Algorithms and geometric constructions

Vladimir A. Uspenskij (1930–2018),
Alexander Shen

LIRMM CNRS & University of Montpellier

CiE 2018

Владимир Андреевич Успенский (27.11.1930–27.06.2018)

What is an algorithm?

What is an algorithm?

- a basic notion (like a notion of a set) or a part of the standard (set-theoretic) framework?

What is an algorithm?

- a basic notion (like a notion of a set) or a part of the standard (set-theoretic) framework?
- Borel, 1912: did not use the word "algorithm" but speak about "the computations that can be really performed" and adds: "I intentionally put aside the question of bigger or smaller practical length of the operation; it is important only that each of the operations can be performed in a finite time by a clear and unambiguous method"

What is an algorithm?

- a basic notion (like a notion of a set) or a part of the standard (set-theoretic) framework?
- Borel, 1912: did not use the word "algorithm" but speak about "the computations that can be really performed" and adds: "I intentionally put aside the question of bigger or smaller practical length of the operation; it is important only that each of the operations can be performed in a finite time by a clear and unambiguous method"
- 1930s: Gödel, Kleene, Church, Turing, Post defined representative classes of algorithms thus making possible the proofs of algorithmic undecidability

Church–Turing thesis

Church–Turing thesis

- computable functions = λ-calculable, general recursive functions, Turing/Post machines etc.

Church–Turing thesis

- computable functions = λ-calculable, general recursive functions, Turing/Post machines etc.
- Post: "Actually the work already done by Church and others carries this identification considerably beyond the working hypothesis stage. But to mask this identification under a definition hides the fact that a fundamental discovery in the limitations of mathematicizing power of Homo Sapiens has been made and blinds us to the need of its continual verification."

Church–Turing thesis

- computable functions = λ-calculable, general recursive functions, Turing/Post machines etc.
- Post: "Actually the work already done by Church and others carries this identification considerably beyond the working hypothesis stage. But to mask this identification under a definition hides the fact that a fundamental discovery in the limitations of mathematicizing power of Homo Sapiens has been made and blinds us to the need of its continual verification."
- later: Kolmogorov–Uspensky, Gurevich

Church–Turing thesis

- computable functions = λ-calculable, general recursive functions, Turing/Post machines etc.
- Post: “Actually the work already done by Church and others carries this identification considerably beyond the working hypothesis stage. But to mask this identification under a definition hides the fact that a fundamental discovery in the limitations of mathematicizing power of Homo Sapiens has been made and blinds us to the need of its continual verification.”
- later: Kolmogorov–Uspensky, Gurevich
- thesis makes sense only for people who understood the intuitive notion of algorithm before learning how to program.

Church–Turing thesis

- computable functions = λ-calculable, general recursive functions, Turing/Post machines etc.
- Post: “Actually the work already done by Church and others carries this identification considerably beyond the working hypothesis stage. But to mask this identification under a definition hides the fact that a fundamental discovery in the limitations of mathematicizing power of Homo Sapiens has been made and blinds us to the need of its continual verification.”
- later: Kolmogorov–Uspensky, Gurevich
- thesis makes sense only for people who understood the intuitive notion of algorithm before learning how to program.
- our topic: the evolution of a notion of geometric construction (special class of algorithms)

Ruler and compass constructions

Ruler and compass constructions

- studied since ancient times (Euclid)

Ruler and compass constructions

- studied since ancient times (Euclid)
- classical problems (circling the square, angle trisection, doubling the cube)

Ruler and compass constructions

- studied since ancient times (Euclid)
- classical problems (circling the square, angle trisection, doubling the cube)
- impossibility proven in 19th century

Ruler and compass constructions

- studied since ancient times (Euclid)
- classical problems (circling the square, angle trisection, doubling the cube)
- impossibility proven in 19th century without a definition of a geometric construction

Ruler and compass constructions

- studied since ancient times (Euclid)
- classical problems (circling the square, angle trisection, doubling the cube)
- impossibility proven in 19th century without a definition of a geometric construction
- proofs reproduced in popular expositions

Ruler and compass constructions

- studied since ancient times (Euclid)
- classical problems (circling the square, angle trisection, doubling the cube)
- impossibility proven in 19th century without a definition of a geometric construction
- proofs reproduced in popular expositions
- but definitions are rare

Ruler and compass constructions

- studied since ancient times (Euclid)
- classical problems (circling the square, angle trisection, doubling the cube)
- impossibility proven in 19th century without a definition of a geometric construction
- proofs reproduced in popular expositions
- but definitions are rare
- and correct definitions are even rarer

Ruler and compass constructions

- studied since ancient times (Euclid)
- classical problems (circling the square, angle trisection, doubling the cube)
- impossibility proven in 19th century without a definition of a geometric construction
- proofs reproduced in popular expositions
- but definitions are rare
- and correct definitions are even rarer
- but what is the problem?

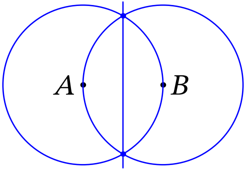
What is a geometric construction?

What is a geometric construction?

- input: objects (points, lines, circles), e.g., two points

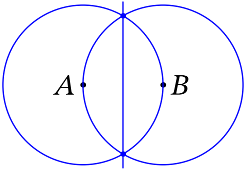
What is a geometric construction?

- input: objects (points, lines, circles), e.g., two points
- output: some other objects, e.g., perpendicular bisector



What is a geometric construction?

- input: objects (points, lines, circles), e.g., two points
- output: some other objects, e.g., perpendicular bisector



- operations:
  - (ruler) draw a line through two points;
  - given three points A, B, C, draw a circle with center A and radius BC;
  - add the intersection points of existing curves

Tao (2011):

Define a configuration to be a finite collection $\mathcal{C}$ of points, lines, and circles in the Euclidean plane. Define a construction step to be one of the following operations to enlarge the collection $\mathcal{C}$:

- (Straightedge) Given two distinct points A, B in $\mathcal{C}$, form the line $\overline{AB}$ that connects A and B, and add it to $\mathcal{C}$.
- (Compass) Given two distinct points A, B in $\mathcal{C}$, and given a third point O in $\mathcal{C}$ (which may or may not equal A or B), form the circle with centre O and radius equal to the length $|AB|$ of the line segment joining A and B, and add it to $\mathcal{C}$.
- (Intersection) Given two distinct curves γ, γ' in $\mathcal{C}$ (thus γ is either a line or a circle in $\mathcal{C}$, and similarly for γ'), select a point P that is common to both γ and γ' (there are at most two such points), and add it to $\mathcal{C}$.

We say that a point, line, or circle is constructible by straightedge and compass from a configuration $\mathcal{C}$ if it can be obtained from $\mathcal{C}$ after applying a finite number of construction steps.

Arbitrary points

Arbitrary points

- problem: given a circle, construct its center

Arbitrary points

- problem: given a circle, construct its center
- unsolvable according to this definition

Arbitrary points

- problem: given a circle, construct its center
- unsolvable according to this definition
- for a trivial reason: no operation to start with

Arbitrary points

- problem: given a circle, construct its center
- unsolvable according to this definition
- for a trivial reason: no operation to start with
- operation: “choose an arbitrary point”

Arbitrary points

- problem: given a circle, construct its center
- unsolvable according to this definition
- for a trivial reason: no operation to start with
- operation: “choose an arbitrary point”
- if adding an arbitrary point is allowed, then all points become constructible

Arbitrary points

- problem: given a circle, construct its center
- unsolvable according to this definition
- for a trivial reason: no operation to start with
- operation: "choose an arbitrary point"
- if adding an arbitrary point is allowed, then all points become constructible
- problem acknowledged

Arbitrary points

- problem: given a circle, construct its center
- unsolvable according to this definition
- for a trivial reason: no operation to start with
- operation: “choose an arbitrary point”
- if adding an arbitrary point is allowed, then all points become constructible
- problem acknowledged
- “points for which we do not have affine or metric information” (Bieberbach)

Arbitrary points

- problem: given a circle, construct its center
- unsolvable according to this definition
- for a trivial reason: no operation to start with
- operation: "choose an arbitrary point"
- if adding an arbitrary point is allowed, then all points become constructible
- problem acknowledged
- "points for which we do not have affine or metric information" (Bieberbach)
- "What we do not include in our analysis are arbitrary elements that are used in some constructions. Their role is not so simple, however, as it is sometimes thought" (Tietze)

Game definition

Game definition

- Alice and Bob play a game with full information

Game definition

- Alice and Bob play a game with full information
- initial configuration: the set of given points/lines/circles

Game definition

- Alice and Bob play a game with full information
- initial configuration: the set of given points/lines/circles
- at each step Alice sees everything and may ask Bob to add
  - a line that goes through two given points;
  - a circle with center O and radius AB (if O, A, B are already in the configuration);
  - an intersection points of two objects in the configuration;
  - an arbitrary point in a given open set

Game definition

- Alice and Bob play a game with full information
- initial configuration: the set of given points/lines/circles
- at each step Alice sees everything and may ask Bob to add
  - a line that goes through two given points;
  - a circle with center O and radius AB (if O, A, B are already in the configuration);
  - an intersection points of two objects in the configuration;
  - an arbitrary point in a given open set [the only operation where Bob has a choice]

Game definition

- Alice and Bob play a game with full information
- initial configuration: the set of given points/lines/circles
- at each step Alice sees everything and may ask Bob to add
  - a line that goes through two given points;
  - a circle with center O and radius AB (if O, A, B are already in the configuration);
  - an intersection points of two objects in the configuration;
  - an arbitrary point in a given open set [the only operation where Bob has a choice]
- Alice wins when (if) the desired object appear in the current configuration

Game definition

- Alice and Bob play a game with full information
- initial configuration: the set of given points/lines/circles
- at each step Alice sees everything and may ask Bob to add
  - a line that goes through two given points;
  - a circle with center O and radius AB (if O, A, B are already in the configuration);
  - an intersection points of two objects in the configuration;
  - an arbitrary point in a given open set [the only operation where Bob has a choice]
- Alice wins when (if) the desired object appear in the current configuration
- definition: *a problem (for a given input configuration) is solvable by ruler and compass = Alice has a winning strategy in the corresponding game*

Negative definition (Akopyan, Fedorov, 2017)

Negative definition (Akopyan, Fedorov, 2017)

- The construction problem (with given input configuration) is *unsolvable*

Negative definition (Akopyan, Fedorov, 2017)

- The construction problem (with given input configuration) is *unsolvable* if there exists a set $\mathcal{L}$ of geometric objects such that:
  - $\mathcal{L}$ contains all the input objects;
  - $\mathcal{L}$ is closed under three basic operations (line through given points, circle with given center and radius, intersection points)
  - the set of points in $\mathcal{L}$ is a dense subset of the plane
  - the desired output object is *not* in $\mathcal{L}$.

Negative definition (Akopyan, Fedorov, 2017)

- The construction problem (with given input configuration) is *unsolvable* if there exists a set $\mathcal{L}$ of geometric objects such that:
  - $\mathcal{L}$ contains all the input objects;
  - $\mathcal{L}$ is closed under three basic operations (line through given points, circle with given center and radius, intersection points)
  - the set of points in $\mathcal{L}$ is a dense subset of the plane
  - the desired output object is *not* in $\mathcal{L}$.
- equivalent definition

Negative definition (Akopyan, Fedorov, 2017)

- The construction problem (with given input configuration) is *unsolvable* if there exists a set $\mathcal{L}$ of geometric objects such that:
  - $\mathcal{L}$ contains all the input objects;
  - $\mathcal{L}$ is closed under three basic operations (line through given points, circle with given center and radius, intersection points)
  - the set of points in $\mathcal{L}$ is a dense subset of the plane
  - the desired output object is *not* in $\mathcal{L}$.
- equivalent definition
- one direction obvious: if such a set $\mathcal{L}$ exists, then Bob may use only objects from $\mathcal{L}$ and prevent Alice from winning

Negative definition (Akopyan, Fedorov, 2017)

- The construction problem (with given input configuration) is *unsolvable* if there exists a set $\mathcal{L}$ of geometric objects such that:
  - $\mathcal{L}$ contains all the input objects;
  - $\mathcal{L}$ is closed under three basic operations (line through given points, circle with given center and radius, intersection points)
  - the set of points in $\mathcal{L}$ is a dense subset of the plane
  - the desired output object is *not* in $\mathcal{L}$.
- equivalent definition
- one direction obvious: if such a set $\mathcal{L}$ exists, then Bob may use only objects from $\mathcal{L}$ and prevent Alice from winning
- another direction: essentially proven by Akopyan and Fedorov (though they avoid using formal definitions!)

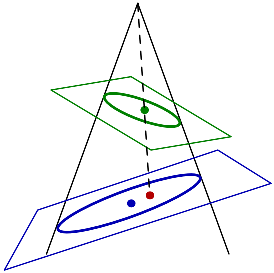
The importance of being formal

The importance of being formal

- classical result: one cannot construct the center of a given circle using only a ruler

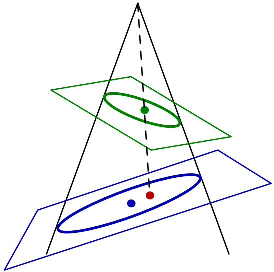
The importance of being formal

- classical result: one cannot construct the center of a given circle using only a ruler
- proof: construction on the green plane is mapped to a construction on the blue plane and produces a red point, not a true center (blue)



The importance of being formal

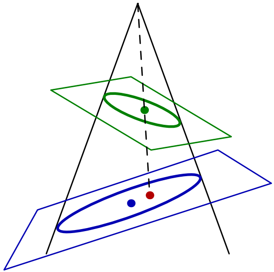
- classical result: one cannot construct the center of a given circle using only a ruler
- proof: construction on the green plane is mapped to a construction on the blue plane and produces a red point, not a true center (blue)



- attributed to David Hilbert by Cauer who proved a stronger "theorem" saying the one cannot construct centres of two disjoint circles

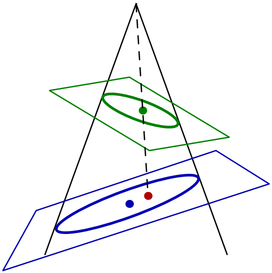
The importance of being formal

- classical result: one cannot construct the center of a given circle using only a ruler
- proof: construction on the green plane is mapped to a construction on the blue plane and produces a red point, not a true center (blue)



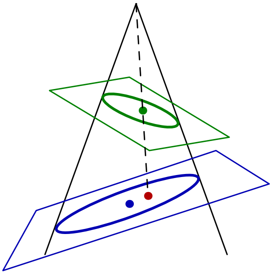
- attributed to David Hilbert by Cauer who proved a stronger "theorem" saying the one cannot construct centres of two disjoint circles ERROR!

What is the problem?



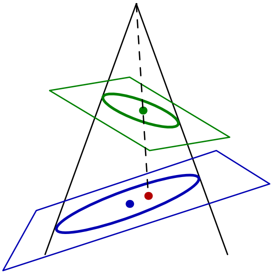
What is the problem?

- should the construction specify explicitly the answer point (or it is enough that this point is among the points constructed)?

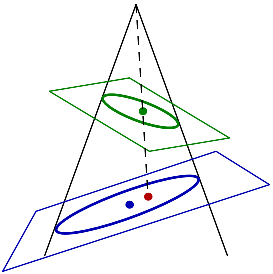


What is the problem?

- should the construction specify explicitly the answer point (or it is enough that this point is among the points constructed)?
- if not, no contradiction (blue point is an image of another green point)

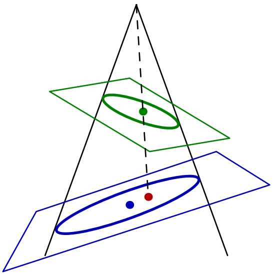


What is the problem?



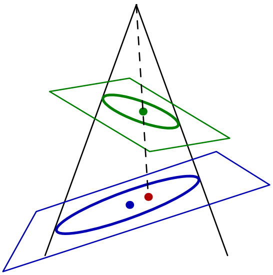
- should the construction specify explicitly the answer point (or it is enough that this point is among the points constructed)?
- if not, no contradiction (blue point is an image of another green point)
- if yes, tests should be allowed (otherwise we cannot construct the incenter of a given triangle without excenters), and the blue tests give not the same results as the green ones

What is the problem?



- should the construction specify explicitly the answer point (or it is enough that this point is among the points constructed)?
- if not, no contradiction (blue point is an image of another green point)
- if yes, tests should be allowed (otherwise we cannot construct the incenter of a given triangle without excenters), and the blue tests give not the same results as the green ones
- proof does not work, but the statement is true (for the game definition); it is important that many projective transformations exist

What is the problem?



- should the construction specify explicitly the answer point (or it is enough that this point is among the points constructed)?
- if not, no contradiction (blue point is an image of another green point)
- if yes, tests should be allowed (otherwise we cannot construct the incenter of a given triangle without excenters), and the blue tests give not the same results as the green ones
- proof does not work, but the statement is true (for the game definition); it is important that many projective transformations exist
- Cauer's "theorem" is plainly false (Akopyan–Fedorov): for some pairs of disjoint circles one *can* construct the center (but for other pairs one cannot)

Conlusions

Conlusions

- formal definitions precede negative results?

Conlusions

- formal definitions precede negative results?
- not the case for geometric constructions

Conlusions

- formal definitions precede negative results?
- not the case for geometric constructions
- not a safe practice

Conlusions

- formal definitions precede negative results?
- not the case for geometric constructions
- not a safe practice

Thanks!