Maximum Series-Parallel Subgraph

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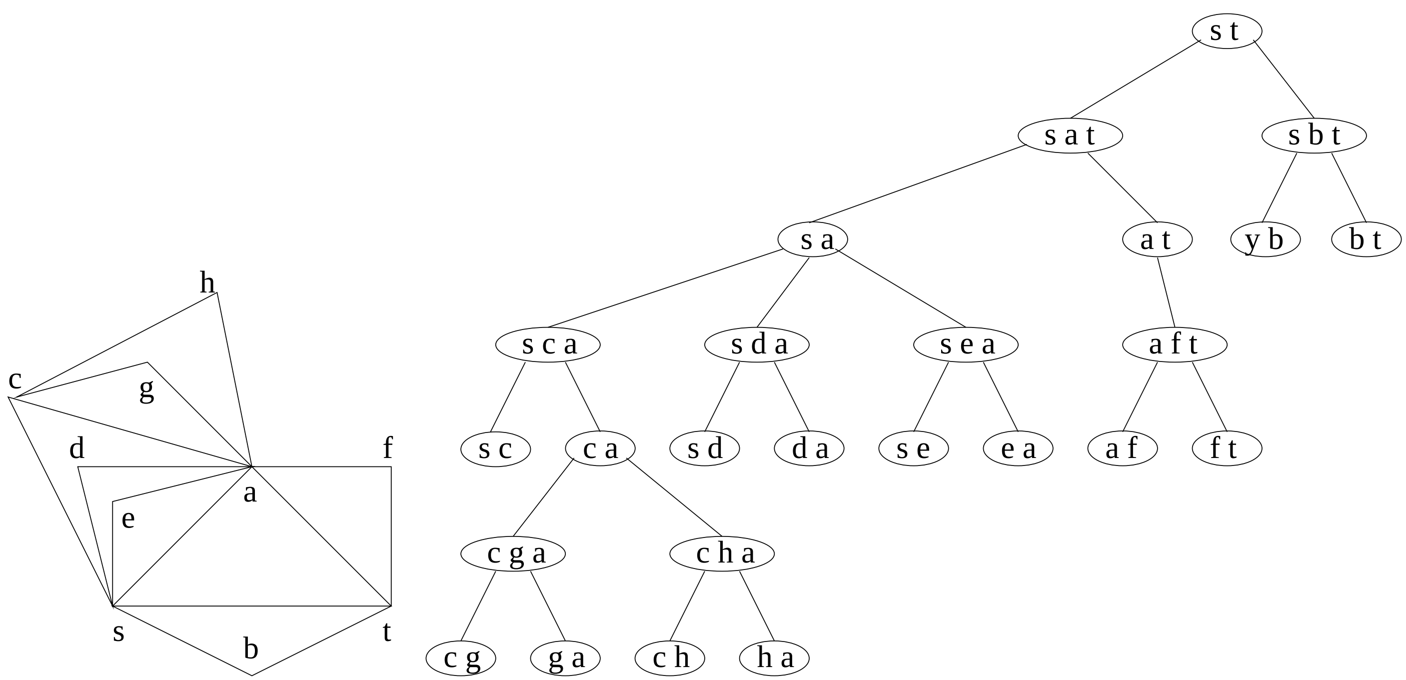
SP graphs

Simple graph is series-parallel:

- No K_4 minor/subdivision
- Partial 2-tree. Maximal SP graph is a 2-tree
- Arises from a forest by adding parallel edges, subdividing edges, and at the end removing parallel edges to keep the graph simple.

Treewidth decompostion of maximal SP graphs



The problem , and known results

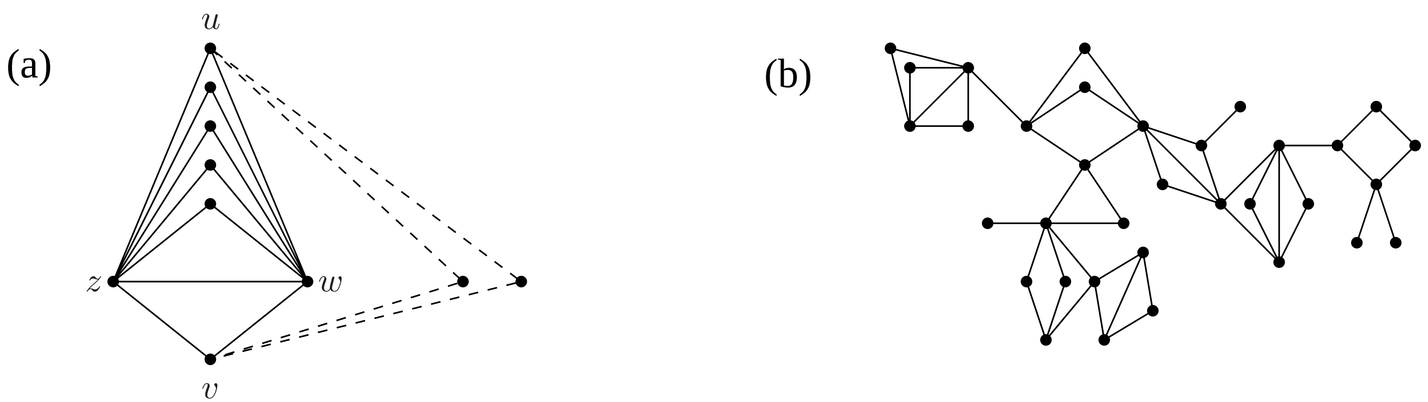
On input (V, E), find maximum $F \subseteq E$ such that (V, F) is a SP graph. Known to be NP-Hard, approximation 1/2 trivial as for SP graphs, $|E| \leq 2|V| - 3$.

Also, weighted: Maximum Spanning Tree achieves 1/2 of OPT (from arboricity).

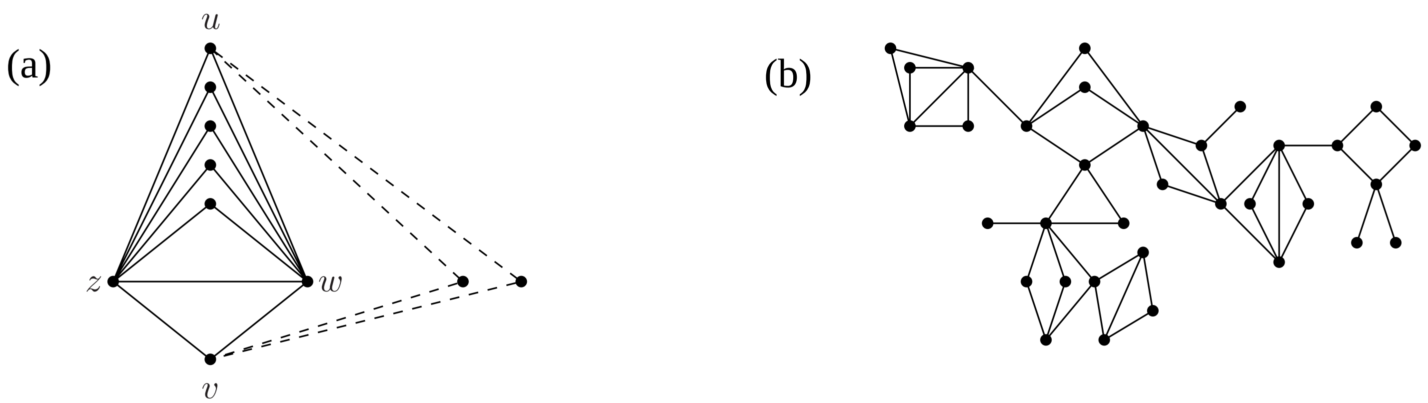
All our graphs from are simple (except when called multigraphs).

New results

- 7/12 approximation using *spruce structures*
- Maximum spruce structure is a 2/3-approximation
- Computing maximum spruce structure NP-Hard



A **spruce** has two bases and several tips. A tip is adjacent exactly to the two bases. If bases are adjacent, the spruce is complete.



A **spruce structure** has blocks spruces or bridges.

If blocks of finite size, the approach fails.

$gain(S) :=$ cyclomatic number.

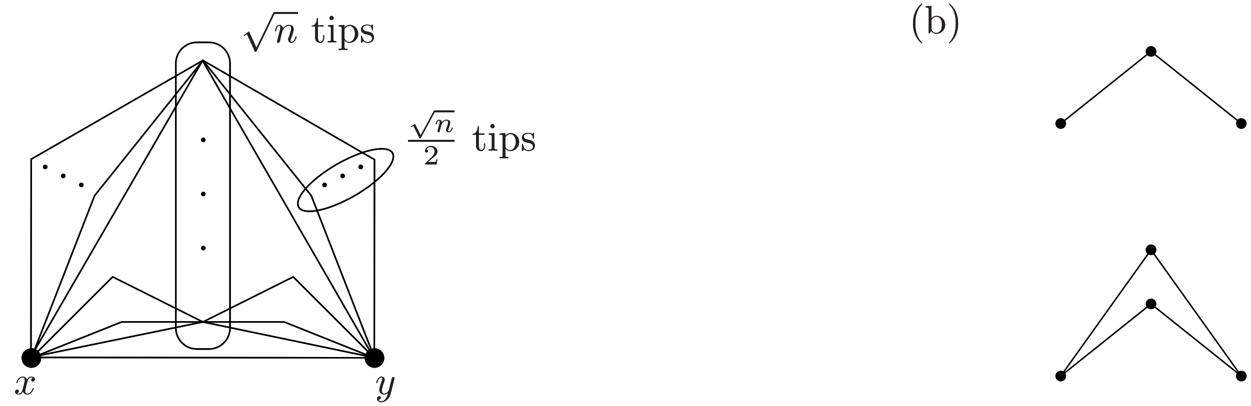
For complete spruces, $gain$ is the number of tips; adjusted gain $\widehat{gain} := gain$.

For incomplete spruces, $gain$ is one less then the number of tips; adjusted gain $\widehat{gain} := gain - 1$.

Selecting spruces of large gain without discarding fails (a).

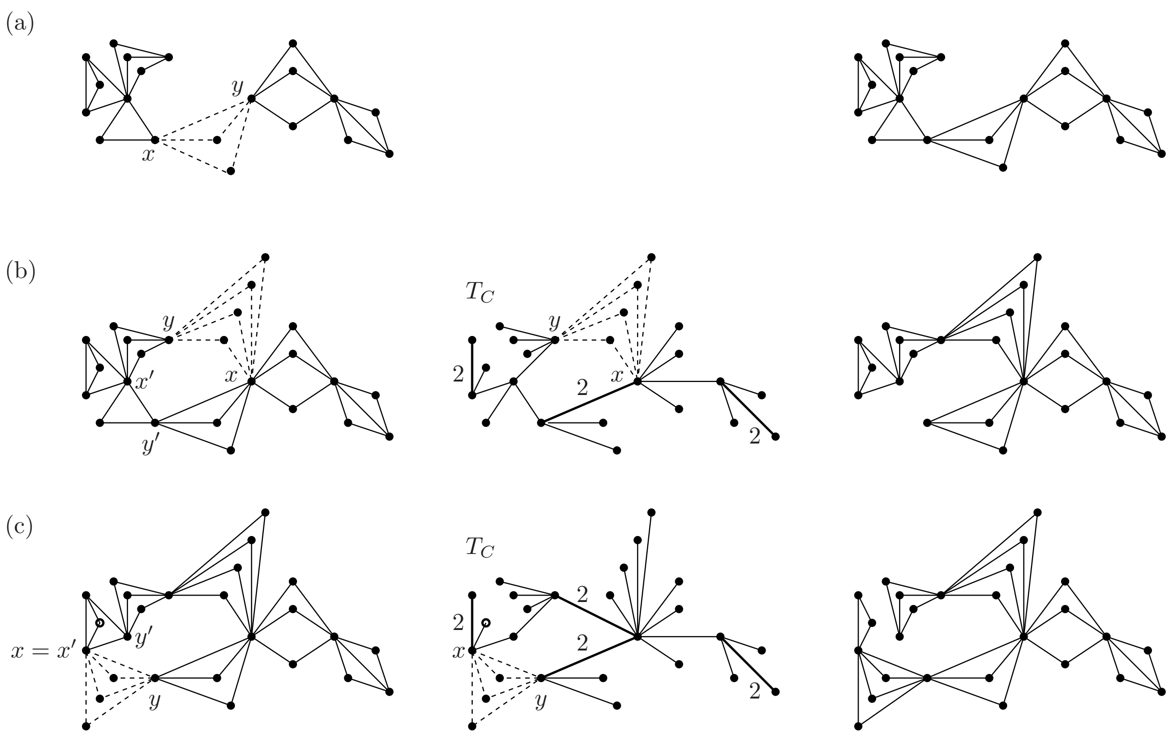
(a)

(b)



Degenerate spruces have $\widehat{gain} \leq 0$. Only two types (b).

Local improvement examples



What we add

Spruces with tips isolated vertices.

What do we remove

We keep a weighted forest, with weights 1 between each tip and one of the bases, and $\widehat{gain}$ between bases. Trees: T_C

$index_Q(x, y)$ is an edge in T_C of minimum weight in the path in T_C from x to y. If x and y are in different components of Q, then let $index_Q(x, y)$ be undefined and consider its weight to be zero.

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Construct-Spruce-Structure (G)
  Q ← ∅
  while there are x and y such that S_Q(x, y) is defined
        and gain^(S_Q(x, y)) > w(index_Q(x, y)) do
      if index_Q(x, y) is undefined
          then Q ← Q ∪ {S_Q(x, y)}
          else let x′ and y′ be the endpoints of index_Q(x, y)
              let S′ be the spruce in Q containing x′ and y′
              Q ← Q \ {S′} ∪ {S_Q(x, y)}
              if x′ or y′ is a tip of S′
                  then let z be between x′, y′, a tip of S′
                      let {e, f} be the edges of S′ touching z
                      S ← S′ − {e, f}
                      if S is not degenerate nor single edge
                          then Q ← Q ∪ {S}
  add bridges to Q to obtain a connected spanning subgraph of G
  return Q
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Running time analysis

Increasing $gain$ would be good, but doesn't quit work as we use $\widehat{gain}$.

Define $\Phi(Q) = 3\, gain(Q) + c(Q)$, where $c(Q)$ is the number of components of Q when Q is seen as a spanning subgraph of G.

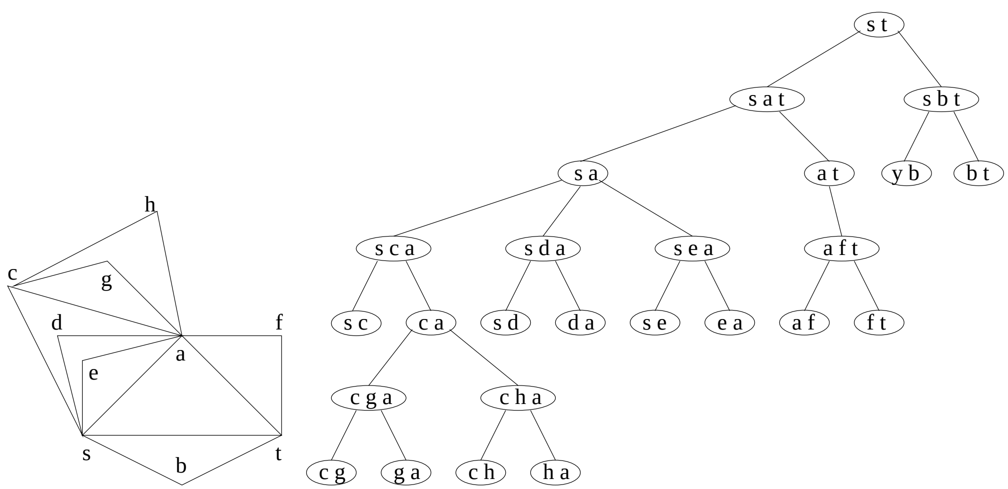
Lemma 1 *Every iteration of the algorithm increases the parameter* Φ*.*

Approximation ratio ideas - to beat $1/2$

significant = constant fraction of n

1. If significantly many vertices in our structure, we win
2. If OPT has significantly less than $2n$ edges, we win
3. If none of the above, the spruces of OPT have significant $\widehat{gain}$

The spruces of OPT



These spruces do not share tips. Some $\widehat{gain}$ is lost, when the tips are not isolated in our structure. Still, enough left.

Approximation ratio ideas - beating $1/2$

From OPT, construct **weighted** SP graph with $\widehat{gain}$ on edges.

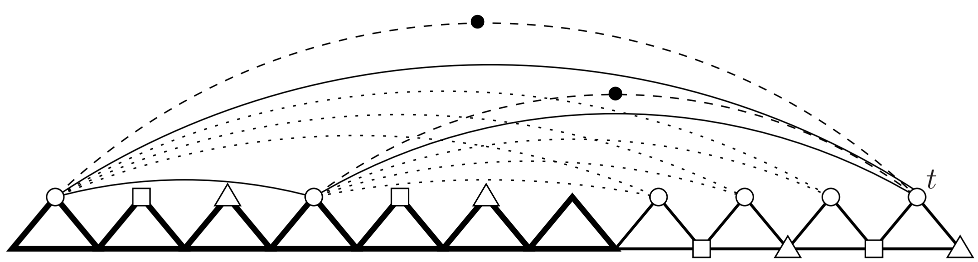
Compare to our weighted forest.

We have a maximum spanning forest in the union of the two graphs!

Thus our $\widehat{gain}$ is $1/2$ of what that of OPT.

Therefore we have significant $\widehat{gain}$.

7/12 **tight**

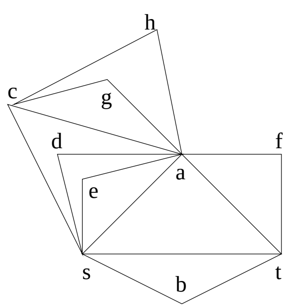


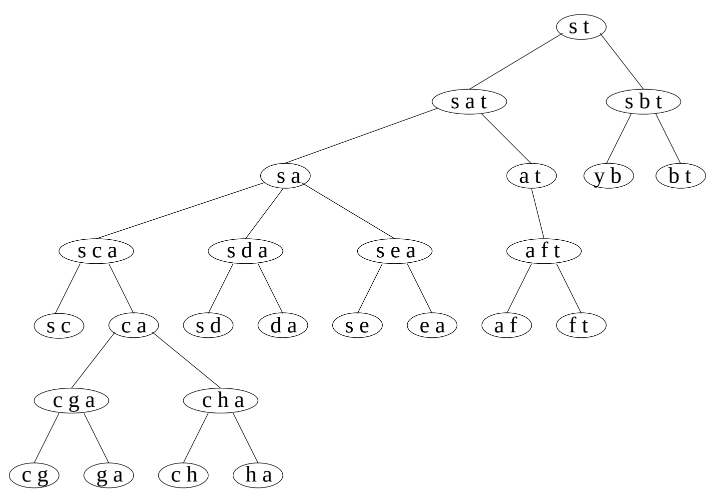
Maximum spruce structure

The spruces of a SP graph have $n-2$ tips. We get 1/3 of that by partitioning those into three spruce structures.

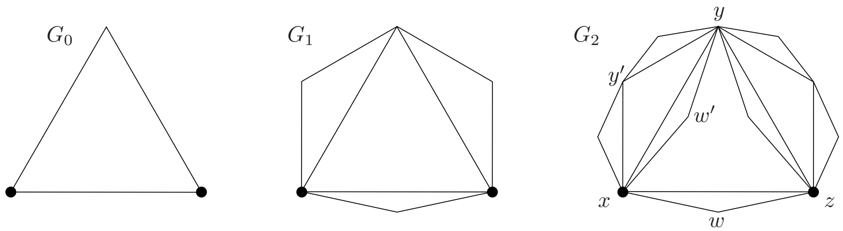
Turns out if no parent-child, no two siblings, it is a spruce structure.

Color root 1 and for every node, color its children with the two other colors.





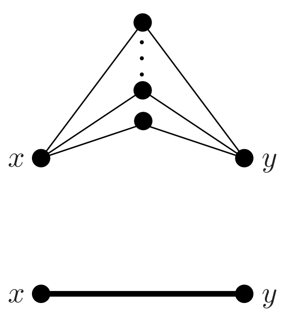
Tight example for maximum spruce structure



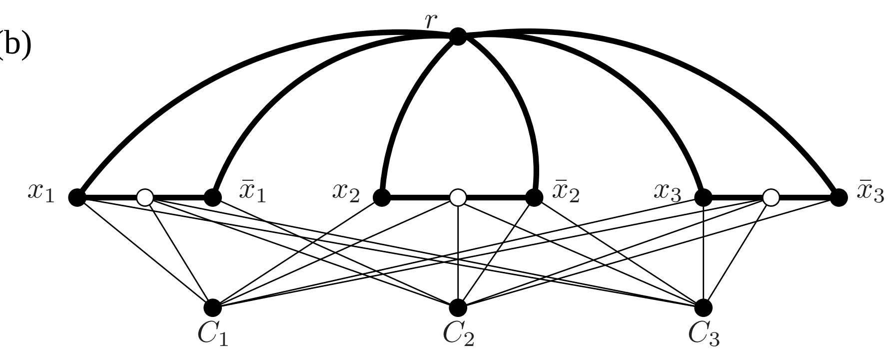
Contructed inductively.

The NP reduction for Maximum Spruce Structure

(a)



(b)



Conclusions

We do a very little better when we pick spruce maximizing $\widehat{gain} - index$.

Completely new approach? Also for Maximum Planar Subgraph, at 4/9 since 1996.

Weighted?