Game arguments in computability theory

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Outline

“A large part of computability theory is essentially about games”
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Three illustrations:
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Three illustrations:

- Unique programs
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- Unique programs
- Gap between conditional complexity and total conditional complexity
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Three illustrations:

- Unique programs
- Gap between conditional complexity and total conditional complexity
- Finding strings with equal distances
Universal functions

Partial function $U : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
**Universal functions**

Partial function $U : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$

Represented by a table:

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<thead>
<tr>
<th>$n$</th>
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Empty cells mean undefined values.

Function is computable iff the table can be filled by a (non-terminating) algorithmic process.

$n$-th row corresponds to unary function $U_n$:

$U(n, x)$

Computable $U$ is universal if every unary (partial) computable function appears among $U_n$. 


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Technical: usually the program $n$, the input $x$, and the output $U(n, x)$ are binary strings; this does not matter
Programmer’s view

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Probably the most practical discovery of XXth century
Unique numberings

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Does such a thing exist?

Imagine that we do not know the answer. Even then, we can easily reduce the question to the other one: who wins in some game.
The game

Two players, Alice and Bob, fill two tables (each player has its own one)
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Why the game is relevant?

If Alice can win by some computable strategy, unique numberings do not exist.

Let Alice play against Bob that ignores her and (step by step) fills B-table with a universal function that provides a unique numbering.

Alice actions are computable, so her function is computable, and she loses.

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To avoid interference, each $C_i$ first should “reserve” a row in $B$-table (empty at that moment, and not reserved by anybody else), and work with it; others do not touch this row
So what?

Naïve picture: each assistant $C_i$, when hired, selects and reserves a row in $B$-table that is not reserved yet (by others), and then faithfully copies $i$-th row in $A$-table into this reserved row.
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Simplified game

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Winning condition for the limit state: Bob wins if (1) every $A$-row appears as a live $B$-row, and (2) all live $B$-rows are different

Strategy:
each assistant $i$ keeps a counter $N_i$: how many $B$-rows he has killed

if $i$-th $A$-row differs from all previous $A$-rows if we look at places $1: \ldots : N_i$ only, copies its content into the reserved row, otherwise kills it (increasing $N_i$), chooses a new reserved row and copies $i$-th $A$-row there
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- if $i$-th $A$-row differs from all previous $A$-rows if we look at places $1 \ldots N_i$ only, copies its content into the reserved row, otherwise kills it (increasing $N_i$), chooses a new reserved row and copies $i$-th $A$-row there
Why the strategy works

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- either he changes the reserved row infinitely many times, killing the previous one again and again ($N_i \to \infty$); in this case no trace among the live rows of limit $B$-table is left;

- or starting from some moment, $C_i$ never kills his reserved row and just copies the contents of $i$-th row of $A$ into it; then $i$-th row of $A$ is among the live rows in $B$-table.
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We need to prove: the second case happens iff $i$-th row in the limit $A$-table differs from all the previous rows in it. (Then only the first occurrence of each $A$-row will be copied into $B$-table, and all live $B$-rows are different.)
Why the strategy works (continued)

We should prove that two things are not possible:

1. The $i$-th row is different from all previous rows in the limit $A$-table, but $N_i \neq 1$.
2. Indeed, look at all places where the $i$-th row differs from the preceding ones. Let $N_i$ be an upper bound for them. Wait until the first $N_i$ rows stabilize up to position $N_i$ and until $N_i$ exceeds $N_i$. Then $A_i$ sees that the $i$-th row is different from preceding ones in one of the first $N_i$ places. Why would he kill the reserved row and increase $N_i$?

$N_i$ has finite limit value, but the $i$-th row appears earlier in the limit $A$-table.

Let $N_i = \lim_{n \to \infty} N_i$; wait until rows 1: : : $i$st stabilize up to position $N_i$, and $N_i$ reaches $N_i$. Then $A_i$ sees the coincidence, why doesn't he increase $N_i$?
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- $N_i$ has finite limit value, but $i$-th row appears earlier in the limit $A$-table. Let $N = \lim N_i$; wait until rows $1 \ldots i$ stabilize up to position $N$, and $N_i$ reaches $N$. Then $S_i$ sees the coincidence, why doesn’t he increase $N_i$?
How to win without killing rows

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First, we agree that each $A$-row can contain only even number of non-empty cells, so cells are filled in pairs (we postpone adding a number until another one arrives).
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This may destroy the winning condition for “odd” $A$-rows (with finite odd number of non-empty cells = functions with finite domain of odd cardinality). We add all such functions (they can be easily enumerated) to $B$-table one by one: a special assistant looks for the first missing one and adds it in a fresh row, then again, etc. No clashes with copied rows.
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Finally, instead of killing a row, we can fill some other cells in it to get an odd row that has not appeared yet, with the same result. (This should be coordinated with the previous step, so no row is added twice.)
What is wrong with unique numberings?

I Are the universal functions (programming languages) with uniqueness property useful?

I No. The problem is that "s-m-n theorem" does not work for them.

I In programming terms: there is no compiler from a normal language to such a strange one (Gödel property for a universal function) for every function V(m;x) there exists a total computable c(m) such that V(m;x) = U(c(m);x) for all m and x (both sides are defined or undefined at the same time, and equal if defined).

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- No. The problem is that “s-m-n theorem” does not work for them.
- In programming terms: there is no compiler from a normal language to such a strange one.
- (Gödel property for a universal function) for every function $V(m, x)$ there exists a total computable $c$ (compiler) such that $V(m, x) = U(c(m), x)$ for all $m$ and $x$ (both sides are defined or undefined at the same time, and equal if defined).
- Another classical result: all universal functions with Gödel property are isomorphic (and do not have the uniqueness property).
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Depends on the programming language.

There is an optimal one that makes complexity minimal (up to $O(1)$ additive term).

Fix an optimal one and denote complexity by $C(x)$.

Defined up to $O(1)$ additive term.

Conditional complexity of $x$ given $y$: the minimal length of a program that transforms input $y$ to output $x$ (for an optimal language). Notation: $C(x|y)$.

Let us replace minimal length by minimal complexity; then any language is OK (optimality not needed).

Total conditional complexity: minimal complexity of a total program that transforms $y$ to $x$; notation $CT(x|y)$. 

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I Evidently, conditional complexity does not exceed total conditional complexity. How big the difference could be?

The difference could be maximal: for a given $n$ there exist strings $x$ and $y$ of length $n$ such that $C(x:jy) = O(1)$ but $CT(x:jy)$ is $n$. (Two extreme cases.)

(Digression) $CT$ attracted attention recently (Bauwens, Vereshchagin). If information distance between $x$ and $y$ is small ($C(x:jy) < 0$ and $C(y:jx) < 0$), still $x$ and $y$ can have different properties. But if $CT(x:jy) > 0$ and $CT(y:jx) > 0$, then $x$ is mapped to $y$ by a simple computable permutation, so $x$ and $y$ are very similar.
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- Alice wins if in the limit there exists some $y \in Y$ such that $a(y)$ is defined and different from all $b_i(y)$
Alice fills the upper row (with elements of $X$, step by step)

Bob adds new (completely filled) rows to the table, at most $2^n$ rows

Alice's goal: some element of her row does not appear again in the $B$-column under it

Bob's goal: every element of $A$-table appears in one of $B$-rows in the same column

Alice's strategy: choose a free cell, put an element that does not exist in its column yet, and wait until Bob answers with a new row

Alice wins: there are enough elements of $X$, and she has more moves than Bob
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- by construction $CT(x|y) \geq n$
3: Muchnik–Vyugin’s result

$I(C(xjy))$ measures how "far" is $x$ from information that exists in $y$. $I(C(xjy))$ is not symmetric: $C(xjy)$ and $C(yjx)$ can be very different. If we need one number as a distance, we can consider $C(xjy) + C(yjx)$, but pair $C(xjy)$; $C(yjx)$ is more informative.

Question: for a given $x$ and $n$, can we find $y$ such that both $C(xjy)$ and $C(yjx)$ are $n + O(1)$?

Some conditions needed: if $C(x) \ll n$, then $C(xjy) \ll n$. If $C(x) > 2n$, then such a $y$ exists with $O(\log n)$ instead of $O(1)$ this is easy: take the shortest program for $x$, and replace $n$ first bits in it by random ones. Topological argument replaces $2n$ by $n + O(\log n)$.
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Game proof: the game is straightforward, but the strategy is quite complicated.
Andrej Muchnik (1958-2007)