NAFIT: New Algorithmic Forms of Information Theory
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Measuring information

- **Classical information theory**: For a random variable that has \( n \) values with probabilities \( p_1, \ldots, p_n \), its Shannon entropy is defined as \( \sum p_i \log(1/p_i) \).

- **Algorithmic information theory**: The Kolmogorov complexity of a finite object \( x \) is the minimal length of a program that produces \( x \).

\[ K_S(x) = \min \{ l(p) : S(p) = n(x) \} \]

Randomness for finite objects

- For all of these two sequences was obtained by a physical random process:
  - 101001110010101110110111001010000110001
  - 101010101010101010101010101010101010101

A page from Kolmogorov’s autograph

Randomness for infinite sequences

- **Martin-Löf**: infinite sequences may be random or non-random
- **Non-random sequence**: a sequence that violates some effective law of probability theory (statement that is true for all sequences except for some effectively null set)
- **Schnorr characterization**: sequence is random if no martingale (gambling system) wins infinitely much against it
- **Levin–Schnorr characterization**: sequence is random iff its prefixes are incompressible

Local rules and global complexity

- **Nature**: local interaction in crystals produces nice periodic structures we observe; we believe that local interaction produces nice aperiodic structures in quasicrystals
- **Mathematical model of local interaction**: tiles
- **Many tile sets are known that produce aperiodic tilings**
- **A technique for constructing robust tilings is developed based on computer science tools (Kleene’s fixed-point theorem)**

Randomness paradox

A factory produces decks of cards. To make them ready for use, after the printing machine there is a shuffling machine that puts cards in a random order. The management wants to add a quality control unit that checks whether the shuffling machine does its job correctly. Should the quality control reject some decks of cards as ‘badly shuffled’?

- **yes**, to prevent an angry customer saying that he bought a deck and found the cards in an increasing order;
- **no**, rejecting some orders destroys the randomness which requires that all appearing order equally often.

On-line randomness

- **Randomness depends on context**
- **Example**: each match in a football tournament starts with coin tossing (it determines who starts the game)
- **bad**: if the outcome of today’s coin tossing can be computed from the contents of yesterday’s newspaper
- **normal**: if it can be computed from the tomorrow’s newspaper
- **mathematical definition**: on-line randomness and on-line complexity; generalization of Levin–Schnorr characterization of randomness for the online case; generalization of martingale characterization of randomness for the online case; relation to randomness with respect to the class of measures; non-additivity for on-line complexity (with possible application to the causality problem).

Online randomness: formal definition

- **Formally**: \( x_1, b_1, x_2, b_2, x_3, b_3, \ldots \); here \( x_i \) are strings representing context information, \( b_i \) are presumable random bits
- **Example**: \( b_i = \Phi(x_i) \) is bad and \( b_i = \Phi(x_i+1) \) is OK
- **On-line measure**: No probability assumption on \( x_i \); assumed conditional probabilities for \( b_i \)
- **On-line effectively null sets**: for every \( \varepsilon > 0 \) one can algorithmically find a covering by interval with upper probability less than \( \varepsilon \)
- **Upper probability can be defined as the upper bound of probabilities wrt all distributions where conditional probability of each next bit is 1/2**

On-line complexity

- **complexity** of \( (x_1 \rightarrow b_1; x_2 \rightarrow b_2; \ldots) \) is the minimal length of an on-line program that gets \( x_1 \), prints \( b_1 \), then gets \( x_2, b_2, \) etc.
- **Theorem**: in the sequence of bits \( x_1, y_1, x_2, y_2, \ldots \)
  \[ K(x_1 \rightarrow y_1; x_2 \rightarrow y_2; \ldots) + K(x_1; y_1 \rightarrow x_2; y_2 \rightarrow x_3; \ldots) \]
  cannot be smaller than the complexity of \( K(x_1; y_1; x_2; y_2; \ldots) \) but can significantly exceed the latter, and also \( K(y_1 \rightarrow x_1; y_2 \rightarrow x_2; \ldots) \)
- **Philosophers can interpret these results as a mathematical way to reconstruct causality from dependence**

An object that combines structure and randomness

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