On module-composed graphs

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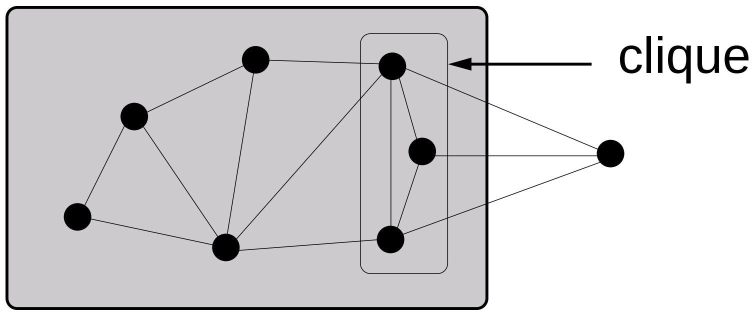
Motivation

Well known graph classes defined by special vertex orderings / elimination orderings:

- chordal graphs



$$G: \quad v_1, \quad v_2, \quad \ldots, \quad v_{i-1}, \qquad v_i, \qquad v_{i+1}, \ldots, v_n$$

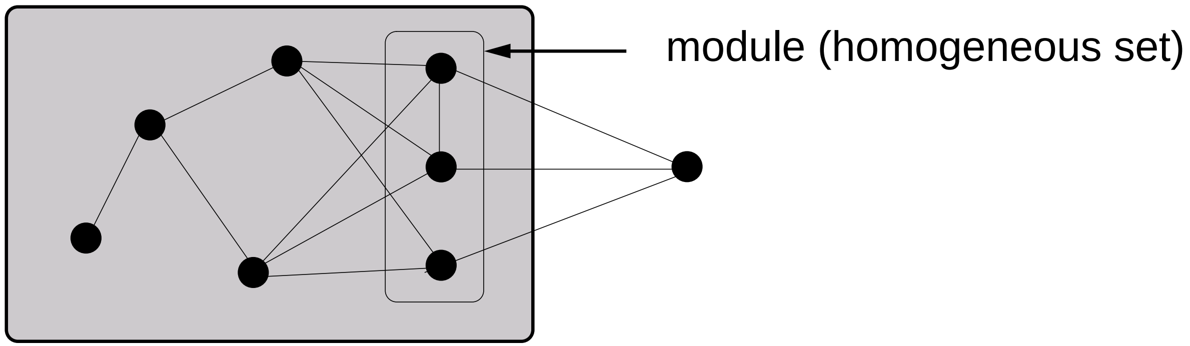
The vertex ordering $v_1, \ldots, v_n$ is denoted as perfect elimination ordering for G.

- k-trees
- distance hereditary graphs
- co-graphs

Module-composed graphs

We introduce a new graph class defined by the existence of an elimination ordering:

module-composed graphs



$G: \quad v_1, \quad v_2, \quad \ldots, \quad v_{i-1}, \qquad v_i, \qquad v_{i+1}, \ldots, v_n$

The vertex ordering $v_1, \ldots, v_n$ is denoted as module-sequence for G.

Basic Properties

1. If graph G is module-composed, then every induced subgraph of G is module-composed.

   (We can remove an arbitrary subset $S \subseteq V$ from a module-sequence and obtain a module-sequence for graph $G[V - S]$.)

2. Given two module-composed graphs G_1, G_2, the disjoint union $G_1 \cup G_2$ is module-composed.

   (The concatenation of two module-sequences for G_1 and G_2 leads a module-sequence for $G_1 \cup G_2$.)

3. Given a module-composed graph G, the addition of a dominating vertex v leads a module-composed graph.

   (Since the whole vertex set of G is a trivial module, we can extend a module-sequence for G by v.)

4. Given a module-composed graph G, the addition of a pendant vertex v leads a module-composed graph.

   (Since every single vertex of G is a trivial module, we can extend a module-sequence for G by v.)

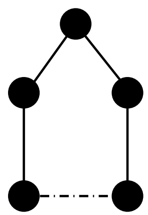
Basic Properties II

5. Graph $G = (V, E)$ is a module-composed graph, if and only if there exists at least one $v \in V$ such that $N(v)$ is a module in graph $G - v$ and for every such vertex v graph $G - v$ is a module-composed graph.

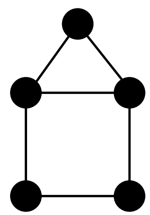
($\Rightarrow$ Let v be the last vertex in a module-sequence for G, then by definition $N(v)$ is a module in graph $G - v$. By (1.) for every $v \in V$ induced subgraph $G - v$ is a module-composed graph.

$\Leftarrow$ Since $G - v$ is a module-composed graph there is some module-sequence for graph $G - v$ and since $N(v)$ is a module in graph $G - v$, we can extend this sequence by v for a module-sequence for G.)

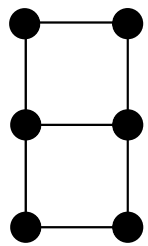
6. Module-composed graphs do not contain one of the following graphs as an induced subgraph:



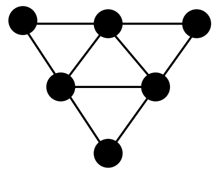
C_n, $n \geq 5$
(i.e. holes)



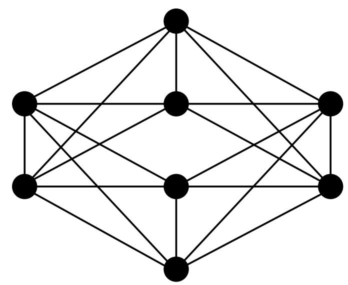
house



domino



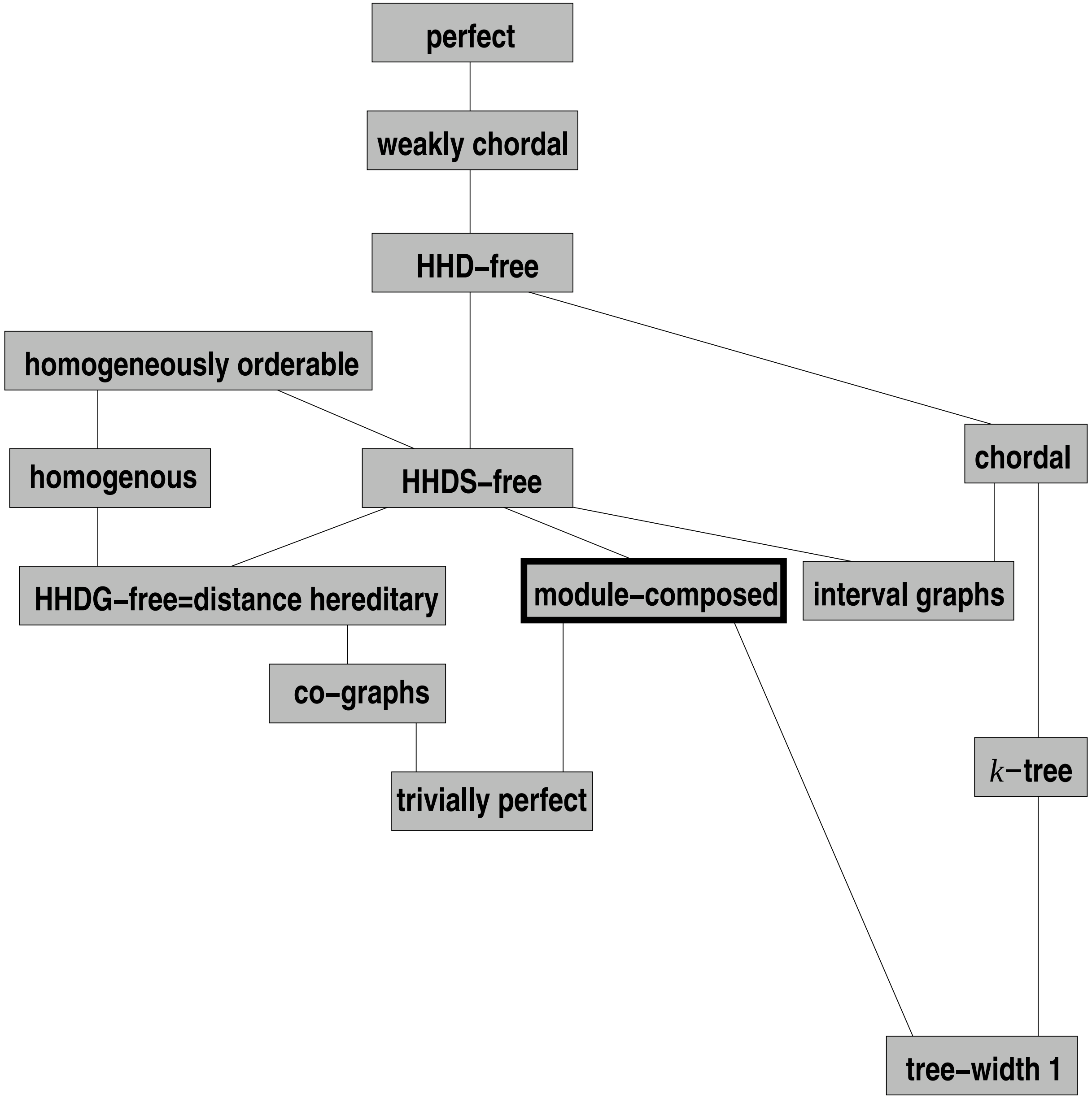
3-sun



co-$2C_4$

(None of these graphs G contains a vertex v such that $N(v)$ is a module in graph $G - v$.)

Graph class inclusions I



Recognizing module-composed graphs

Theorem Given a graph $G = (V, E)$, one can decide in time $\mathcal{O}(|V| \cdot (|V| + |E|))$ whether G is module-composed, and in the case of a positive answer, construct a module-sequence for G.

Proof (Sketch)

(1) **for** $i = 1$ **to** $|V|$ **do**
(2) construct a modular decomposition T_G for G;
(3) using T_G find a vertex v by a case distinction, such that $N(v)$ is a module in $G - v$;
(4) $G = G - v$;
(5) **return** module-sequence or the answer NO

Correctness: basic property (5.)

Running time:

(2) there exist $\mathcal{O}(|V| + |E|)$ algorithms for constructing a modular decomposition

(3) if $N(v)$ is a module in $G - v$, then v is either a child of the root of T_G or v is a special grandchild of the root of T_G and can be found in time $\mathcal{O}(|V|)$

Problem Given a modular decomposition for G, can we find a modular decomposition for $G - v$ in time less than $\mathcal{O}(|V| + |E|)$, e.g. in $\mathcal{O}(|V|)$?

Easy problems on module-composed graphs

Since module-composed graphs are HHD-free, we know by a result of Jamison and Olariu 1988.

Theorem For every module-composed graph which is given together with a module-sequence the size of a largest independent set, the size of a largest clique, the chromatic number and the minimum number of cliques covering the graph can be computed in linear time.

Remark The set of all module-composed graphs has unbounded tree-width.

(Every complete graph is module-composed.)

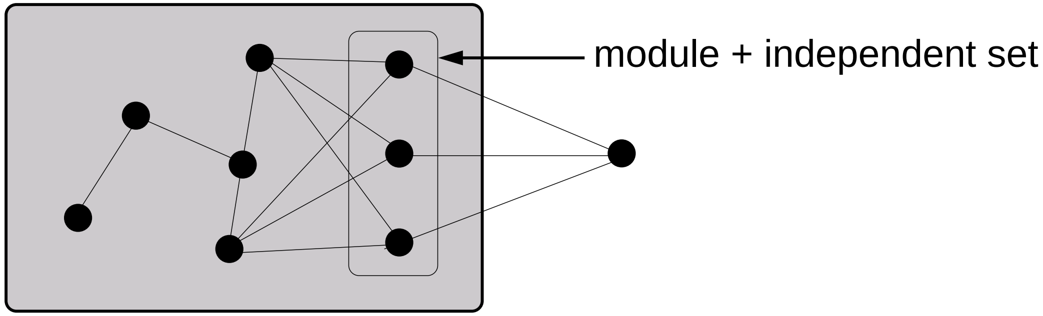
Remark The set of all module-composed graphs has unbounded clique-width.

(Every graph which can be constructed from a single vertex by a sequence of one vertex extentions by a dominating vertex or a pendant vertex is module-composed. But the set of all such defined graphs has unbounded clique-width [Rao 2008].)

Independent module-composed graphs

We additionally introduce a restricted version of module-composed graphs:

independent module-composed graphs



$$G: \quad v_1, \quad v_2, \quad \ldots, \quad v_{i-1}, \qquad v_i, \qquad v_{i+1}, \ldots, v_n$$

The vertex ordering $v_1, \ldots, v_n$ is denoted as independent module-sequence for G.

Characterizations of independent module-composed graphs

Theorem Let G be some graph. The following conditions are equivalent.

1. G is independent module-composed.
2. G is bipartite module-composed.
3. G is bipartite distance hereditary.
4. G is domino, hole, and odd-cycle-free.
5. G can be generated by a pruning sequence without true twins.

Proof (Sketch)

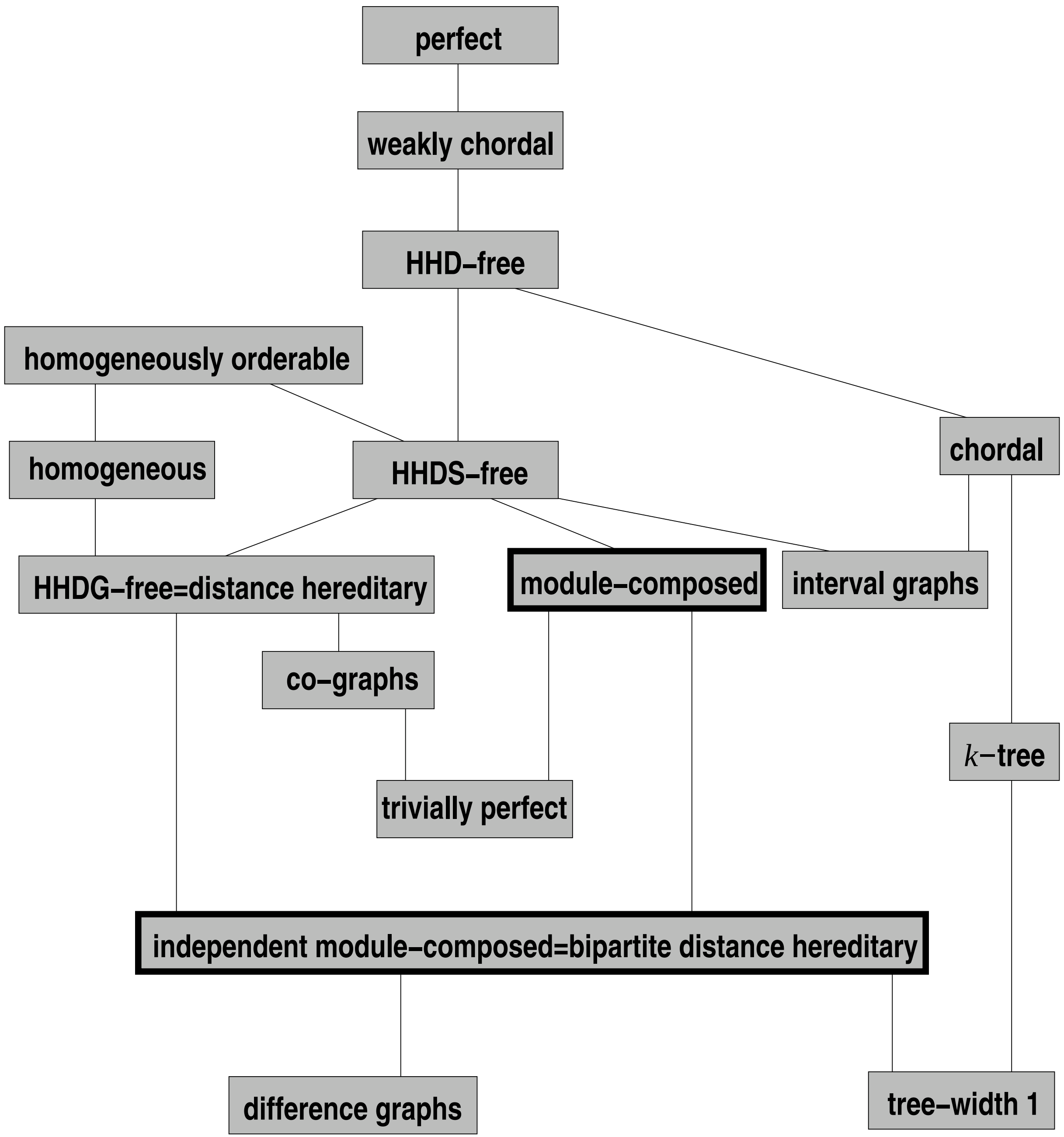
$(3) \Leftrightarrow (4) \Leftrightarrow (5)$ well known results

$(2) \Rightarrow (1)$ For bipartite graphs the neighbourhood of every vertex is an independent set.

$(1) \Rightarrow (4)$ Domino, hole, and odd-cycles are not independent module-composed graphs.

$(5) \Rightarrow (2)$ A pruning sequence without true twins can be transformed into a module-sequence by moving false twins directly after its pair vertex.

Graph class inclusions II



Recognizing independent module-composed graphs

Theorem Given a graph $G = (V, E)$ one can decide in time $\mathcal{O}(|V| + |E|)$ whether G is an independent module-composed graph and in the case of a positive answer, construct an independent module-sequence.

Proof (Sketch)

decision

- by the given characterization for independent module-composed graphs
- by a BFS (Breadth First Search)

construction of a sequence

- a pruning sequence without true twins can be transformed into an independent module-sequence
- a BFS ordering can be transformed into an independent module-sequence
- a reverse Lex-BFS (Lexicographic Breadth First Search) ordering is even an independent module-sequence

Thank you for your attention!