Finding induced paths of given parity in claw-free graphs

Pim van 't Hof

joint work with Marcin Kamiński and Daniël Paulusma

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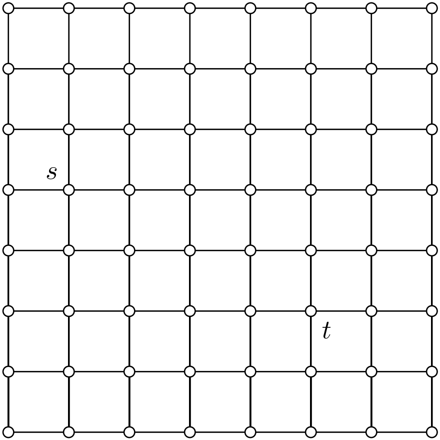
Finding paths of given parity



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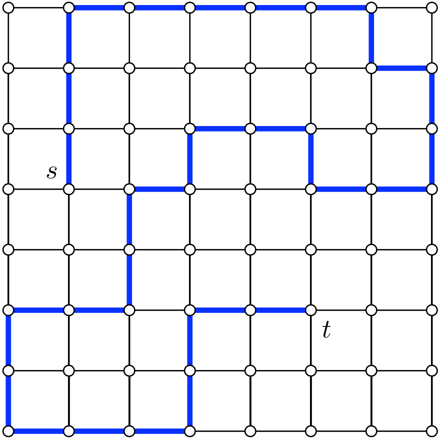

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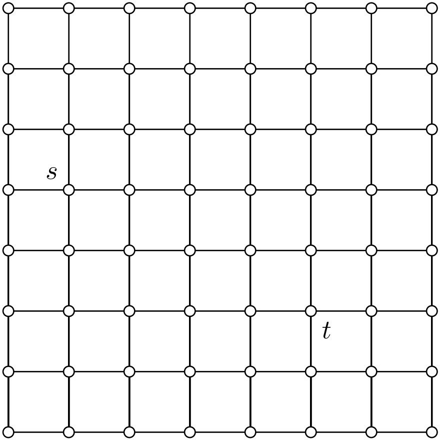

Finding paths of given parity

ODD PATH

Instance: A graph G and two vertices s and t.
Question: Does G have an odd path from s to t?

EVEN PATH

Instance: A graph G and two vertices s and t.
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Finding paths of given parity is easy

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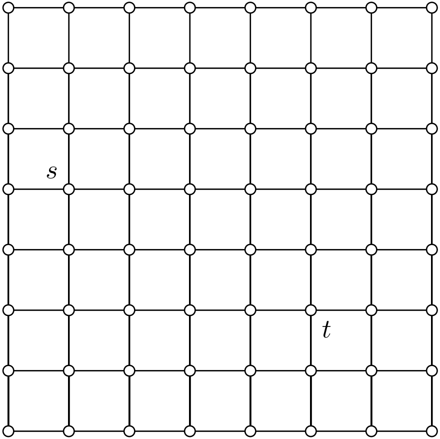
Theorem (LaPaugh & Papadimitriou, 1984)

Both the ODD PATH *problem and the* EVEN PATH *problem can be solved in linear time.*

Finding induced paths of given parity



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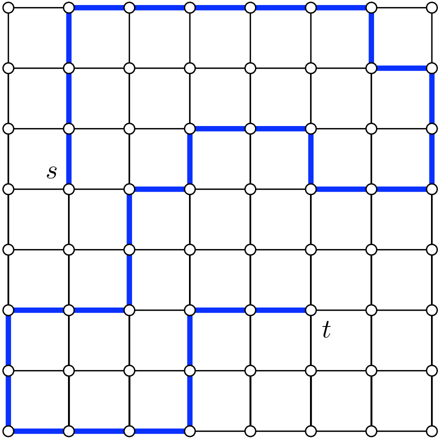

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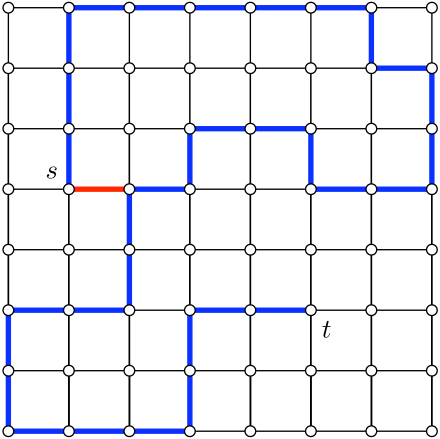

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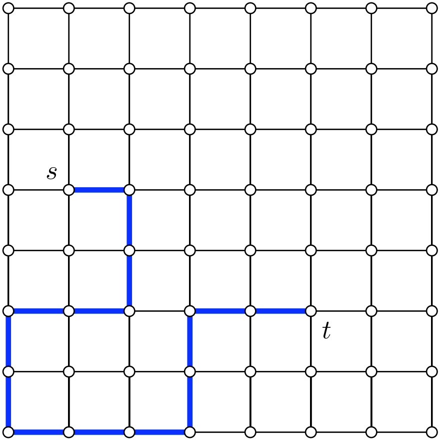

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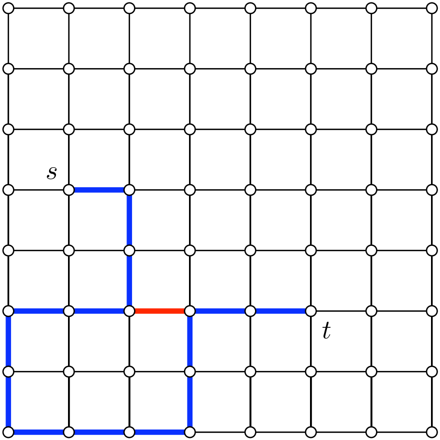

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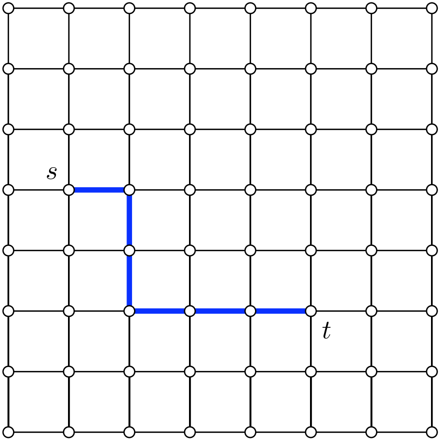

Finding induced paths of given parity

Odd Induced Path

Instance: A graph G and two vertices s and t.
Question: Does G have an odd induced path from s to t?

Even Induced Path

Instance: A graph G and two vertices s and t.
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Finding induced paths of given parity is hard

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Instance: A graph G and two vertices s and t.
Question: Does G have an odd induced path from s to t?

Even Induced Path

Instance: A graph G and two vertices s and t.
Question: Does G have an even induced path from s to t?

Theorem (Bienstock, 1991)

The Odd Induced Path *problem is NP-complete.*

The Parity Path problem

Parity Path

Instance: A graph G and two vertices s and t.
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Theorem (LaPaugh & Papadimitriou, 1984)

Both the Odd Path *problem and the* Even Path *problem can be solved in linear time.*

Corollary

The Parity Path *problem can be solved in polynomial time for the class of line graphs.*

The Parity Path problem

Theorem ("folklore")

The Parity Path *problem can be solved in polynomial time for the class of perfect graphs.*

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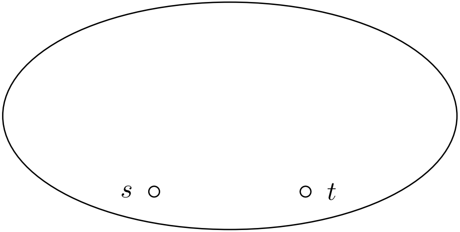

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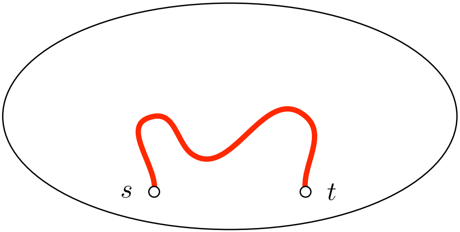

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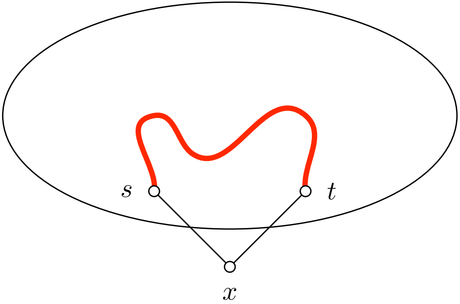

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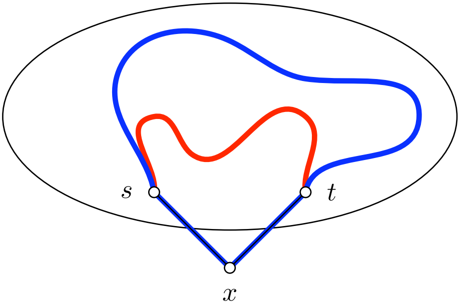

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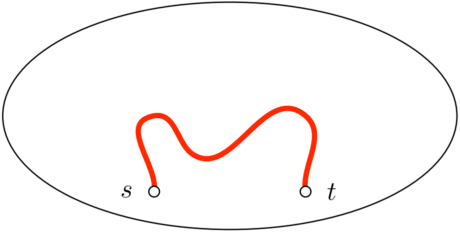

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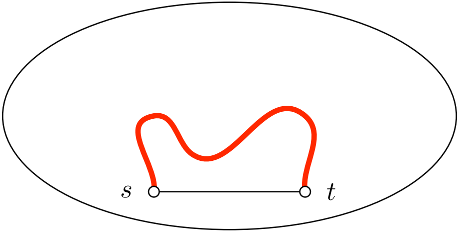

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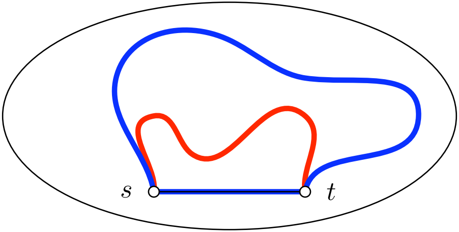

The PARITY PATH problem

The PARITY PATH can be solved in polynomial time for:

- line graphs
- perfect graphs

The Parity Path problem

The Parity Path can be solved in polynomial time for:

- line graphs
- perfect graphs
- perfectly orientable graphs
- circular-arc graphs
- chordal graphs
- comparability graphs
- cocomparability graphs
- permutation graphs
- planar perfect graphs

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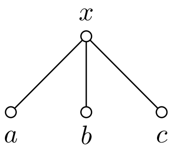
We show that claw-free graphs can be added to this list.

Claw-free graphs

A claw

Claw-free graphs

A claw is the graph $(\{a, b, c, x\}, \{xa, xb, xc\})$.



A graph is claw-free if it does not contain a claw as an induced subgraph.

Solving PARITY PATH for claw-free graphs

We now present an algorithm that solves the following two problems in polynomial time if the input graph G is claw-free.

ODD INDUCED PATH

Instance: A graph G and two vertices s and t.
Question: Does G have an odd induced path from s to t?

EVEN INDUCED PATH

Instance: A graph G and two vertices s and t.
Question: Does G have an even induced path from s to t?

As an immediate result, we can solve the PARITY PATH problem in polynomial time for the class of claw-free graphs.

Outline of the algorithm

Input: a claw-free graph G and two vertices s and t.

- Preprocess G to obtain a graph H

Solve Odd Induced Path and Even Induced Path for H

- H is not perfect
  - H contains an odd hole
  - H contains an odd antihole
- H is perfect
  - H has no clique cutset
    - H is elementary
    - H is peculiar
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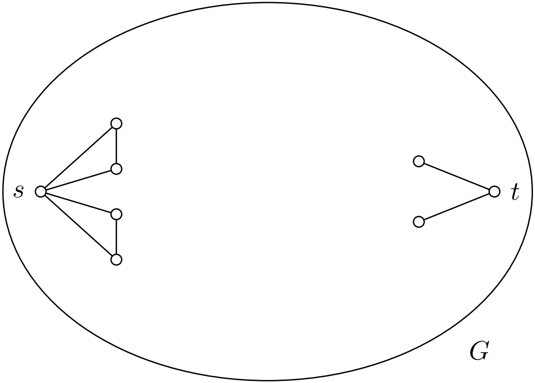
Preprocessing the input graph

Step 1: **Making s and t simplicial.**



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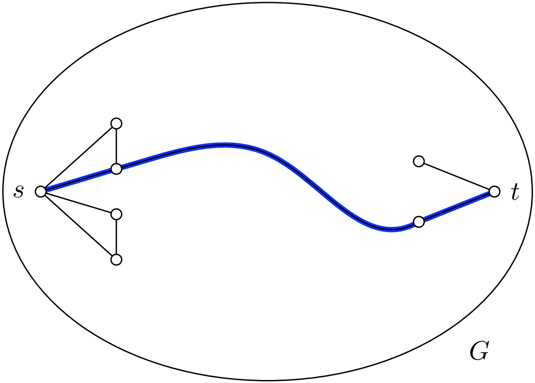

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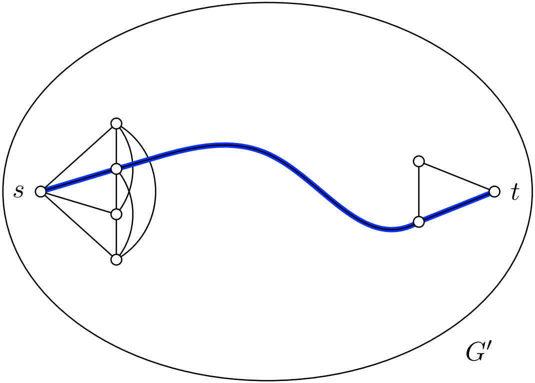
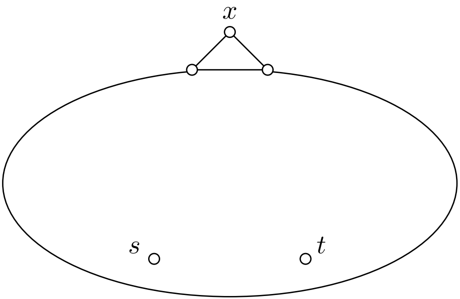


Preprocessing the input graph

Step 2: **Cleaning the graph G'.**

A vertex x is called irrelevant for s and t if it does not lie on any induced path from s to t.



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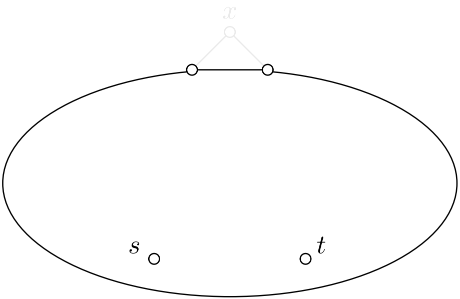
Step 2: **Cleaning the graph G'.**

A graph is clean for s and t if it does not contain irrelevant vertices.



Preprocessing the input graph

Step 2: **Cleaning the graph G'.**

THREE-IN-A-PATH

Instance: A graph G and three vertices v_1, v_2, v_3 of G.
Question: Does G have an induced path containing v_1, v_2, v_3?

Theorem (Derhy & Picouleau, 2009)

The THREE-IN-A-PATH *problem is NP-complete.*

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The Three-in-a-Path *problem is NP-complete.*

Corollary

In general, the problem of determining if a vertex x is irrelevant for two vertices s and t is NP-complete.

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THREE-IN-A-TREE

Instance: A graph G and three vertices v_1, v_2, v_3 of G.
Question: Does G have an induced tree containing v_1, v_2, v_3?

Theorem (Chudnovsky & Seymour, submitted)

The THREE-IN-A-TREE *problem is solvable in polynomial time.*

Preprocessing the input graph

Step 2: **Cleaning the graph G'.**

Three-in-a-Tree

Instance: A graph G and three vertices v_1, v_2, v_3 of G.
Question: Does G have an induced tree containing v_1, v_2, v_3?

Theorem (Chudnovsky & Seymour, submitted)

The Three-in-a-Tree *problem is solvable in polynomial time.*

Corollary

We can "clean" the graph G' in polynomial time.

Preprocessing the input graph

We obtain a graph H such that

- H is claw-free;
- s and t are simplicial vertices of H;
- H is clean for s and t.

Observation

G, s, t is a YES*-instance of* ODD INDUCED PATH *(respectively* EVEN INDUCED PATH*) if and only if H, s, t is a* YES*-instance.*

Outline of the algorithm

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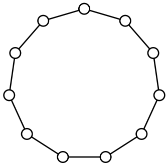
Lemma

Let H be a claw-free graph that is clean for two simplicial vertices s and t. If H contains an odd hole, then there exists both an odd and an even induced path from s to t.



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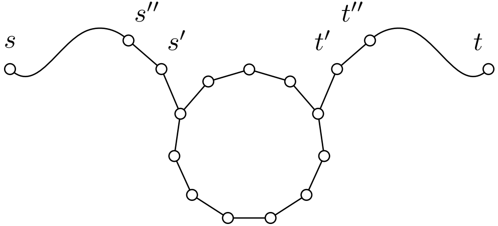

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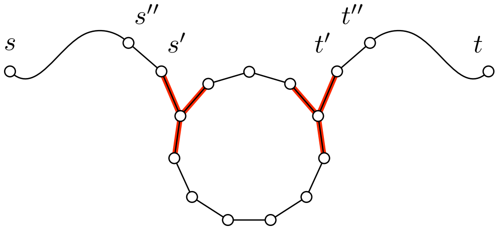

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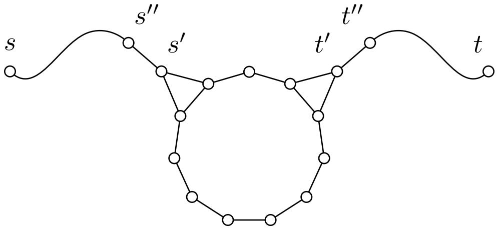

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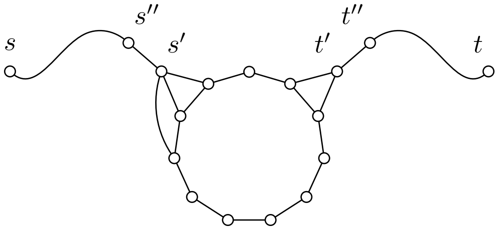

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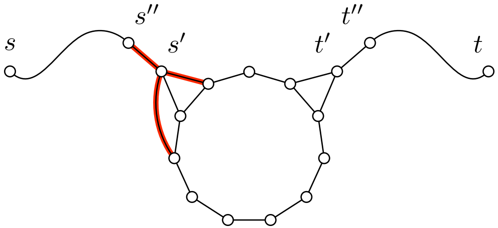

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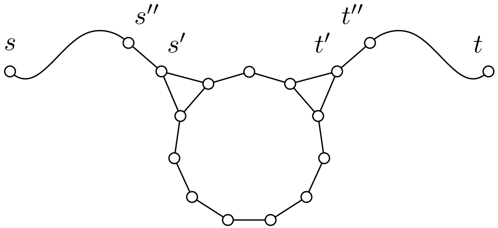

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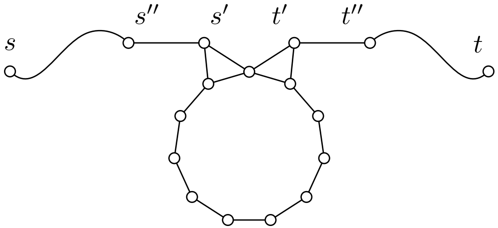

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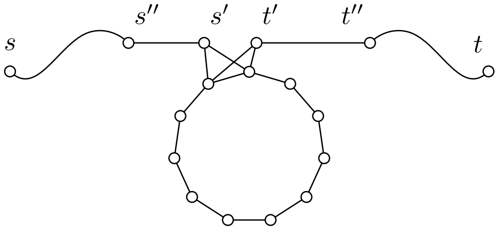

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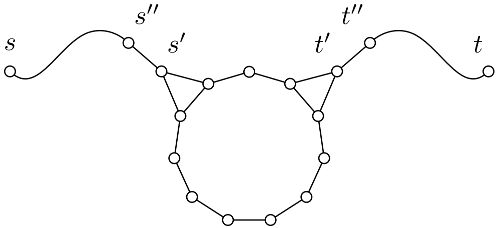

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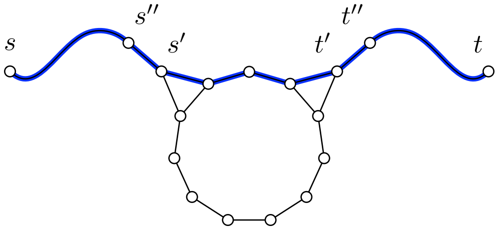

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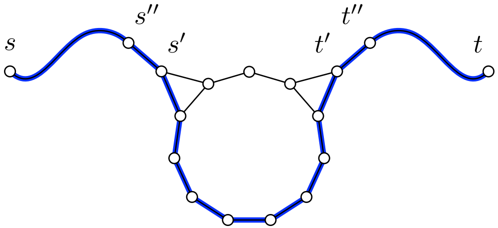

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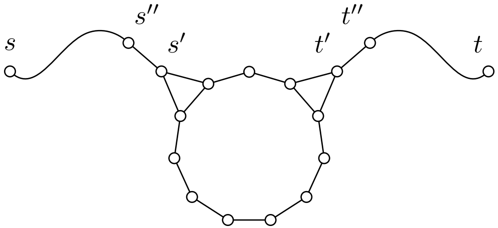

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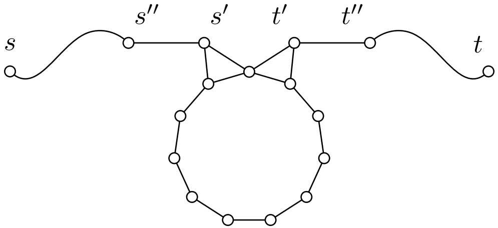

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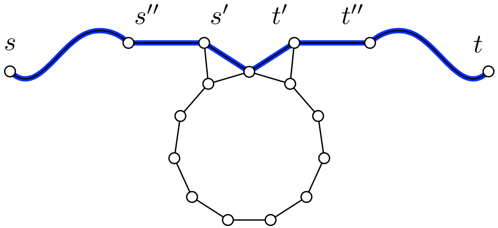

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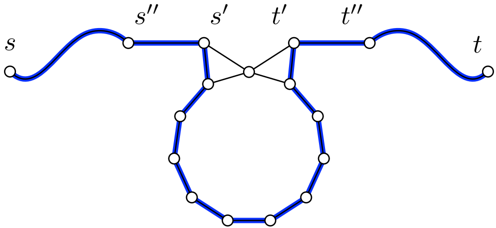

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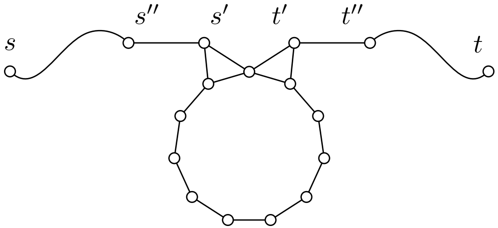

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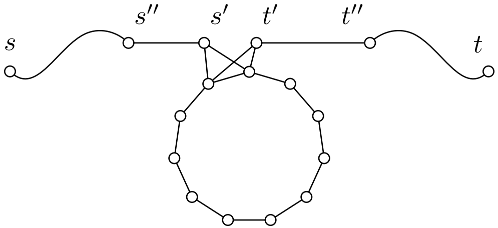

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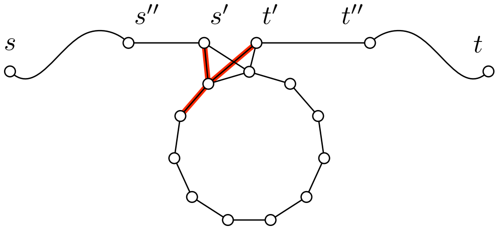

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Lemma

Let H be a claw-free graph that is clean for two simplicial vertices s and t. Then H does not contain an odd antihole of length more than 5.

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  - H has a clique cutset

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Theorem ("folklore")

The PARITY PATH *problem can be solved in polynomial time for the class of perfect graphs.*

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SHORTEST ODD INDUCED PATH

Instance: A graph G and two vertices s and t.
Task: Find a shortest odd induced path from s to t.

SHORTEST EVEN INDUCED PATH

Instance: A graph G and two vertices s and t.
Task: Find a shortest even induced path from s to t.

H is perfect

Theorem (Chvátal & Sbihi, 1988)

A perfect claw-free graph with no clique cutset is either elementary or peculiar.

Outline of the algorithm

Input: a claw-free graph G and two vertices s and t.

- Preprocess G to obtain a graph H

Solve Odd Induced Path and Even Induced Path for H.

- H is not perfect
  - H contains an odd hole
  - H contains an odd antihole
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Elementary graphs

Definition

A graph H is *elementary* if its edges can be colored with two colors such that every induced path on three vertices has its two edges colored differently. We call such a coloring an *elementary coloring* of H.

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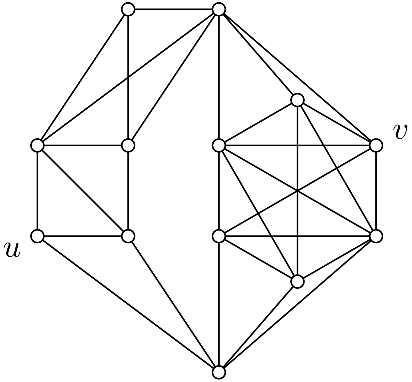
Lemma

We can determine in polynomial time if a graph H is elementary. If it is, an elementary coloring of H can be found in polynomial time.

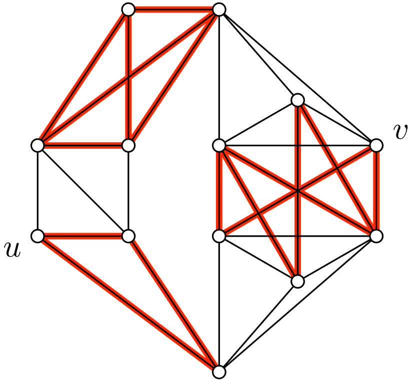
Solving Odd Induced Path for elementary graphs

An elementary graph H.



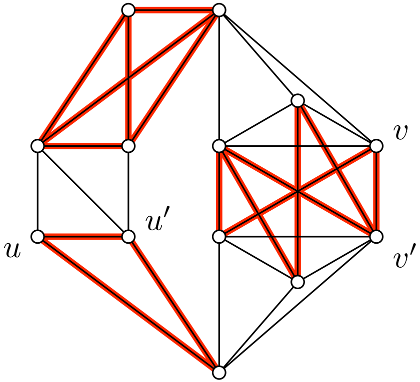
Solving ODD INDUCED PATH for elementary graphs

An elementary coloring φ of H.



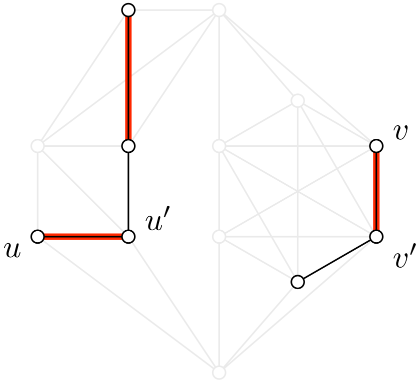
Solving Odd Induced Path for elementary graphs

Neighbors u' of u and v' of v with $\varphi(uu') = \varphi(vv')$.



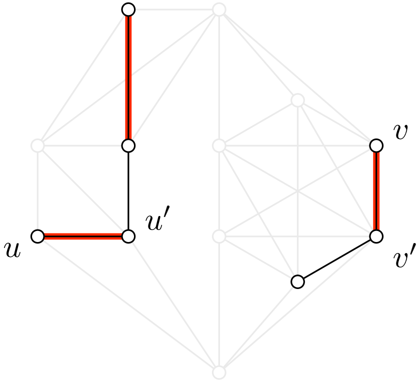
Solving Odd Induced Path for elementary graphs

The graph $H_{u'v'}$ (all other neighbors of u and v deleted).



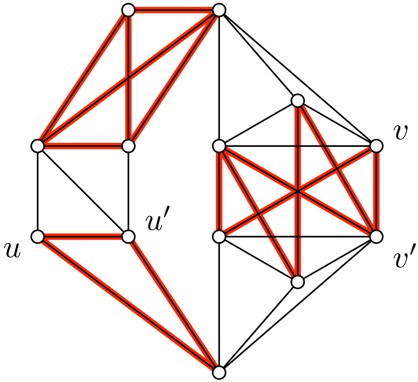
Solving Odd Induced Path for elementary graphs

Vertices u and v are *not* in the same component of $H_{u'v'}$.



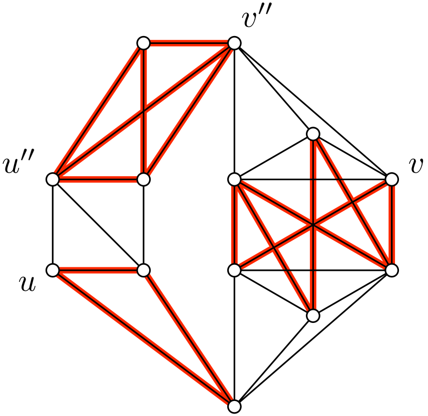
Solving Odd Induced Path for elementary graphs

Neighbors u' of u and v' of v with $\varphi(uu') = \varphi(vv')$.



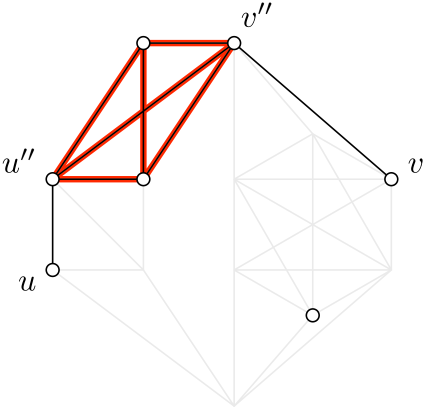
Solving ODD INDUCED PATH for elementary graphs

Neighbors u'' of u and v'' of v with $\varphi(uu'') = \varphi(vv'')$.



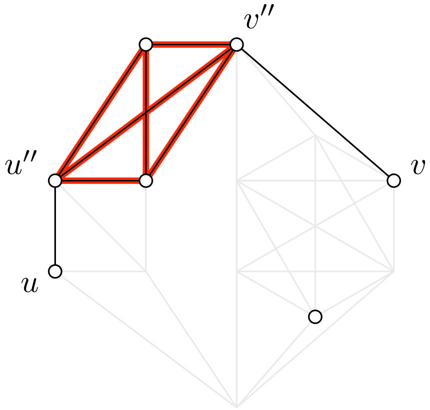
Solving ODD INDUCED PATH for elementary graphs

The graph $H_{u''v''}$ (all other neighbors of u and v deleted).



Solving ODD INDUCED PATH for elementary graphs

Vertices u and v *are* in the same component of $H_{u''v''}$.



H is elementary

Since we only have to consider $\mathcal{O}(n^2)$ graphs $H_{u'v'}$ and we can perform all checks in polynomial time, we have the following.

Lemma

The ODD INDUCED PATH *and* EVEN INDUCED PATH *problems can be solved in polynomial time for elementary graphs.*

Outline of the algorithm

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Peculiar graphs

Peculiar graphs

Definition

A graph H is *peculiar* if it can be obtained from a complete graph K as follows. Partition $V(K)$ into six mutually disjoint non-empty sets $A_i, B_i, i = 1, 2, 3$. For each $i = 1, 2, 3$, remove at least one edge with one end-vertex in A_i and the other end-vertex in B_{i+1}, where the subscripts are taken modulo 3. Finally, add three new mutually disjoint non-empty complete graphs $D_i, i = 1, 2, 3$, such that for each $i = 1, 2, 3$ each vertex in D_i is besides all vertices in D_i adjacent to all vertices in $V(K)\backslash(A_i \cup B_i)$ and to no other vertices.

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Observation

Every peculiar graph is P_{19}-free.

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Lemma

Every peculiar graph is P_6-free but not P_5-free.

H is peculiar

The P_6-freeness of peculiar graphs immediately implies:

Lemma

The Odd Induced Path *and* Even Induced Path *problems can be solved in polynomial time for peculiar graphs.*

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Together with the same result for elementary graphs, this yields:

Theorem

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So how do we deal with clique cutsets?

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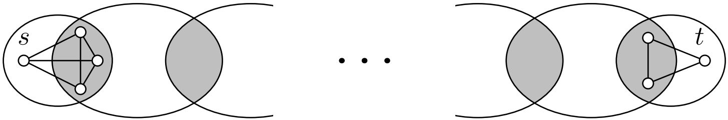

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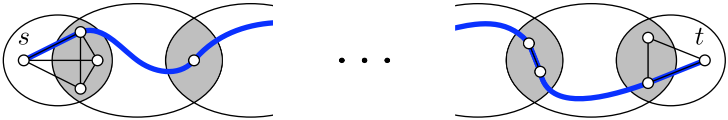

Main results

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Corollary

We can decide in polynomial time if a claw-free graph has an odd (or even) induced cycle through a specified vertex.

What would we like (you) to do

- Find a polynomial-time algorithm for the SHORTEST ODD INDUCED PATH problem for claw-free graphs, or show that the problem is NP-complete.

What would we like (you) to do

- Find a polynomial-time algorithm for the Shortest Odd Induced Path problem for claw-free graphs, or show that the problem is NP-complete.
- Find a polynomial-time algorithm for the Parity Path problem for planar graphs, or show that the problem is NP-complete.

Thank you!

Pim van 't Hof

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