Game arguments in computability theory

Alexander Shen
LIRMM, CNRS & UM2, Montpellier; on leave from
ИППИ РАН, Москва

Outline

"A large part of computability theory is essentially about games"

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- Unique programs

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- Gap between conditional complexity and total conditional complexity

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Three illustrations:

- Unique programs
- Gap between conditional complexity and total conditional complexity
- Finding strings with equal distances

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Computable U is universal if every unary (partial) computable function appears among U_n

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Probably the most practical discovery of XXth century

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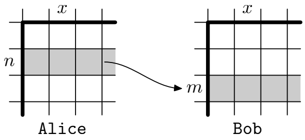
Imagine that we do not know the answer. Even then, we can easily reduce the question to the other one: who wins in some game

The game

Two players, Alice and Bob, fill two tables (each player has its own one)



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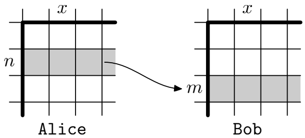

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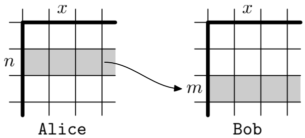

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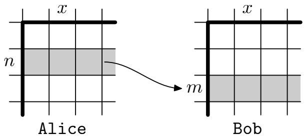

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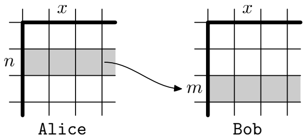

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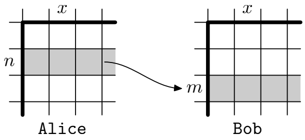

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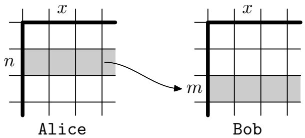

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Bob wins if: (1) each A-row equals some B-row;
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Bob actions are computable; every computable function is a row in A-table (may be, several rows) and therefore is a row in B-table; all B-rows are different

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In fact, Friedberg just proved the existence of unique numberings

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To avoid interference, each C_i first should "reserve" a row in B-table (empty at that moment, and not reserved by anybody else), and work with it; others do not touch this row

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Difficulty: the condition "i-th A-row differs from the previous ones" cannot be checked at any finite step

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- has some B-row reserved by him (the first free one at the beginning or when the previous one is killed)
- if i-th A-row differs from all previous A-rows if we look at places $1 \ldots N_i$ only, copies its content into the reserved row, otherwise kills it (increasing N_i), chooses a new reserved row and copies i-th A-row there

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We need to prove: the second case happens iff i-th row in the limit A-table differs from all the previous rows in it. (Then only the first occurence of each A-row will be copied into B-table, and all live B-rows are different.)

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- N_i has finite limit value, but i-th row appears earlier in the limit A-table. Let $N = \lim N_i$; wait until rows $1 \ldots i$ stabilize up to position N, and N_i reaches N. Then S_i sees the coincidence, why doesn't he increase N_i?

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Finally, instead of killing a row, we can fill some other cells in it to get an odd row that has not appeared yet, with the same result. (This should be coordinated with the previous step, so no row is added twice.)

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- Another classical result: all universal functions with Gödel property are isomorphic (and do not have the uniqueness property).

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- Total conditional complexity: minimal complexity of a total program that transforms y to x; notation $CT(x|y)$

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(Digression) CT attracted attention recently (Bauwens, Vereshchagin). If information distance between x and y is small ($C(x|y) \approx 0$ and $C(y|x) \approx 0$), still x and y can have different properties. But if $CT(x|y) \approx 0$ and $CT(y|x) \approx 0$, then x is mapped to y by a simple computable permutation, so x and y are very similar.

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- Alice wins if in the limit there exists some $y \in Y$ such that $a(y)$ is defined and different from all $b_i(y)$

Game field

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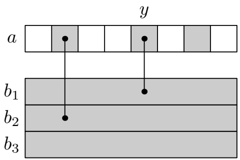
Alice fills the upper row
(with elements of X, step by step)



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Bob adds new (completely filled)
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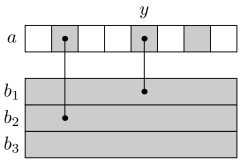

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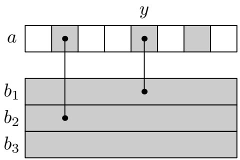

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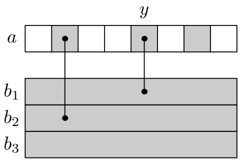

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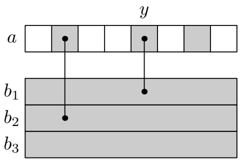

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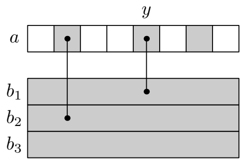

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Alice wins: there are enough elements of *X*, and she has more moves than Bob

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- topological argument replaces $2n$ by $n + O(\log n)$.

More strings

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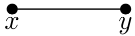
We constructed for a given x some y that has “distance” n in both directions



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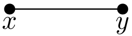
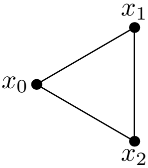


Now for a given x_0 we want to find a "right triangle" $x_0x_1x_2$ where all six "distances" are $n + O(1)$



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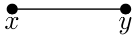

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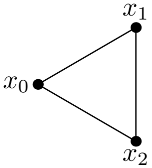

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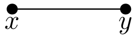

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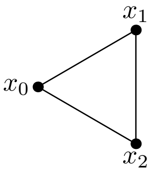

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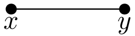

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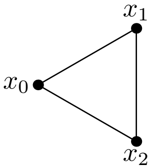

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Game proof: the game is straightforward, but the strategy is quite complicated

Andrej Muchnik (1958-2007)