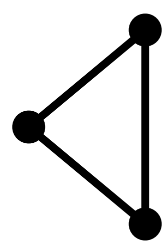

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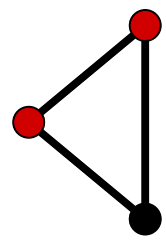

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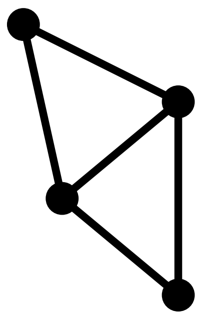

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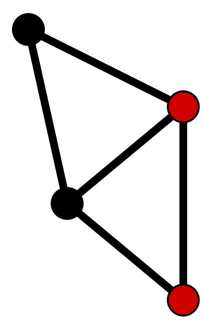

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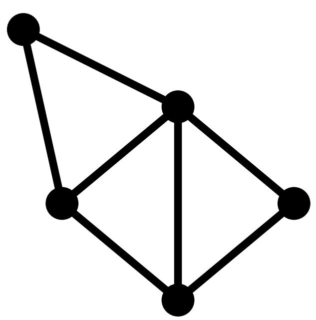

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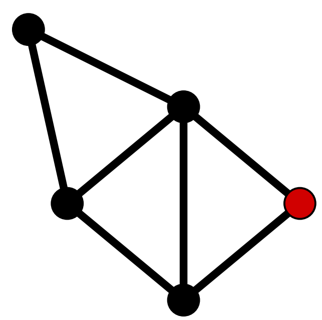

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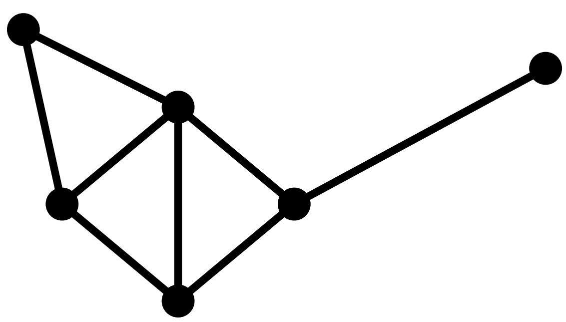

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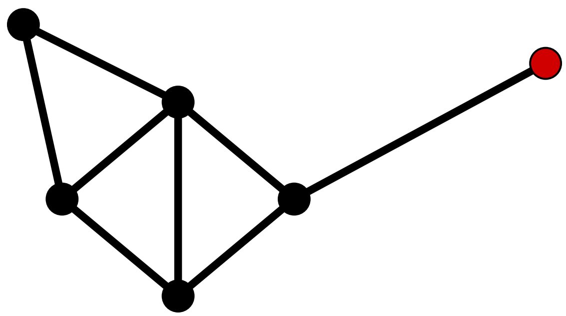

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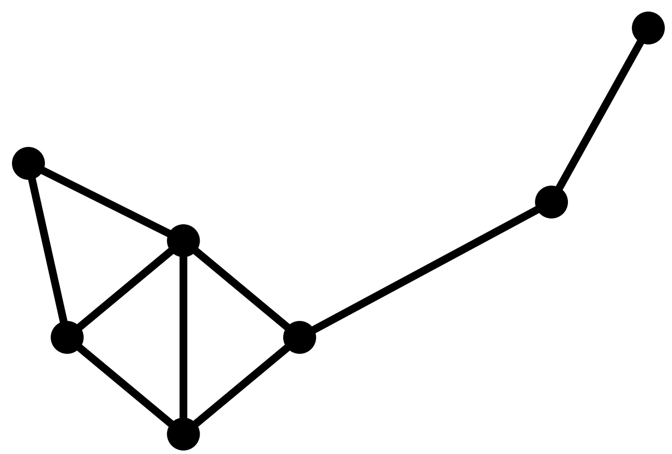

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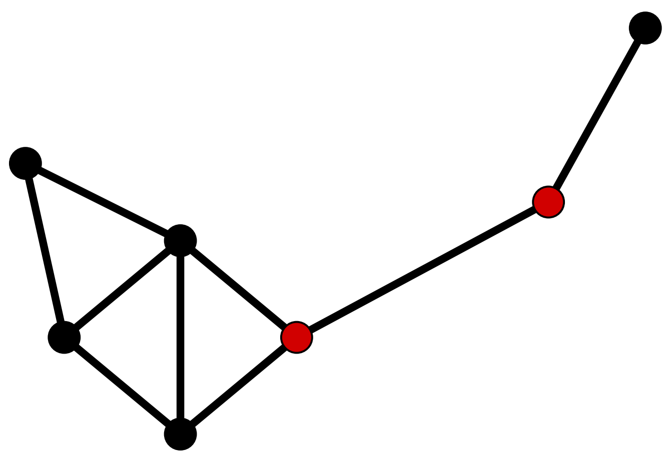

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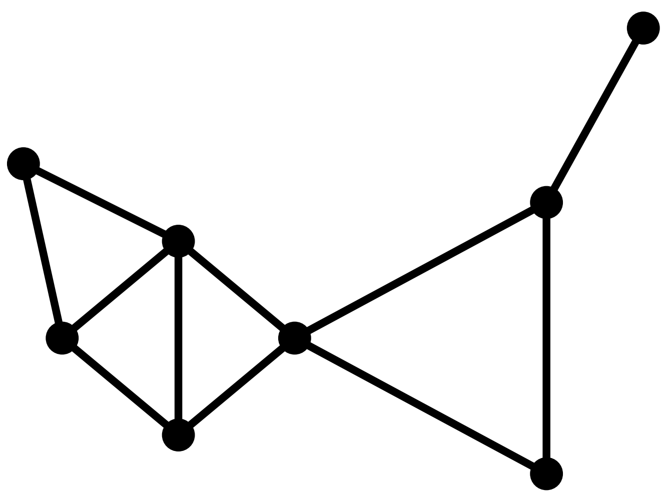

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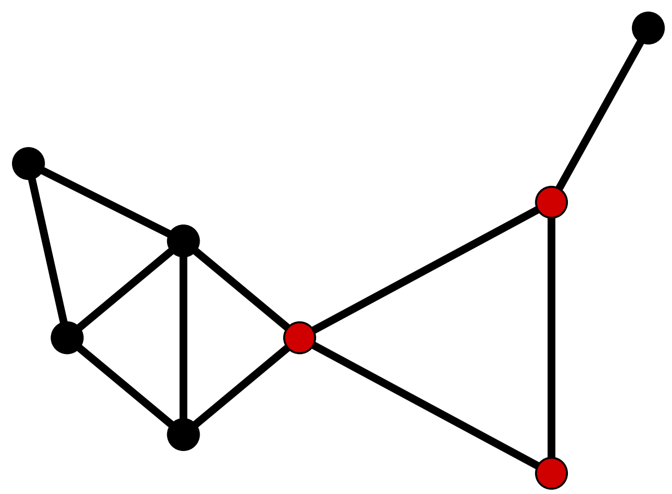

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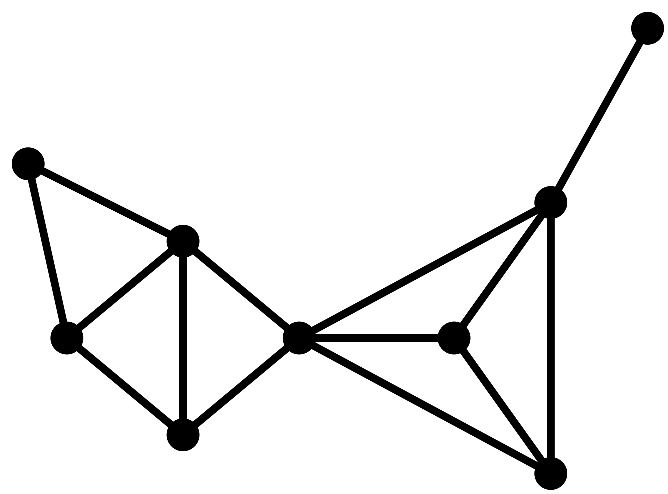

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Alternative characterisations

Theorem [folklore]

Graph is chordal if and only if

it does not contain a chordless cycle of length at least 4 as induced subgraph.

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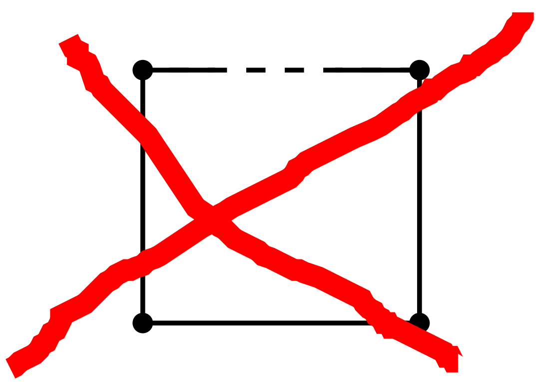
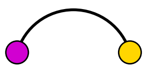
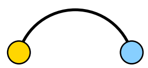
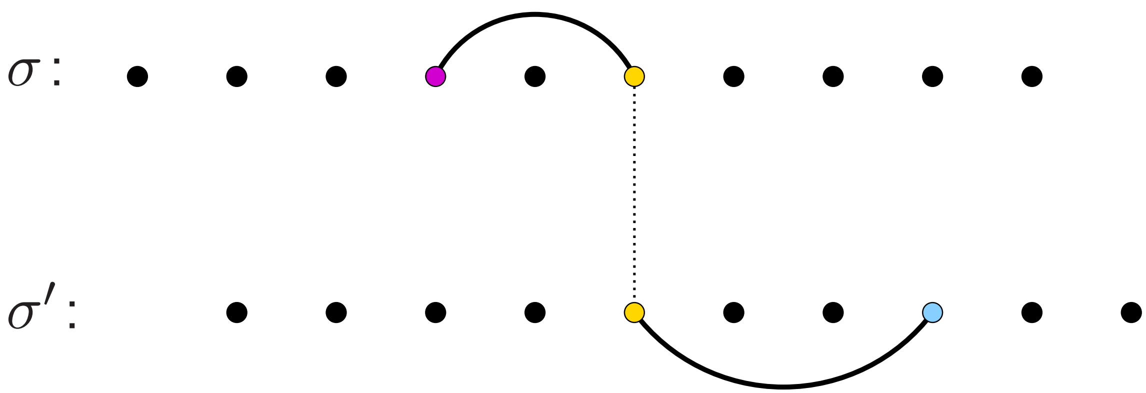


Alternative characterisations

Layout pair (σ, σ') with transitivity property:

for all vertex triples  with  $\prec_\sigma$  and  $\prec_{\sigma'}$ 

if  and 



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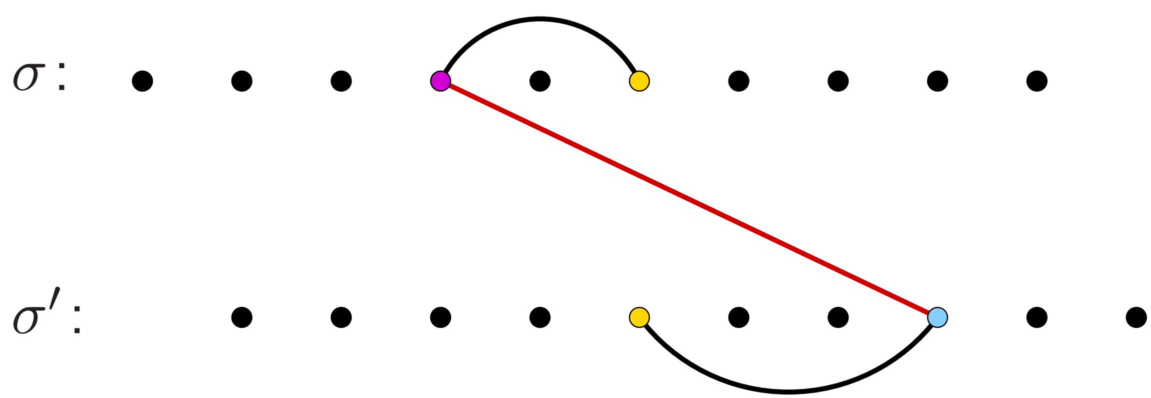
if and then



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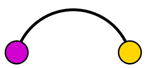
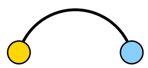
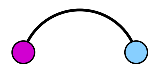
if  and  then 

Theorem [folklore]

Graph is chordal if and only if

it has vertex layout σ such that (σ, σ^R) has transitivity property.

Chordal digraphs

Presentation part

Inductive construction

start from an empty digraph,

- select “clique”
- add new vertex
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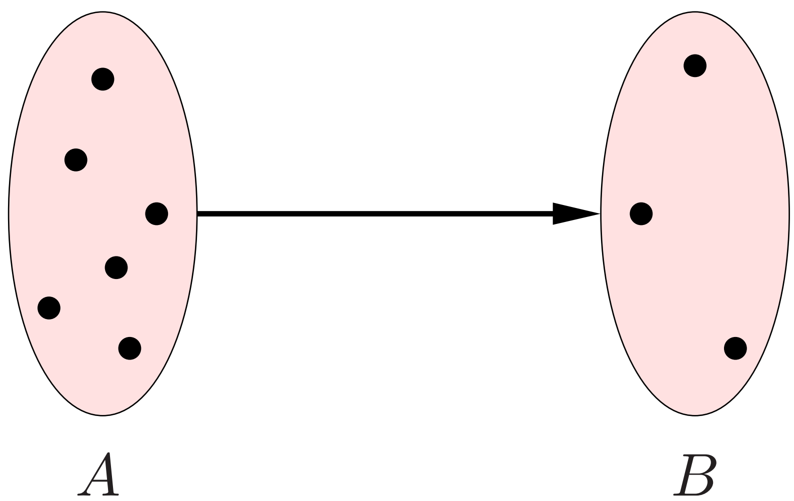
What is **"clique"**?, what is **"adjacent"**?

"Clique"

sets A and B of vertices
with (a, b) arcs for all $a \in A$ and $b \in B$, where $a \neq b$



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Remark

$A = B$ results in a complete digraph

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"Adjacent"

add arcs without creating unnecessary cycles

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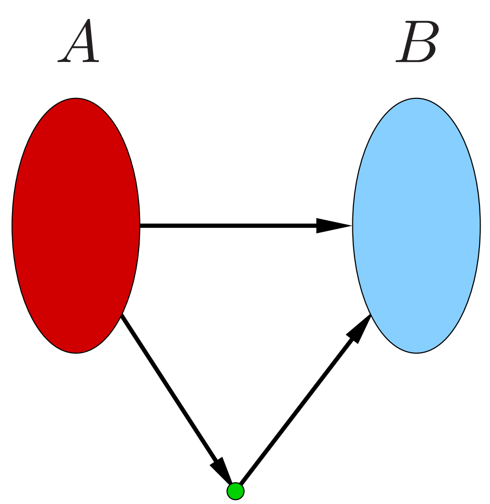

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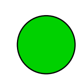

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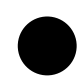

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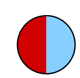

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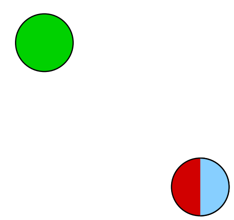

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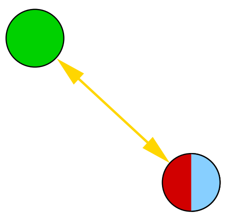

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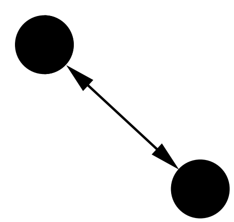

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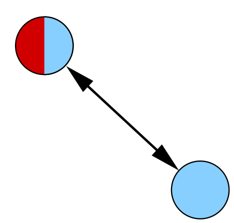

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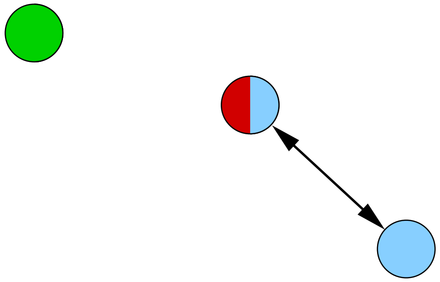

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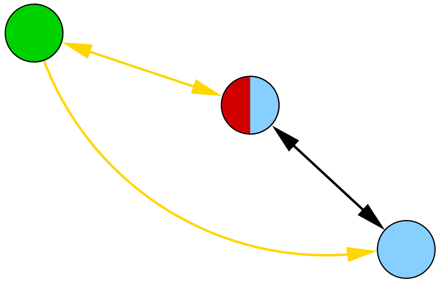

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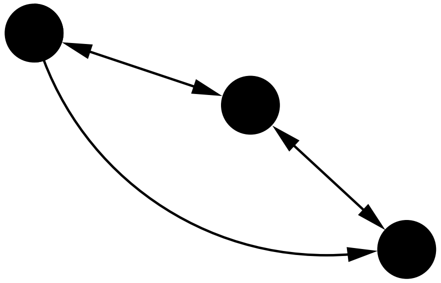

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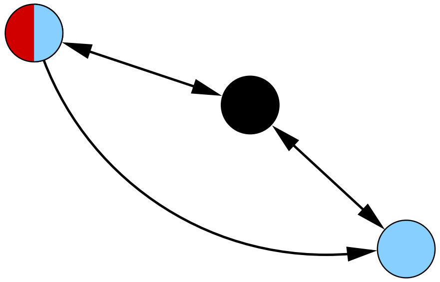

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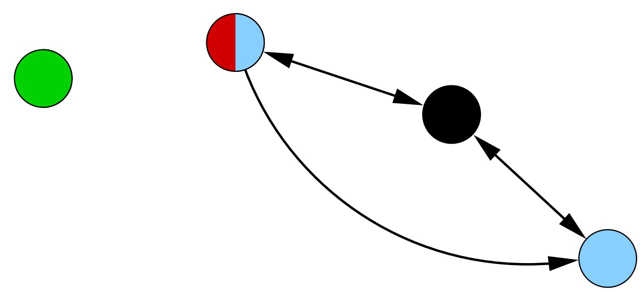

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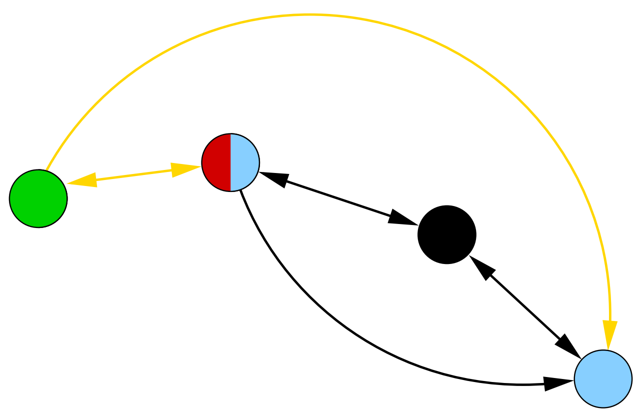

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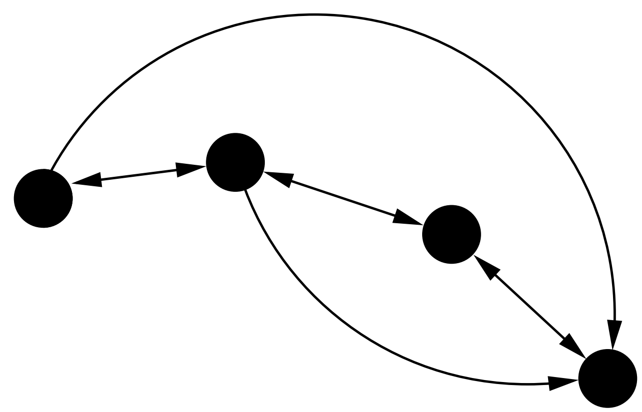

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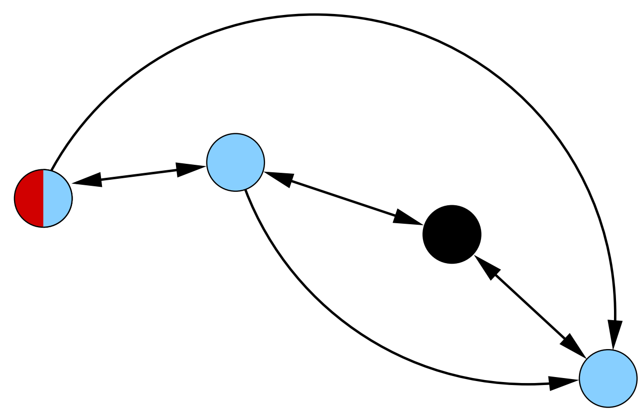

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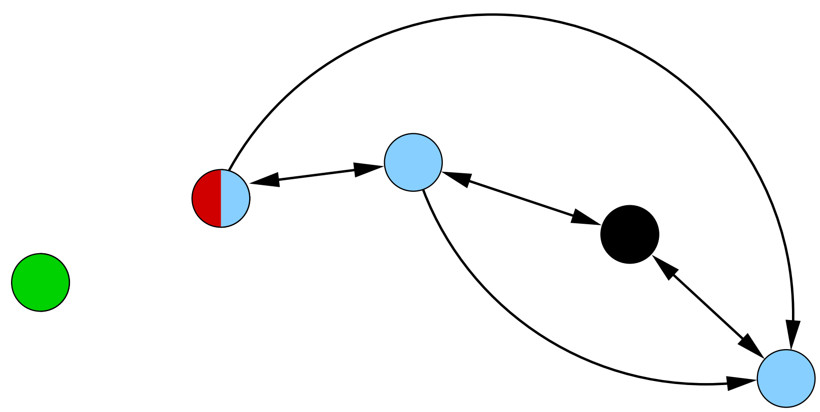

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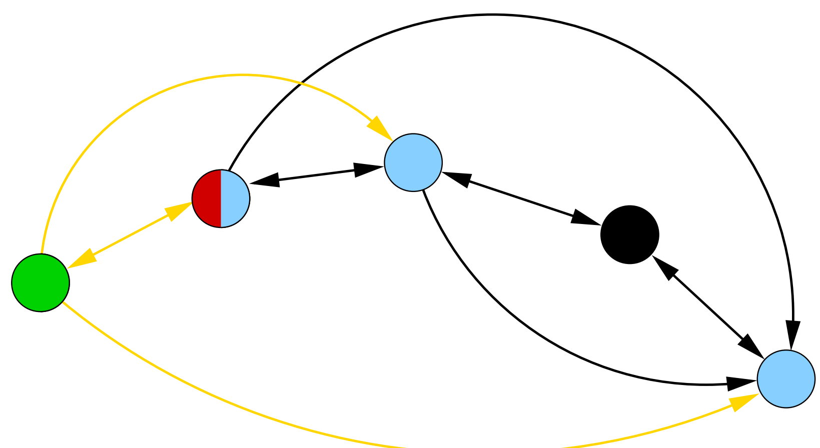

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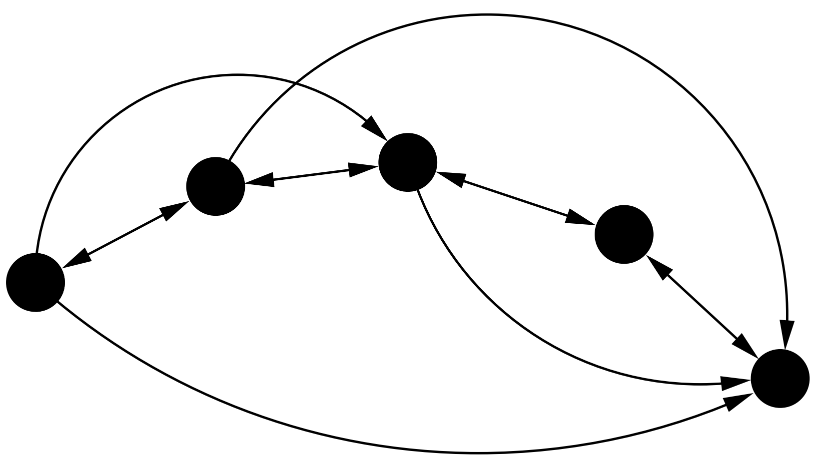

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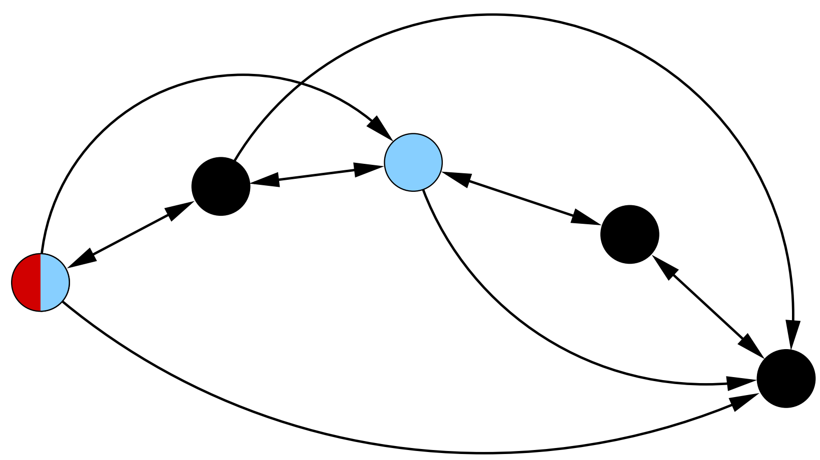
- select "clique"
- add new vertex
- make it "adjacent" to all vertices in the "clique"

Example



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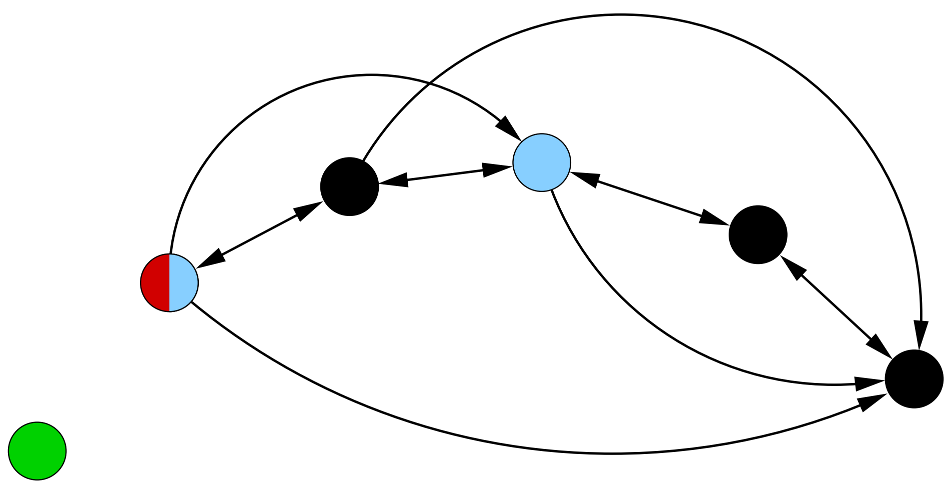

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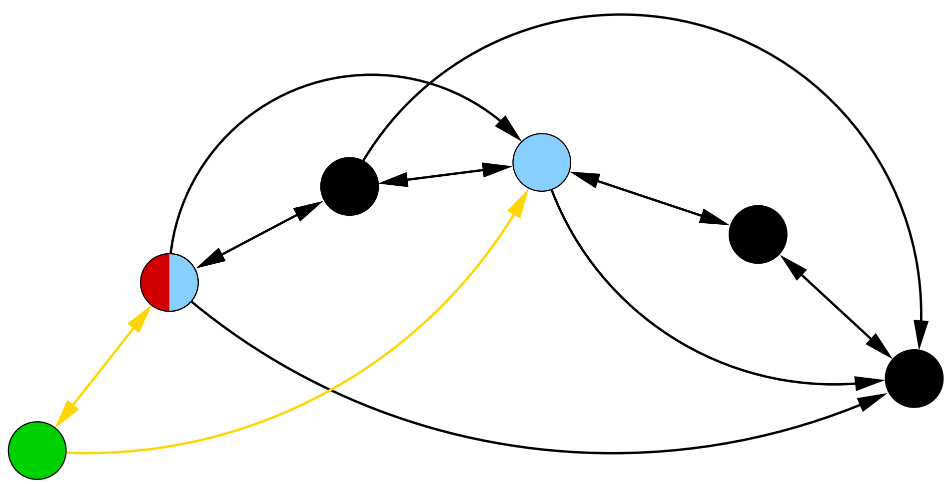

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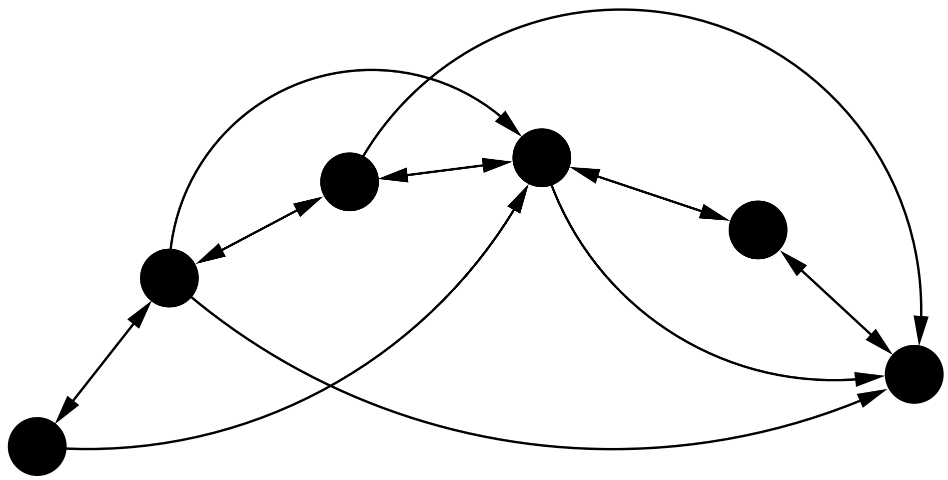
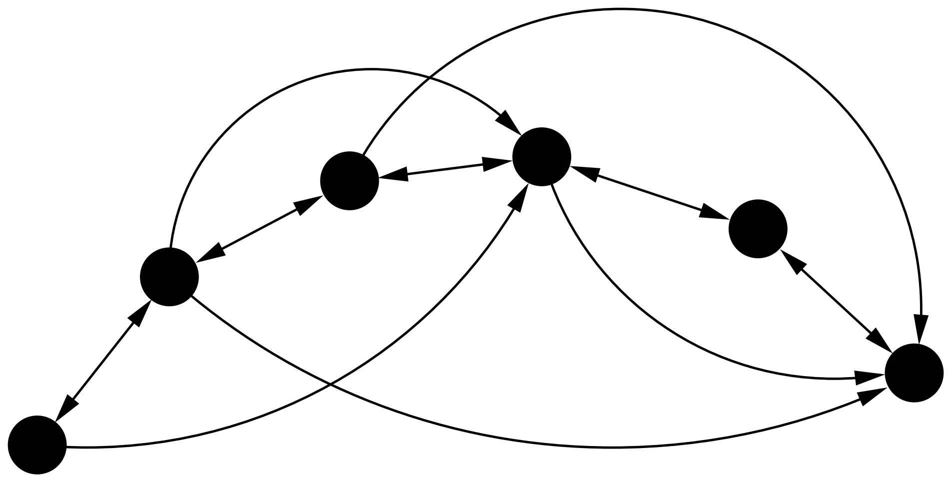


Chordal digraphs



Observation

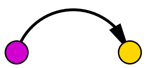
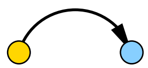
chordal digraphs exist and have non-trivial structure

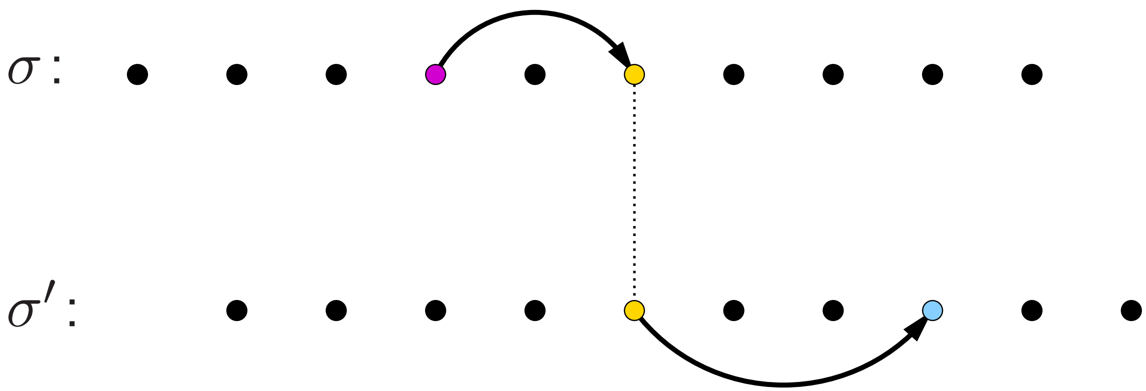
Presentation part

Layout characterisation

Layout pair (σ, σ') with transitivity property:

for all vertex triples  with  $\prec_\sigma$  and  $\prec_{\sigma'}$ 

if  and 



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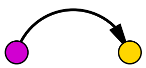
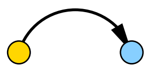
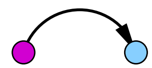
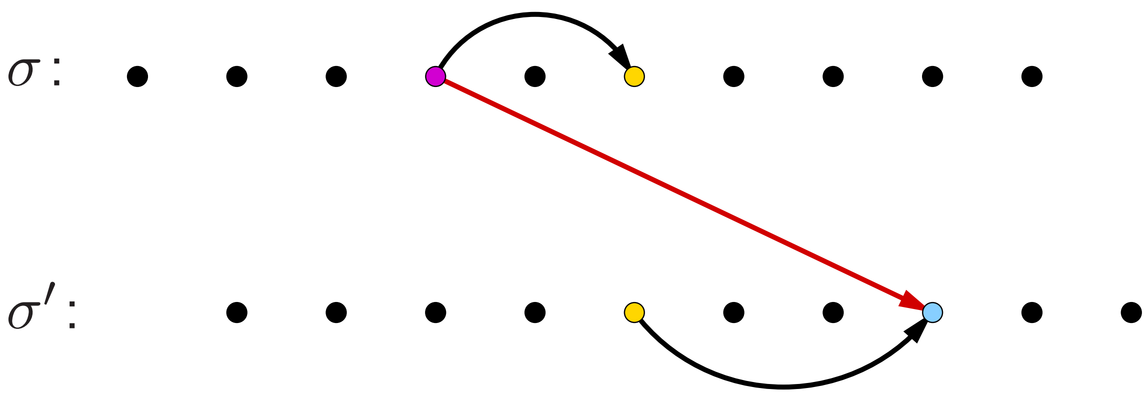
if  and  then 



Characterisations of chordal digraphs

Layout characterisation

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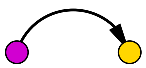
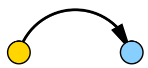
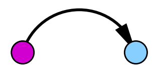
if  and  then 

Theorem

Digraph is chordal if and only if
it has vertex layout σ such that (σ, σ^R) has transitivity property.

Characterisations of chordal digraphs

Theorem

Digraph is chordal if and only if
it has vertex layout σ such that (σ, σ^R) has transitivity property.

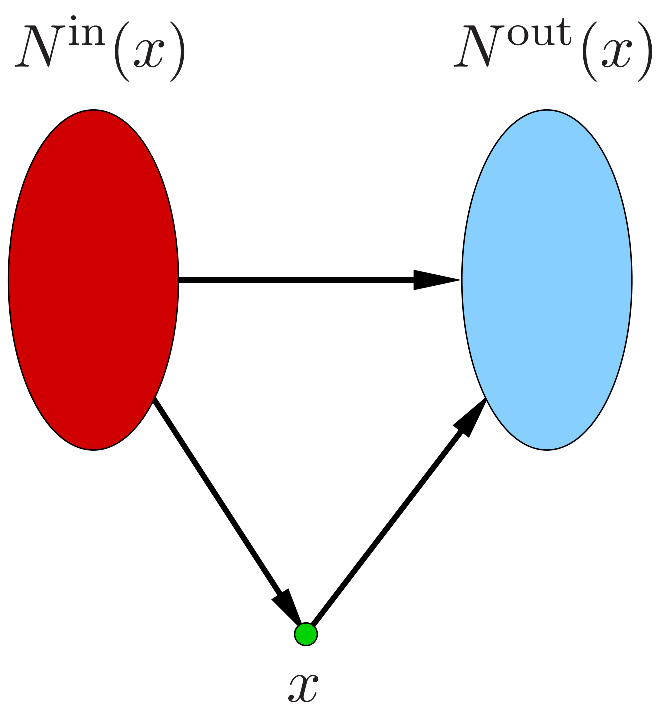
For G digraph and σ vertex ordering,
G acyclic and σ topological $\longrightarrow$ (σ, σ^R) has transitivity property

Theorem

Acyclic digraphs are exactly the chordal digraphs without double-connections.

Elimination characterisation

vertex x is *di-simplicial* if $(N^{\mathrm{in}}(x), N^{\mathrm{out}}(x))$ is "clique"



Characterisations of chordal digraphs

Elimination characterisation

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Digraph is chordal if and only if

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di-simplicial vertex verification in $\mathcal{O}(n^2)$ time

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Remark

di-simplicial vertex verification in $\mathcal{O}(n^2)$ time

chordal digraph recognition in $\mathcal{O}(n^4)$ time

Forbidden subgraphs of semicomplete chordal digraphs

Presentation part

Forbidden subgraphs of semicomplete chordal digraphs

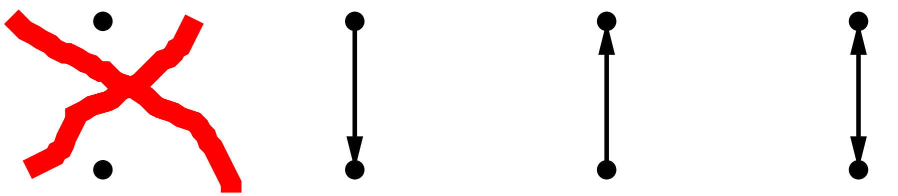
connection between two vertices in a digraph

Forbidden subgraphs of semicomplete chordal digraphs

connection between two vertices in a digraph

In this part:

forbid a connection type



Forbidden subgraphs of semicomplete chordal digraphs

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In this part:

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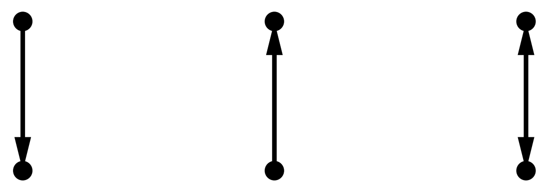
Such digraphs: **semicomplete**

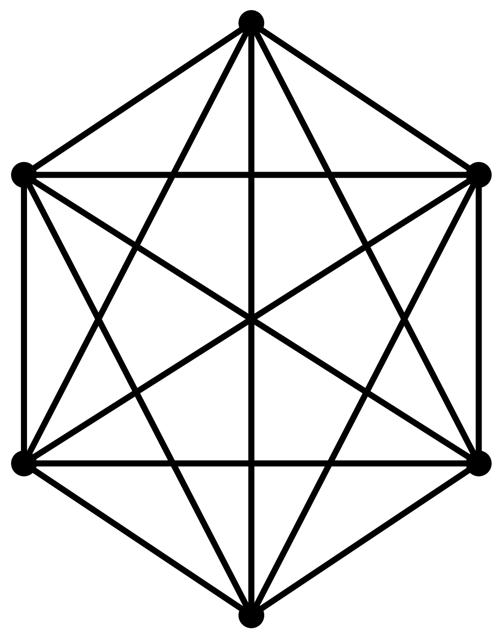


Forbidden subgraphs of semicomplete chordal digraphs

Observation for semicomplete digraphs

number of complete undirected graphs on r vertices: 1

number of semicomplete digraphs on r vertices: > 40 (for $r \geq 5$)



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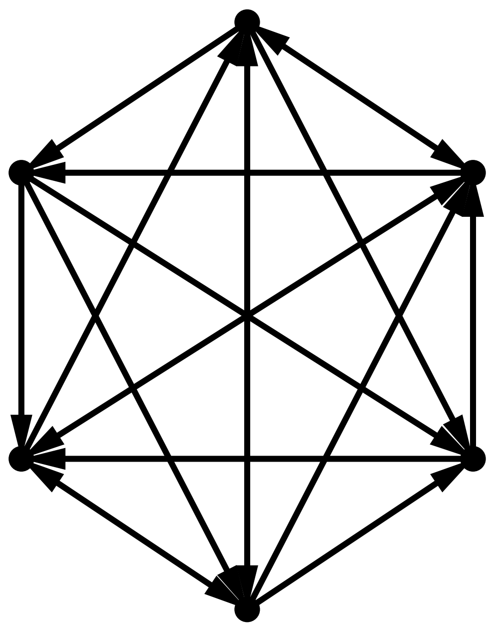

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Observation for semicomplete digraphs

number of complete undirected graphs on r vertices: 1
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Remark

class of semicomplete digraphs at least as rich as
whole class of undirected graphs

Forbidden subgraphs of semicomplete chordal digraphs

first result and connection to chordal undirected graphs

Lemma

For an arbitrary chordal digraph, the restriction to only double-connections induces a chordal undirected graph.

Forbidden subgraphs of semicomplete chordal digraphs

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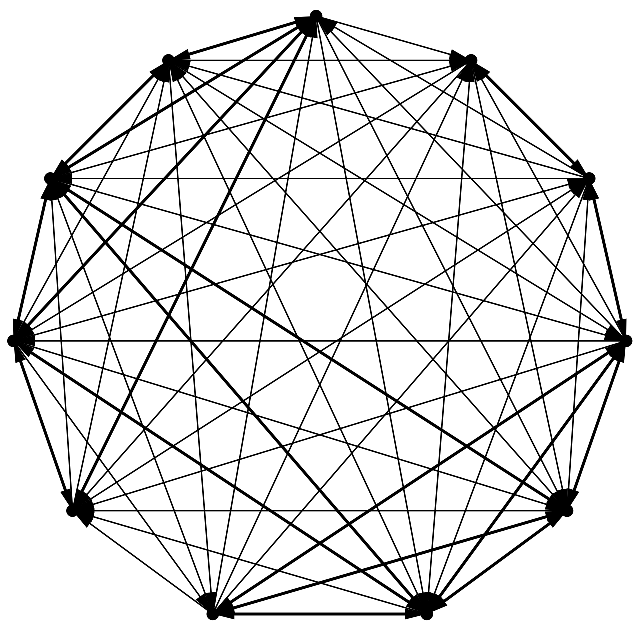

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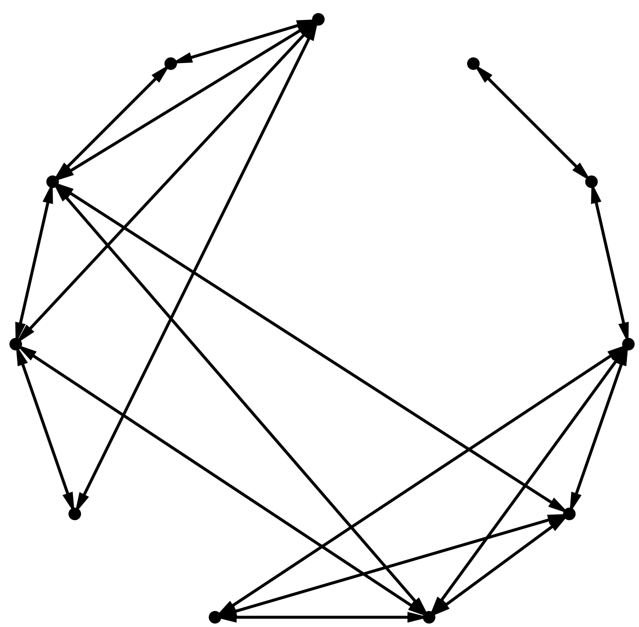

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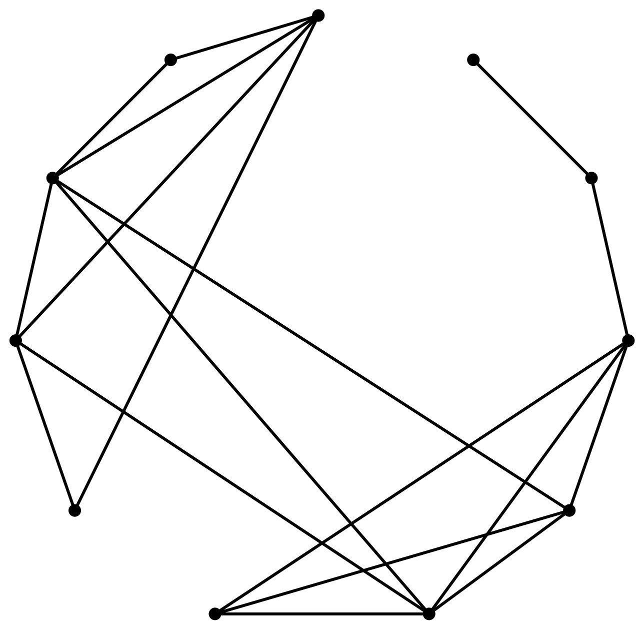

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lemma shows everything that there is to know about structure among double-connected vertices

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first result and connection to chordal undirected graphs

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For an arbitrary chordal digraph, the restriction to only double-connections induces a chordal undirected graph.

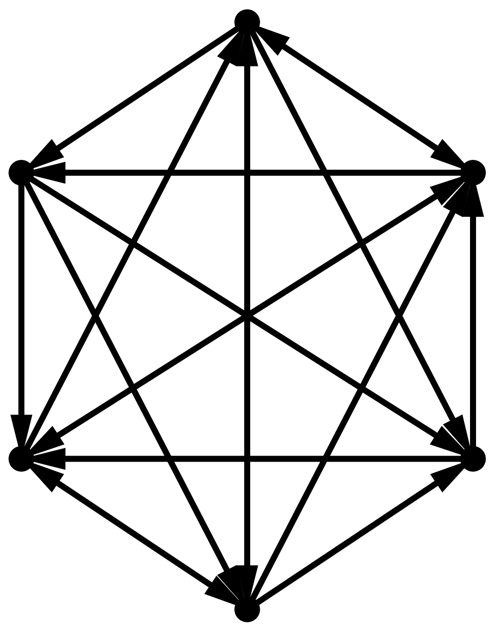
Remark

lemma shows everything that there is to know about structure among double-connected vertices

chordal digraphs with only double-connections correspond exactly to chordal undirected graphs

Forbidden subgraphs of semicomplete chordal digraphs

now, consider single-connections



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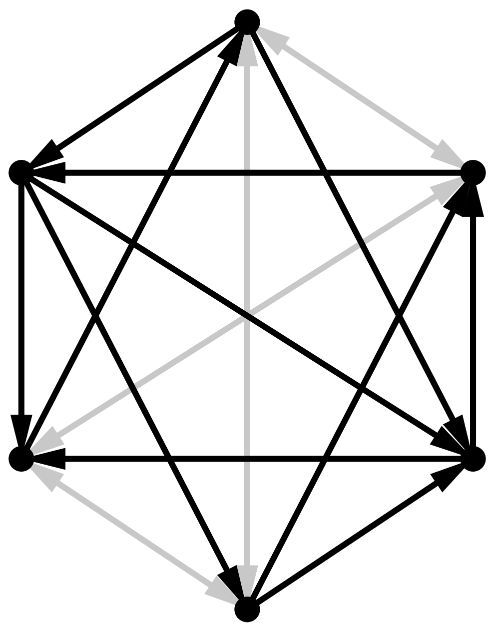
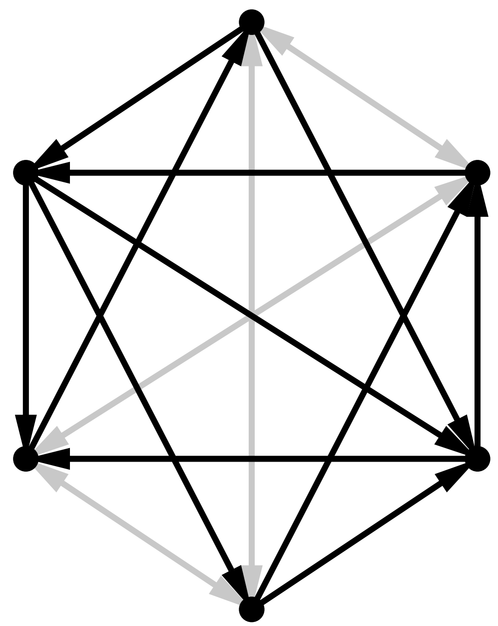


Forbidden subgraphs of semicomplete chordal digraphs

now, consider single-connections

Observation

pairs of non-adjacent vertices are double-connected
so connection type precisely known



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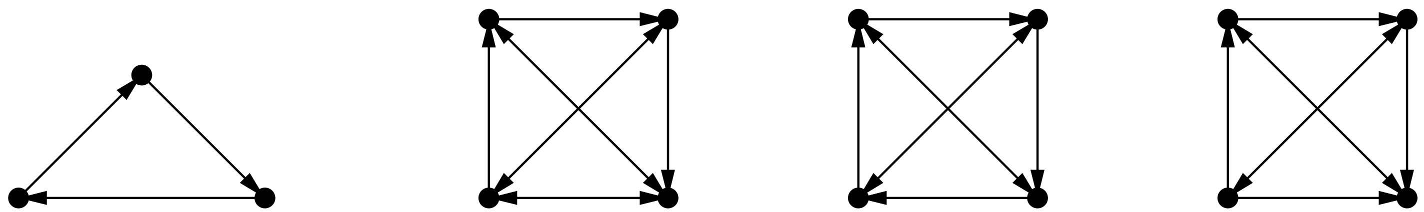
Lemma

Depicted   semicomplete digraphs are not chordal.



Forbidden subgraphs of semicomplete chordal digraphs

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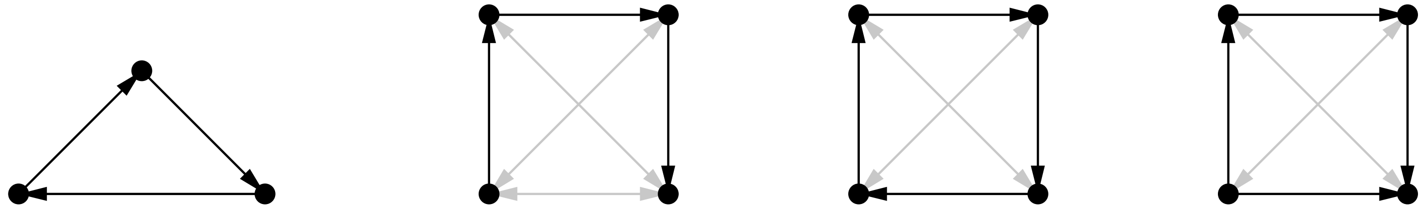
Depicted (restricted) semicomplete digraphs are not chordal.



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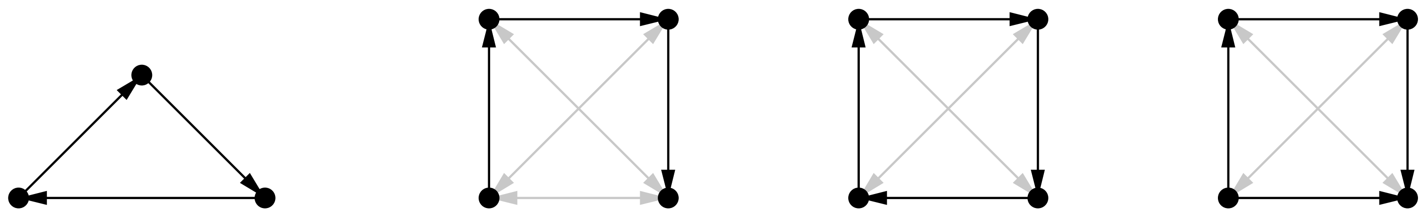
Induced subgraphs of chordal digraphs are chordal.



Forbidden subgraphs of semicomplete chordal digraphs

now, consider single-connections

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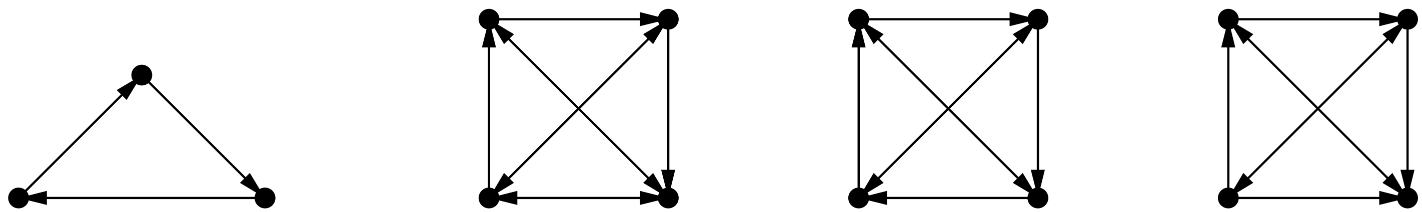
Induced subgraphs of chordal digraphs are chordal.

Lemma

No chordal digraph contains depicted digraphs as induced subgraph.



Forbidden subgraphs of semicomplete chordal digraphs

Lemma

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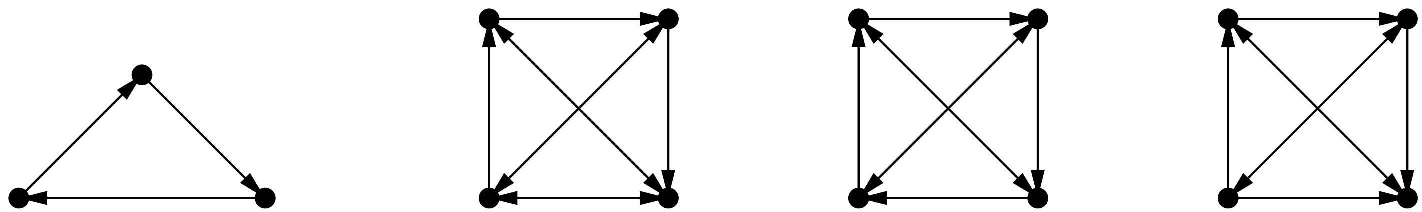


For semicomplete digraphs, also converse holds.

Forbidden subgraphs of semicomplete chordal digraphs

our main result about chordal semicomplete digraphs

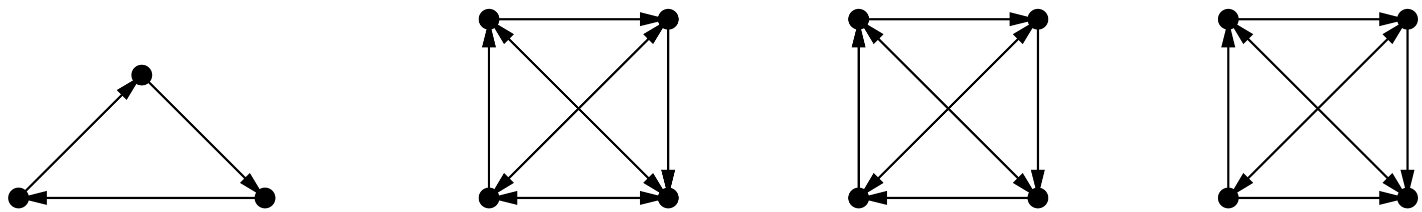
Lemma

For G semicomplete digraph,

if restriction to double-connections chordal

and G not chordal

then G contains one of depicted digraphs as induced subgraph.



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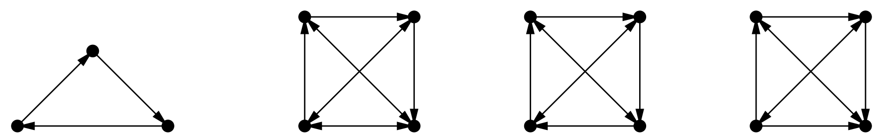


Proof very nice and longer,

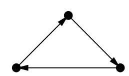
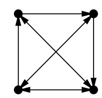
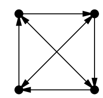
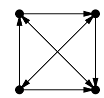
also relies on properties of chordal graphs and simplicial vertices

Forbidden subgraphs of semicomplete chordal digraphs

final characterisation

Theorem

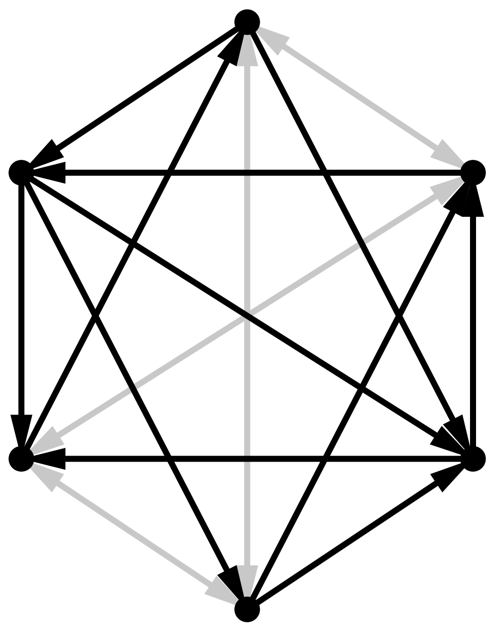
A semicomplete digraph is chordal if and only if

restriction to only double-connections is chordal
and does not contain any of     as induced subgraph.

Forbidden subgraphs of semicomplete chordal digraphs

Example

Does example digraph contain forbidden induced subgraphs?



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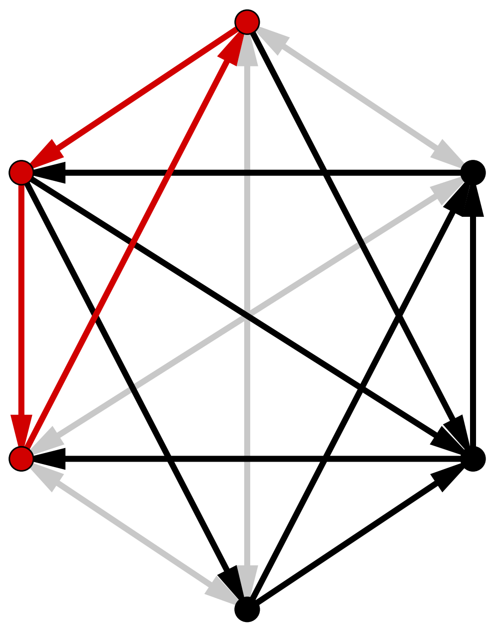

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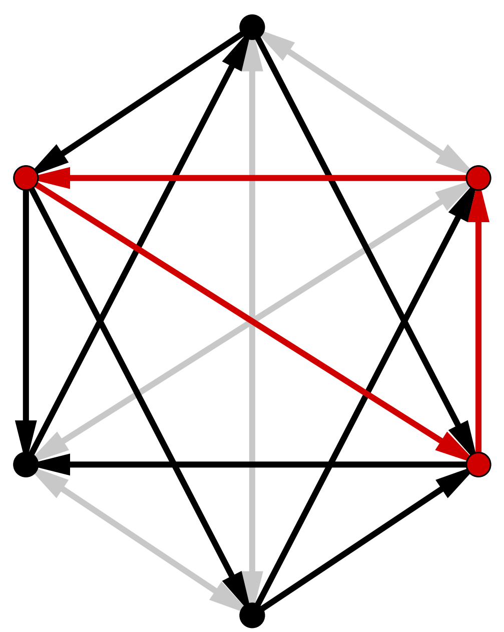

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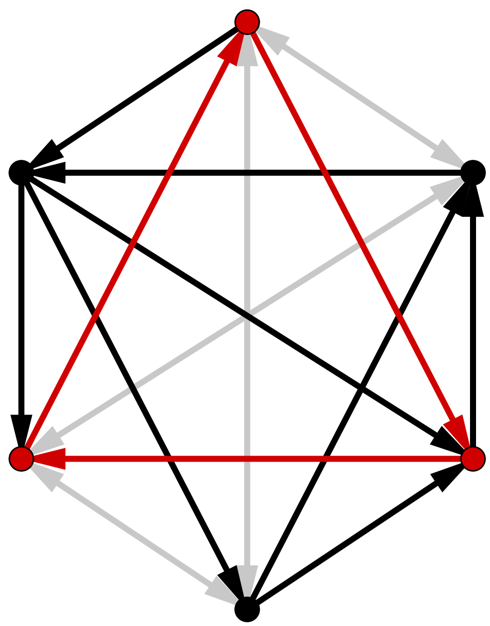


Example

Does example digraph contain forbidden induced subgraphs?

Yes!

Concluding remarks

Presentation part

Concluding remarks

We have seen:

- *without double-connections*
  acyclic digraphs are exactly the chordal digraphs

- *without single-connections*
  chordal undirected graphs “are” exactly the chordal digraphs

- *with single- and double-connections*
  characterisation of a class of chordal digraphs

Concluding remarks

We have seen:

- *without double-connections*
  acyclic digraphs are exactly the chordal digraphs

- *without single-connections*
  chordal undirected graphs "are" exactly the chordal digraphs

- *with single- and double-connections*
  characterisation of a class of chordal digraphs

Main goal

complete list of forbidden subgraphs for chordal digraphs

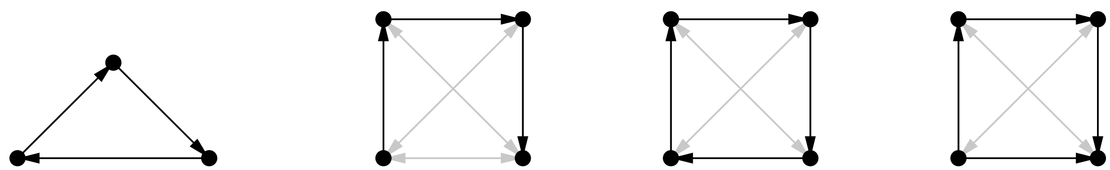
Concluding remarks

Question

Does every chordal undirected graph appear as the double-connection structure of a semicomplete chordal digraph?

In other words,

do



implicitly forbid more structure?

Concluding remarks

chordal undirected graphs have more characterisations such as using minimal separators and cliques

Question

Is there a similar characterisation for chordal digraphs?

What is an appropriate notion of *separator* and *clique*?

Concluding remarks

definition of chordal digraphs is motivated and inspired by Kelly-width definition by Hunter and Kreutzer

$\Longrightarrow$ strong algorithmic connections

Question

Which problems are easy on chordal digraphs?