R. Gandhi[1] and B. Greening[1] and S. Pemmaraju[2] and R. Raman[3]

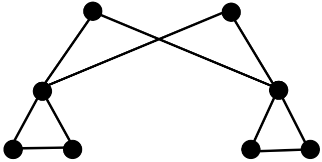
Subcoloring and Hypocoloring Interval Graphs

WG 2009

[1]Rutgers University, Camden. [2]University of Iowa, USA. [3]Max-Planck-Institut für Informatik, Germany

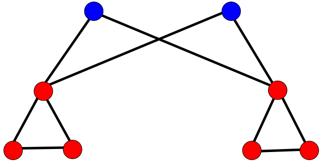
June 24, 2009

Subcoloring



- Let $G = (V, E)$ be a graph.
- A Subcoloring of G is a partition $V_1, \cdots, V_k$ of V, such that each V_i is a union of *disjoint* cliques.

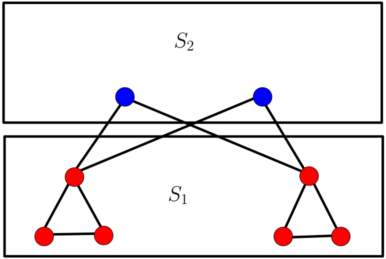
Subcoloring



- Let $G = (V, E)$ be a graph.
- A Subcoloring of G is a partition $V_1, \cdots, V_k$ of V, such that each V_i is a union of *disjoint* cliques.

Subcoloring

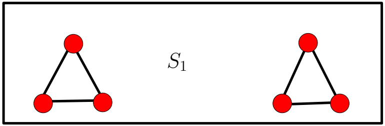
max planck institut informatik



- Let $G = (V, E)$ be a graph.
- A Subcoloring of G is a partition $V_1, \cdots, V_k$ of V, such that each V_i is a union of *disjoint* cliques.

Subcoloring

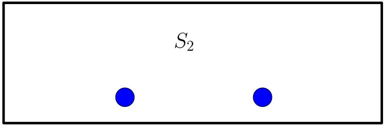
max planck institut informatik



- Let $G = (V, E)$ be a graph.
- A Subcoloring of G is a partition $V_1, \cdots, V_k$ of V, such that each V_i is a union of *disjoint* cliques.

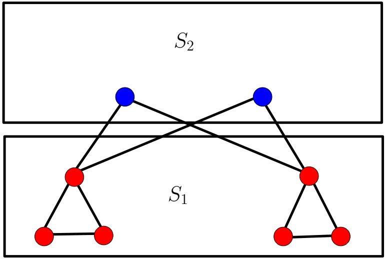
Subcoloring

max planck institut informatik



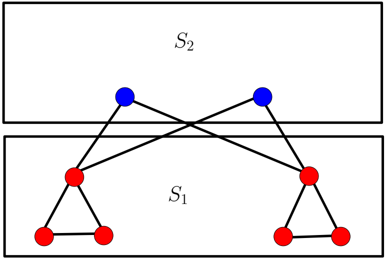
- Let $G = (V, E)$ be a graph.
- A Subcoloring of G is a partition $V_1, \cdots, V_k$ of V, such that each V_i is a union of *disjoint* cliques.

Subcoloring



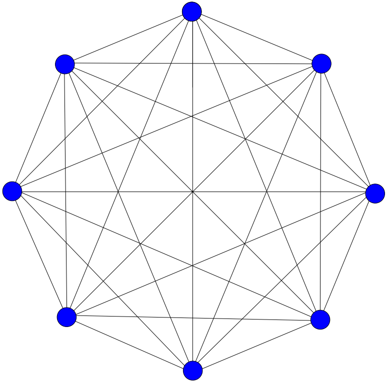
- Let $G = (V, E)$ be a graph.
- A Subcoloring of G is a partition $V_1, \cdots, V_k$ of V, such that each V_i is a union of *disjoint* cliques.

Subchromatic Number



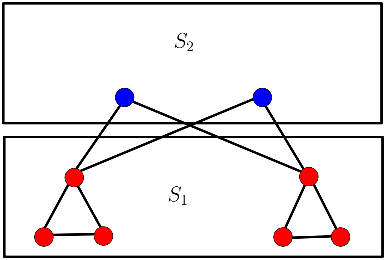
- The subchromatic number of a graph G, $\chi_s(G)$ is the *smallest* k for which such a partition $V_1, \cdots, V_k$ exists.

Example



$\chi_s(K_n) = 1$

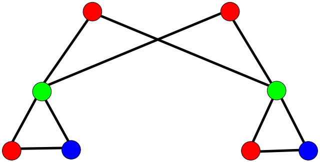
$\chi_s(G) \le \chi(G)$



- $\chi_s(G) \le \chi(G)$
- Infact,
  $\chi_s(G) \le \min\{\chi(G), \chi(\overline{G})\}$

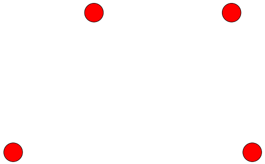
$\chi_s(G) \leq \chi(G)$

max planck institut informatik



- $\chi_s(G) \leq \chi(G)$
- Infact,
  $\chi_s(G) \leq \min\{\chi(G), \chi(\overline{G})\}$

$\chi_s(G) \leq \chi(G)$



- $\chi_s(G) \leq \chi(G)$
- Infact,
  $\chi_s(G) \leq \min\{\chi(G), \chi(\overline{G})\}$

$\chi_s(G) \leq \chi(G)$

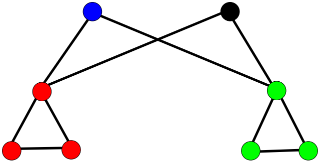
max planck institut informatik

- $\chi_s(G) \leq \chi(G)$
- Infact, $\chi_s(G) \leq \min\{\chi(G), \chi(\overline{G})\}$

$\chi_s(G) \leq \chi(G)$

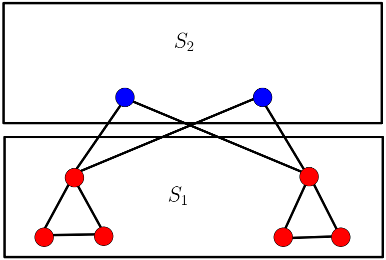
- $\chi_s(G) \leq \chi(G)$
- Infact, $\chi_s(G) \leq \min\{\chi(G), \chi(\overline{G})\}$

$\chi_s(G) \leq \chi(G)$



- $\chi_s(G) \leq \chi(G)$
- Infact, $\chi_s(G) \leq \min\{\chi(G), \chi(\overline{G})\}$

$\chi_s(G) \leq \chi(G)$



- $\chi_s(G) \leq \chi(G)$
- Infact,
  $\chi_s(G) \leq \min\{\chi(G), \chi(\overline{G})\}$

Applications: Approximation Algorithms

- Several Optimization problems on graphs are easier when the underlying graph is a clique, or a disjoint union of cliques.
- If $\chi_s(G)$ is *small*, obtaining a *good* solution to each subcolor class, and picking the best gives a good approximation.

Applications: Maximum Feasible subsystems

$$\begin{array}{c|cccc} & C_1 & C_2 & C_3 & C_4 \\ \hline x_1 & 1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 & 1 \\ x_3 & 1 & 1 & 1 & 0 \end{array}$$

- Given $l \leq Ax \leq u$, $A_{ij} \in \{0, 1\}$, $x \geq 0$
- Find the largest system of inequalities that has a feasible solution.

Applications: Maximum Feasible subsystems

max planck institut informatik

$$
\begin{array}{cc}
 & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$

- Given $l \leq Ax \leq u$, $A_{ij} \in \{0, 1\}$, $x \geq 0$
- Find the largest system of inequalities that has a feasible solution.

Applications: Maximum Feasible subsystems

max planck institut informatik

$$
\begin{array}{cc}
 & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$

- A is a *clique-vertex incidence matrix* if there is a graph G s.t. $x_i \in C_j$ iff $A_{ij} = 1$.
- Consecutive-ones $\Rightarrow$ Interval Graphs.

Applications: Maximum Feasible subsystems

$$\begin{array}{cc} & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

- A is a *clique-vertex incidence matrix* if there is a graph G s.t. $x_i \in C_j$ iff $A_{ij} = 1$.
- Consecutive-ones $\Rightarrow$ Interval Graphs.

Applications: Maximum Feasible subsystems

max planck institut informatik

$$\begin{array}{cc} & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

- Computing a subcoloring gives a partition of the inequalities that are easier to handle.

Applications: Maximum Feasible subsystems

max planck institut informatik

$$\begin{array}{cc} & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

- Computing a subcoloring gives a partition of the inequalities that are easier to handle.

Applications

max planck institut informatik

- Item Pricing: Highway Problem.
- Unspittable Flow on a path.
- Distributed Computing: Cluster graph.

Results

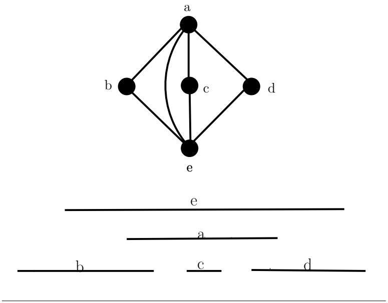
Graph	U.B	L.B	
General	$\frac{n}{\log_2 n/4} + O(\frac{n}{\log_2^2 n})$	$\frac{n}{2\log_2 n+1}$	[Albertson et al 89] [Broersma et al. 03]
Perfect	$\sqrt{2n+1/4}$	$\sqrt{2n-1}$	[Erdös et al. 91]
Chordal	$\log n$	$\log n$	[Broersma et al. 03]

Results

max planck institut informatik

- F-free coloring: Color so that no color class has an induced graph isomorphic to F.
- P_k-free coloring: Each color class does not contain an induced P_k (path with k vertices).
- Subcoloring $= P_3$-free coloring.
- [Fiala et al. 01] Deciding if G has an F-free coloring with $r \geq 2$ colors is NP-hard for triangle-free planar graphs of max. degree 4.
- [Stacho 08] Deciding of a chordal graph has a subcoloring with $r \geq 3$ colors is NP-hard. For $r = 2$, poly. time.
- [Hoàng Le 01] P_4-free coloring of comparability, co-comparability graphs is NP-hard.

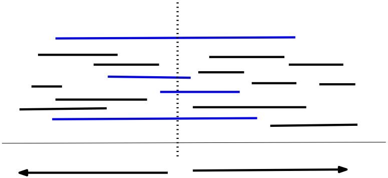
Interval Graphs



- $G = (V, E)$ is an interval graph if the vertices can be represented as an intersection graph of intervals.

Subchromatic Number
Interval Graphs

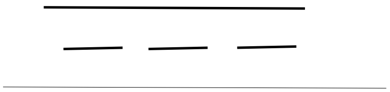
max planck institut informatik



- Pick the middle clique and recurse.
- $\chi_s(G) \leq \lfloor \log_2(n+1) \rfloor$

$K_{1,k}$-free graphs

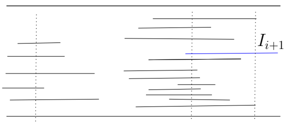
$K_{1,3}$



- [Albertson et al. 89]
- Interval graphs without an induced $K_{1,k}$ have subchromatic number at most $k-1$.

Subcoloring Interval Graphs

Dynamic Programming [Broersma et al.]
Proceed in the order of the left endpoints of the intervals.



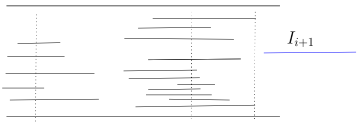
For a color class C, either

- I_{i+1} can be added to the last clique (if it does not intersect the previous clique), or
- I_{i+1} can form a new clique in C.
- Hence we only need $\max(C_{q-1}), \min(C_q), \max(C_q)$ to decide the two cases.

Subcoloring Interval Graphs

max planck institut informatik

Dynamic Programming [Broersma et al.]
Proceed in the order of the left endpoints of the intervals.



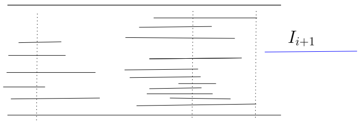
For a color class C, either

- I_{i+1} can be added to the last clique (if it does not intersect the previous clique), or
- I_{i+1} can form a new clique in C.
- Hence we only need $\max(C_{q-1}), \min(C_q), \max(C_q)$ to decide the two cases.

Subcoloring Interval Graphs

max planck institut informatik

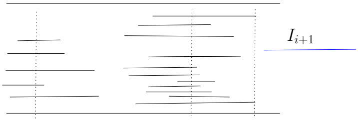
Dynamic Programming [Broersma et al.]



- State specified by a $(3r+1)$-tuple $[k; union', inter, union]$
- where $union', inter, union$ are r-tuples.
- Total states $O(n^{3r+1})$.
- Compute Boolean value $B[k; union', inter, union]$

Subcoloring Interval Graphs

Dynamic Programming [Broersma et al.]

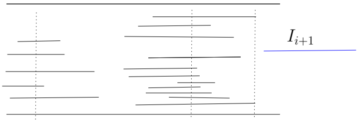


- Compute Boolean value $B[k; union', inter, union]$
- $B[k; union', inter, union] = 1$ iff there is a subcoloring of G_k with r colors s.t.
  - $union_j$ is the right endpoint of inter(K_l),
  - $union_j$ is the right endpoint of union(K_{l-1})
  - $inter_j$ is the right end point of intersection of K_l

Subcoloring Interval Graphs

max planck institut informatik

Dynamic Programming [Broersma et al.]



- Running time $O(r \cdot n^{3r+1})$.
- Since $r = O(\log n)$,
- Running time $O(n^{\log n})$

A 3-approximation algorithm

max planck institut informatik

- We give a 3-approximation algorithm for subcoloring interval graphs.

Binary clique

$BC(1)$

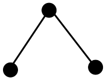
- $\chi_s(BC(k)) \geq k$
- $BC(k)$ can also be realized as an interval graph.

Binary clique

- $\chi_s(BC(k)) \geq k$
- $BC(k)$ can also be realized as an interval graph.

Binary clique

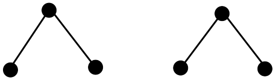
$BC(2)$



- $\chi_s(BC(k)) \geq k$
- $BC(k)$ can also be realized as an interval graph.

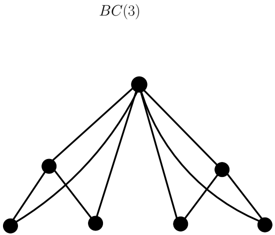
Binary clique

$BC(3)$



- $\chi_s(BC(k)) \geq k$
- $BC(k)$ can also be realized as an interval graph.

Binary clique



- $\chi_s(BC(k)) \geq k$
- $BC(k)$ can also be realized as an interval graph.

Approximation Algorithm

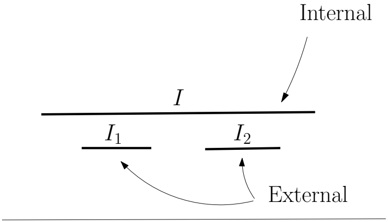
max planck institut informatik

- Phase I: Assign to each interval a *subclique number*, *scn*.
  - Let $S_i = \{I : scn(I) = i\}$
- Phase II: Compute a 3-subcoloring for each S_i.

Approximation Algorithm

max planck institut informatik

Subclique Numbers

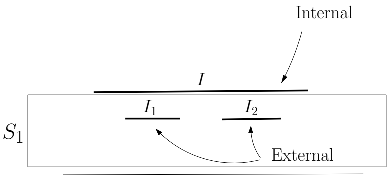


- An interval I is *internal* if $\exists$ $I_1, I_2 \subseteq \mathcal{I}$ s.t.
  - $I_1 \cap I_2 = \emptyset$.
  - $I_1 \subseteq I$ and $I_2 \subseteq I$
- Otherwise I is *external*

Approximation Algorithm

max planck institut informatik

Subclique Numbers



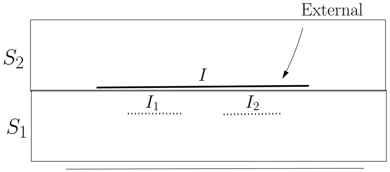
Phase I: Peel off External Intervals.

1. Set $k = 1$
2. While $\mathcal{I} \neq \emptyset$ do
   - $S_k = \{I : I \text{ is } \textit{external}\}$
   - $\mathcal{I} = \mathcal{I} \setminus S_k$
   - $k = k + 1$
3. Return $S_1, \cdots, S_k$

Approximation Algorithm

max planck institut informatik

Subclique Numbers



Phase I: Peel off External Intervals.

1. Set $k = 1$
2. While $\mathcal{I} \neq \emptyset$ do
   - $S_k = \{I : I \text{ is } \textit{external}\}$
   - $\mathcal{I} = \mathcal{I} \setminus S_k$
   - $k = k + 1$
3. Return $S_1, \cdots, S_k$

Approximation Algorithm

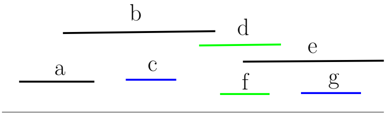
max planck institut informatik

Lemma

If Phase I returns $S_1, \cdots, S_k$, then there exists a $BC(k)$ as an induced subgraph

Approximation Algorithm

Subcoloring $S = S_i$

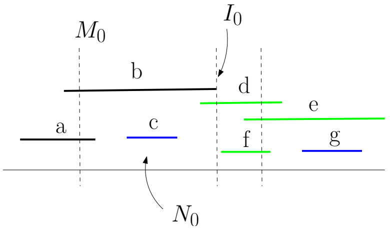


1. M_k = leftmost maximal clique.
2. I_k = Interval in M_k with rightmost endpoint.
3. N_k = intervals not in M_k completely in I_k.
4. $S = S \setminus (N_k \cup M_k)$
5. If $k = 0(mod)2$ $C_0 = C_0 \cup M_k$
6. Else $C_1 = C_1 \cup M_k$
7. $C_2 = C_2 \cup N_k$
8. Return C_0, C_1, C_2

Approximation Algorithm

max planck institut informatik

Subcoloring $S = S_i$



1. M_k = leftmost maximal clique.
2. I_k = Interval in M_k with rightmost endpoint.
3. N_k = intervals not in M_k completely in I_k.
4. $S = S \setminus (N_k \cup M_k)$
5. If $k = 0(mod)2$ $C_0 = C_0 \cup M_k$
6. Else $C_1 = C_1 \cup M_k$
7. $C_2 = C_2 \cup N_k$
8. Return C_0, C_1, C_2

Approximation Algorithm

max planck institut informatik

SubColor(G)

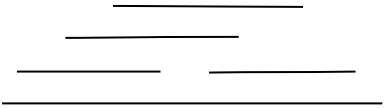
1. Compute $scn(I)$ for each interval.
2. Let $S_i = \{I : scn(I) = i\}$.
3. Subcolor each S_i with at most 3 colors.

Theorem

Algorithm SubColor(G) *is a 3-approximation for subcoloring interval graphs.*

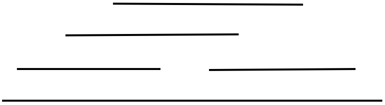
Proper Interval Graphs

max planck institut informatik



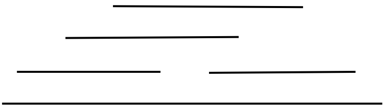
- An interval graph is *proper* if it can be realized such that no interval properly contains another.
- This is equivalent to interval graphs that can be drawn with all intervals of equal length.

Proper Interval Graphs



- Partitioning an interval graph into the fewest number of proper interval graphs.
- SubColor(G) gives a 6-approximation.

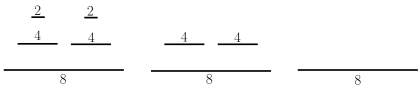
Proper Interval Graphs



Let k= fewest no. of proper interval graphs.

- $k \leq \chi_s(G)$
- A proper interval graph has $\chi_s(G) \leq 2$.
- Hence $\chi_s(G) \leq 2k$.
- Subcolor$(G) \leq 3\chi_s(G) \leq 6k$.

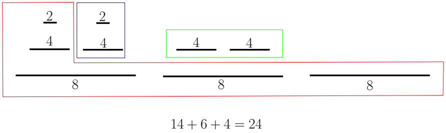
Hypocoloring



Given $G = (V, E)$, $w : V \to \mathbb{N}$,

- Compute a sub-coloring $\{V_1, \ldots, V_k\}$ of G such that
- $\sum_{i=1}^{k} \max_{K \in V_i} w(K)$ is minimized, where
- $w(K) = \sum_{v \in K} w(v)$.

Hypocoloring

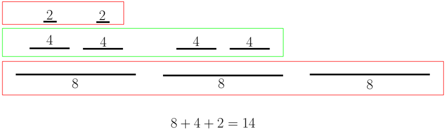


Given $G = (V, E)$, $w : V \to \mathbb{N}$,

- Compute a sub-coloring $\{V_1, \ldots, V_k\}$ of G such that
- $\sum_{i=1}^{k} \max_{K \in V_i} w(K)$ is minimized, where
- $w(K) = \sum_{v \in K} w(v)$.

Hypocoloring

max planck institut informatik



Given $G = (V, E)$, $w : V \rightarrow \mathbb{N}$,

- Compute a sub-coloring $\{V_1, \ldots, V_k\}$ of G such that
- $\sum_{i=1}^{k} \max_{K \in V_i} w(K)$ is minimized, where
- $w(K) = \sum_{v \in K} w(v)$.

Hypocoloring

max planck institut informatik

- Introduced by deWerra, et al. [DeWerra 05]
- NP-hard on bipartite graphs
- NP-hard for triangle-free planar graphs with $\Delta \geq 3$.
- PTAS for graphs of bounded tree-width.

Hypocoloring

max planck institut informatik

- Hypocoloring is NP-complete on interval graphs.
- DSA $\leq$ Hypocoloring $\leq$ Max-Coloring $\leq$ $O(\log n)$ Hypocoloring.
- This gives an $O(\log n)$ approximation algorithm for hypocoloring.

Questions

max planck institut
informatik

- Polynomial time algorithm for subcoloring interval graphs ?
- Constant factor approximations for chordal/perfect graphs ?
- Constant factor approximation for disk graphs ?
- Constant factor approximation for hypocoloring interval graphs ?