Directed Rank-Width and Displit Decomposition

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Graph Complexity Measures

- Tree-width, clique-width or rank-width are interesting.
  - They yield Fixed Parameter Tractable algorithms.
  - They give structural informations on graphs.
- Rank-width is particularly interesting.
  - It is equivalent to clique-width (OS, 06).
  - Recognition of $RWD(\leq k)$ in cubic-time (HO, 07).
  - Characterization by a finite list of graphs to exclude as vertex-minors (O, 05).
  - Algebraic characterization (CK, 07).
  - Rank-width is related to split decomposition (O, 05).

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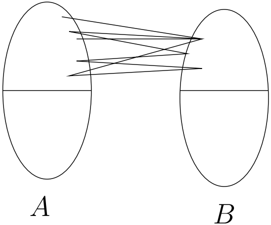
Rank-Width and Split Decomposition

Split decomposition generalizes modular decomposition.

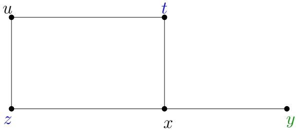
- A split in an undirected graph.
  - $A \cup B = V_G$.
  - $|A|, |B| \geq 2$.
- Prime graph = no split.



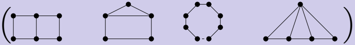
$$\mathrm{rwd}(G) = \max\{\mathrm{rwd}(H) \mid H \text{ induced prime wrt to split decomposition}\}$$

Nice Characterization of $RWD(= 1)$

- y is a pendant vertex.
- z and t are twins.



The following are equivalent

- G has rank-width 1.
- G is obtained by creating twins or adding pendant vertices.
- G is a distance-hereditary graph.
- G is completely decomposable by the split decomposition.
- G is  -free.

- distance hereditary: $d_H(x, y) = d_G(x, y)$ for any connected subgraph H.

Limits

Rank-width is defined only for undirected graphs.

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- There exist several generalizations of rank-width to directed graphs. [Kanté]
  - $GF(4)$-rank-width and bi-rank-width.
- $GF(4)$-rank-width has also good combinatorial properties.
  - Some of the known properties of undirected rank-width can be generalized to $GF(4)$-rank-width.

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  - Some of the known properties of undirected rank-width can be generalized to $GF(4)$-rank-width.

Goal: A characterization of $GF(4)$-$RWD(\leq 1)$ similar to the one of $RWD(\leq 1)$.

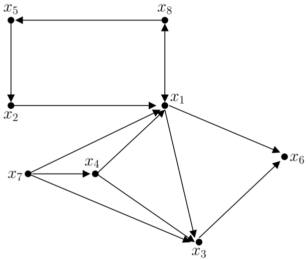
Outline

Representation of a digraph over GF(4)

- 4 elements $\{0, 1, \mathbb{a}, \mathbb{a}^2\}$ in GF(4).
  - $1 + \mathbb{a} + \mathbb{a}^2 = 0.$
  - $\mathbb{a}^3 = 1.$
  - $\alpha + \alpha = 0$ for all $\alpha \in \{0, 1, \mathbb{a}, \mathbb{a}^2\}$.

$M_G =$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1	0	$\mathbb{a}^2$	$\mathbb{a}$	$\mathbb{a}^2$	0	$\mathbb{a}$	$\mathbb{a}^2$	1
x_2	$\mathbb{a}$	0	0	0	$\mathbb{a}^2$	0	0	0
x_3	$\mathbb{a}^2$	0	0	$\mathbb{a}^2$	0	$\mathbb{a}$	$\mathbb{a}^2$	0
x_4	$\mathbb{a}$	0	$\mathbb{a}$	0	0	0	$\mathbb{a}^2$	0
x_5	0	$\mathbb{a}$	0	0	0	0	0	$\mathbb{a}^2$
x_6	$\mathbb{a}^2$	0	$\mathbb{a}^2$	0	0	0	0	0
x_7	$\mathbb{a}$	0	$\mathbb{a}$	$\mathbb{a}$	0	0	0	0
x_8	1	0	0	0	$\mathbb{a}$	0	0	0



Cut-Rank Function

$$\mathit{cutrk}_G(X) = \mathrm{rk}\left(M_G[X, V_G \backslash X]\right)$$

$M_G =$

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x_1	0	$\mathbb{a}^2$	$\mathbb{a}$	$\mathbb{a}^2$	0	$\mathbb{a}$	$\mathbb{a}^2$	1
x_2	$\mathbb{a}$	0	0	0	$\mathbb{a}^2$	0	0	0
x_3	$\mathbb{a}^2$	0	0	$\mathbb{a}^2$	0	$\mathbb{a}$	$\mathbb{a}^2$	0
x_4	$\mathbb{a}$	0	$\mathbb{a}$	0	0	0	$\mathbb{a}^2$	0
x_5	0	$\mathbb{a}$	0	0	0	0	0	$\mathbb{a}^2$
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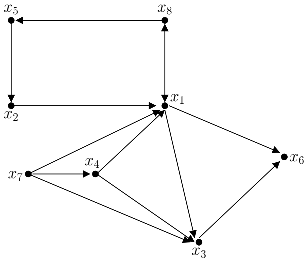
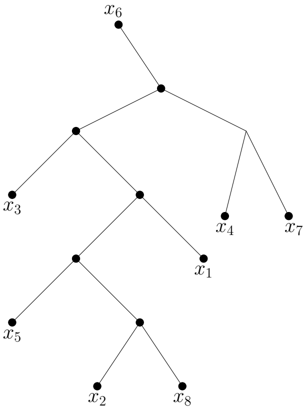


- $\sigma(\mathbb{a}) = \mathbb{a}^2,\ \sigma(\mathbb{a}^2) = \mathbb{a}$: σ is an automorphism
- $M_G[V_G \backslash X, X] = \sigma\left(M_G[X, V_G \backslash X]\right)$
- $\Longrightarrow$ cutrk_G is symmetric and submodular.

Directed Rank-Width

- A layout is a pair $(T, \mathcal{L})$.
  - T is a sub-cubic tree.
  - $\mathcal{L} : V_G \rightarrow L_T$ is a bijection.
- e induces a bipartition $(X_e, V_G \backslash X_e)$ of V_G.
- $wd(e) = cutrk_G(X_e) = cutrk_G(V_G \backslash X_e)$.
- $\mathrm{rwd}(T, \mathcal{L}) = \max_e \{wd(e)\}$.



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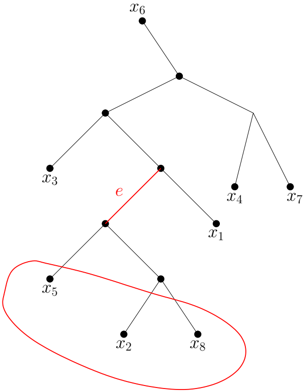

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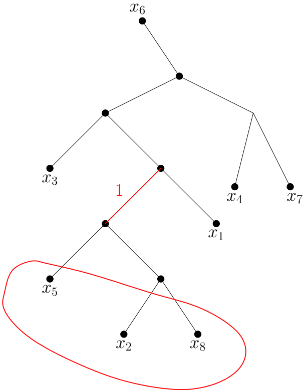

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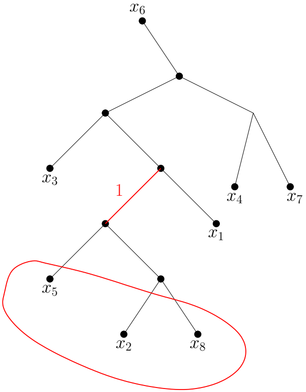

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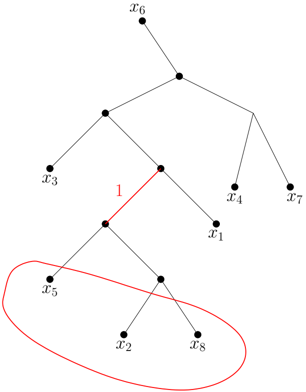

$$\text{rwd}(G) = \min\{\text{rwd}(T, \mathcal{L}) \mid (T, \mathcal{L}) \text{ layout of } G\}$$

Some Properties

- Directed rank-width is equivalent to clique-width.
  - $\mathrm{rwd}(G) \leq \mathrm{cwd}(G) \leq 2 \cdot 4^{\mathrm{rwd}(G)} - 1$.
- Directed $RWD(\leq k)$ recognizable in cubic-time.
- Characterization by a finite list of excluded configurations.
- Algebraic operations for solving MS-definable problems exist.
- It is not related to split decomposition but to displit decomposition.

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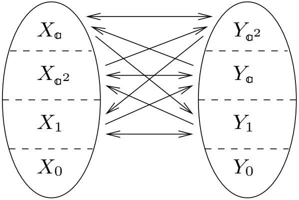
Plan

1. Directed Rank-Width
2. Displit Decomposition
3. Digraphs of Rank-Width 1
   - The Theorem
   - Proof of Theorem

Displit Decomposition

A displit in a graph is a bipartition $\{X, Y\}$ (with $|X| \geq 2 \leq |Y|$) such that $cutrk_G(X) = cutrk_G(Y) = 1$

- Generalize undirected split.
- If a graph has a displit, then it can be decomposed.
- The displit decomposition is the recursive decomposition of a graph by simple displit decompositions, until every graph is prime (i.e. has no displit).



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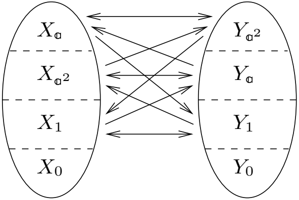

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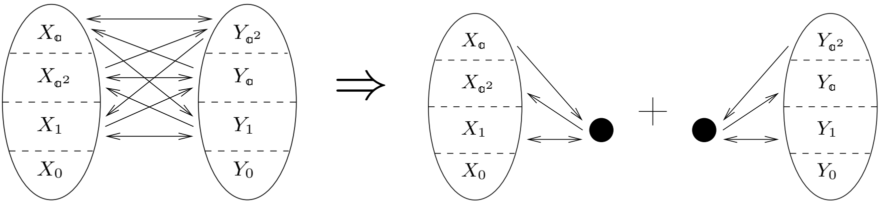

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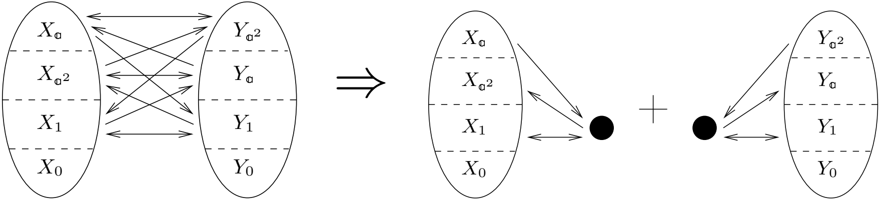

Displit Decomposition Tree

Displit decomposition respects the decomposition frame of Cunningham and Edmonds

Thus it can be represented by a unique undirected tree.

This tree is called the Displit decomposition tree.

Theorem

Displit decomposition tree can be computed in $O(nm)$-time.

Lemma

$$\mathrm{rwd}(G) = \max\{\mathrm{rwd}(H) \mid H \textit{ induced prime wrt to displit decomposition}\}$$

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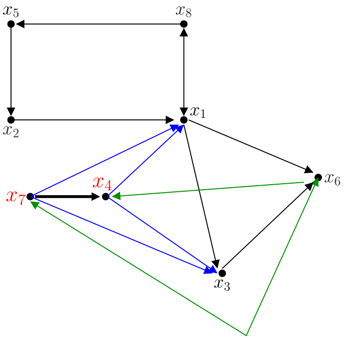
Plan

Dtwins

$$x \text{ and } y \text{ are dtwins if } (A = N^+_{G-y}(x),\ B = N^-_{G-y}(x))$$

- $N^+_{G-x}(y) = A,\ N^-_{G-x}(y) = B$



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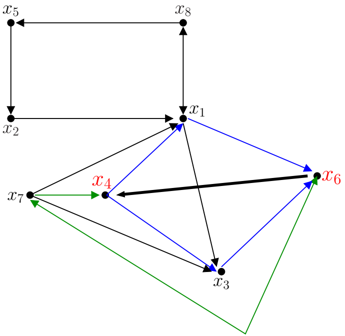

- $N^{+}_{G-x}(y) = A,\ N^{-}_{G-x}(y) = B$
- $N^{+}_{G-x}(y) = B,\ N^{-}_{G-x}(y) = (B \backslash A) \cup (A \backslash B)$

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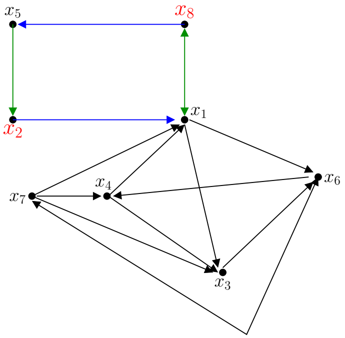

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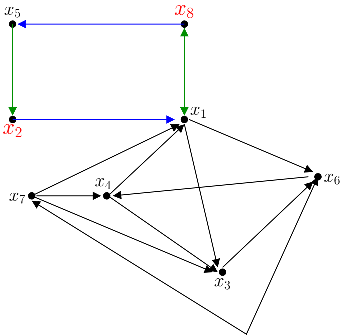


Proposition

$\{\{x, y\}, -\}$ is a displit if and only if x and y are dtwins or x is pendant to y or y is pendant to x.

The Main Theorem

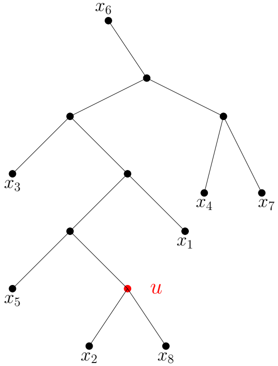
Let G be a connected digraph with at least 2 vertices. Then the following conditions are equivalent:

1. G is completely decomposable by the displit decomposition.
2. G can be obtained from a single vertex by creating dtwins or adding pendant vertices.
3. G has rank-width 1.
4. For every $W \subseteq V$ with $|W| \geq 4$, $G[W]$ has a non-trivial displit.
5. $u(G)$ is distance-hereditary and for every $W \subseteq V$ with $|W| \leq 5$, $\mathrm{rwd}(G[W]) \leq 1$.

- $u(G) = G$ without directions on arcs.
- completely decomposable = no prime nodes.
- Condition 1 gives an $O(nm)$-time algorithm.
- Condition 5 gives a characterization by forbidden induced subgraphs.

$1 \implies 2$: By Induction

Completely Decomposable $\implies$ sum of dtwins and pendant vertices



- Let $(T, \mathcal{L})$ a displit decomposition of G.
- u is degenerated (or linear).
- x_2 and x_8 are dtwins.

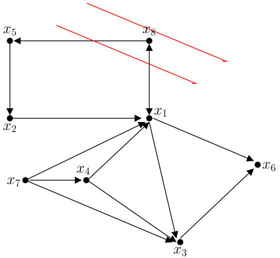
Inductive hypothesis: $G - x_2$ sum of dtwins and pendant vertices.

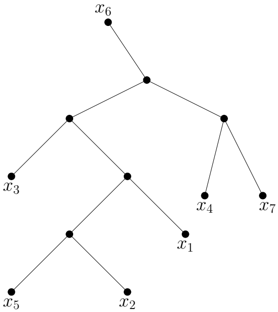
$\implies$ G also.

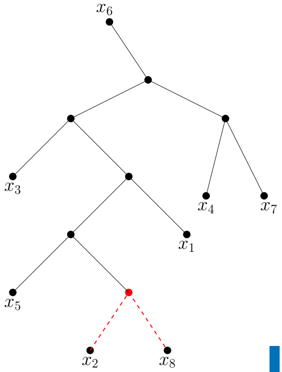
$2 \Longrightarrow 3$: By Induction

Sum of dtwins and pendant vertices $\Longrightarrow \mathrm{rwd}(G) = 1$

- x_8 is the last added vertex.
- Inductive hypothesis: $G - x_8$ has rank-width 1.
- $\Longrightarrow$ G has rank-width 1 since x_2 and x_8 forms a dtwin.



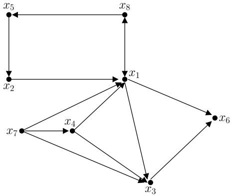


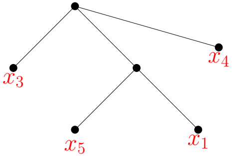


$3 \Longrightarrow 4$

rwd$(G) = 1 \Longrightarrow G[W]$ has a displit, $|W| \geq 4$

- $G[W]$ has rank-width at most 1.
- $\{\{x_3, x_4\}, \{x_5, x_1\}\}$ is a displit.





$3 \Longrightarrow 4$, $4 \Longrightarrow 1$

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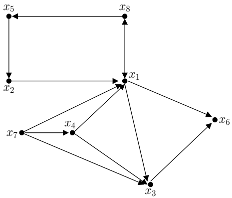

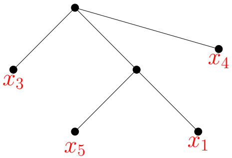

$\neg$ completely decomposable $\Longrightarrow\ \exists\ G[W]$ prime, $|W| \geq 4$

- G not completely decomposable $\Longrightarrow\ \exists$ a prime node u.
- u adjacent with at least 4 vertices of G (otherwise u is degenerated).
- $\Longrightarrow\ \exists$ an induced subgraph of G which is prime.

$3 \iff 5$

$\mathrm{rwd}(G) = 1 \implies u(G)$ distance hereditary and $\mathrm{rwd}(G[W]) \leq 1,\ |W| \leq 5$

- $\mathrm{rwd}(G) = 1 \implies \mathrm{rwd}(u(G)) = 1$.
- $\implies u(G)$ is distance-hereditary graph.

$u(G)$ distance hereditary and $\mathrm{rwd}(G[W]) \leq 1,\ |W| \leq 5 \implies \mathrm{rwd}(G) = 1$

- $\mathrm{rwd}(G) > 1$ and $u(G)$ is distance hereditary.
- Let W st $\mathrm{rwd}(G[W]) > 1$ be minimal (wrt this property).
- One can prove the following (working on the split decomposition tree of $u(G)$):
  - $u(G[W])$ has no pendant vertex,
  - if $u(G[W])$ has a false twin, then $G[W]$ has at most 4 vertices,
  - if $u(G[W])$ has no false twin and no pendant vertex, then $u(G)$ is complete,
  - if $u(G[W])$ is complete, then $G[W]$ has at most 5 vertices.
- $\implies \exists\ W$ st $\mathrm{rwd}(G[W]) > 1$ with $|W| \leq 5$.

Conclusion and Perspectives

Conclusion

- Nice characterization of digraphs of rank-width 1.
- Extends to digraphs with labels (from fields) on the arcs.

Remarks

- A notion of directed split by Cunningham.
- bi-rank-width 2 $\equiv$ completely decomposable by directed split decomposition.

Perspectives

- Similar characterization for bi-rank-width 2.
- A better combinatorial characterization of digraphs of rank-width 1.