Time-bounded Kolmogorov complexity provides an obstacle to soficness of multidimensional shifts

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- **Def** (A **shift** X in dimension 1) :
  - An alphabet Σ
  - A set F of forbidden patterns
  - X : the set of all configurations in $\Sigma^{\mathbb{Z}}$ without any pattern from F

- **Def** : An **admissible pattern** of a shift X is a pattern that appears somewhere in a configuration of X

- **Def** (A shift X in dimension 1) :
  - An alphabet Σ
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  - X : the set of all configurations in $\Sigma^{\mathbb{Z}}$ without any pattern from F

- **Type 1** : SFT, can be defined by a finite set F

Ex. SFT : alternated black and white squares

$\Sigma = \{\square, \blacksquare\}$, $F = \{\square\square, \blacksquare\blacksquare\}$

$X = \{\ldots \square\blacksquare\square\blacksquare\square \ldots, \ldots \blacksquare\square\blacksquare\square\blacksquare \ldots\}$

- **Def** (A shift X in dimension 1) :
  - An alphabet Σ
  - A set F of forbidden patterns
  - X : the set of all configurations in $\Sigma^{\mathbb{Z}}$ without any pattern from F

- **Type 2** : sofic shift, can be defined by a regular language F

Ex. sofic shift : even number of white squares

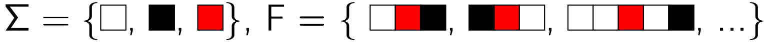
$\Sigma = \{□, ■\}$, $F = \{$ ■□ (□□)* ■ $\}$

X={...,···□■$\underbrace{\text{□□□···□}}_{\text{nombre pair}}$■□···,...}

- **Def** (A shift X in dimension 1) :
  - An alphabet Σ
  - A set F of forbidden patterns
  - X : the set of all configurations in $\Sigma^{\mathbb{Z}}$ without any pattern from F

- **Type 3** : **effective** shift, can be defined by a computable set F

Ex. effective shift : the mirror shift



- **Def** (A shift X in dimension 1) :
  - An alphabet Σ
  - A set F of forbidden patterns
  - X : the set of all configurations in $\Sigma^{\mathbb{Z}}$ without any pattern from F

- **Type 2 (equivalent definition)** : sofic shift.

  There exists :
  - An SFT $Y \subseteq \Delta^{\mathbb{Z}}$
  - A mapping $\phi : \Delta \to \Sigma$
  - $X = \phi(Y)$

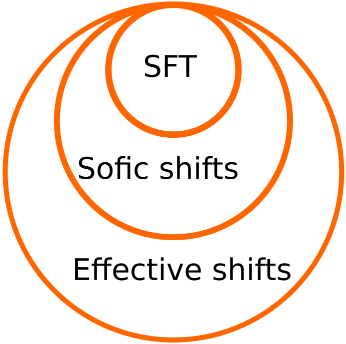
Ex. sofic shift: Y={..., ···b■aba···b■a···,...}

nombre pair

$\Sigma = \{\square, \blacksquare\}$ X={..., ···□■□□□···□■□···,...}

- How to distinguish between sofic and effective shifts ?

  - Pumping lemma
  - Particular properties of sofic shift (entropy)



- Properties of SFT and sofic shifts are quite well understood in dimension 1

- Can we generalize those definitions in higher dimensions ?

  - Straightforward for SFT and effective shifts
  - More difficult for sofic shifts

- **Def** (A **shift** X in dimension $d > 1$, $d = 2$) :
  - An alphabet Σ
  - A set F of forbidden patterns
  - X : the set of all configurations in $\Sigma^{\mathbb{Z}^d}$ without any pattern from F

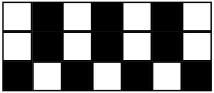
- **Def** (A shift X in dimension $d > 1$, $d = 2$) :
  - An alphabet Σ
  - A set F of forbidden patterns
  - X : the set of all configurations in $\Sigma^{\mathbb{Z}^d}$ without any pattern from F

- **Type 1** : **SFT**, can be defined by a finite set F

Ex. SFT : horizontally alternated black and white squares

$\Sigma = \{$□, ■$\}$, F $=\{$ □□, ■■ $\}$

...

X={..., ...  ... ,...}

...

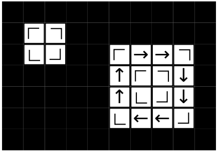
- **Def** (A shift X in dimension $d > 1$, $d = 2$) :
  - An alphabet Σ
  - A set F of forbidden patterns
  - X : the set of all configurations in $\Sigma^{\mathbb{Z}^d}$ without any pattern from F

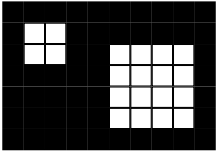
- **Type 2** : **sofic** shift. There exists :
  - an SFT $Y \subseteq \Delta^{\mathbb{Z}^d}$
  - A mapping $\phi : \Delta \to \Sigma$
  - $X = \phi(Y)$

Ex. sofic shift :

white squares

$\Sigma = \{\square, \blacksquare\}$

Y = 

X= 

- **Def** (A shift X in dimension $d > 1$, $d = 2$) :
  - An alphabet Σ
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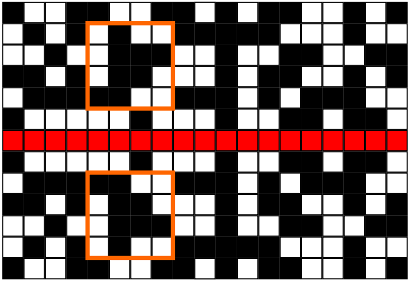
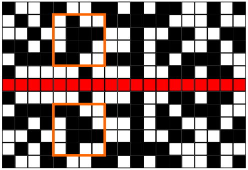
  - X : the set of all configurations in $\Sigma^{\mathbb{Z}^d}$ without any pattern from F
- **Type 3** : **effective** shift, can be defined by a computable set F

Ex. effective shift : the mirror shift

$\Sigma = \{\square, \blacksquare, \textcolor{red}{\blacksquare}\}$



Ex. effective shift:
the mirror shift
$\Sigma = \{\square, \blacksquare, \textcolor{red}{\blacksquare}\}$



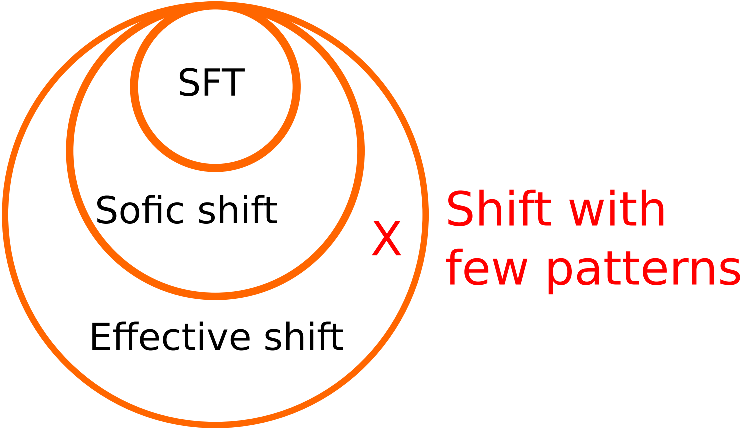
Claim

This shift is not sofic

An intuition behind the usual proof : Quadratic information (description of $n \times n$-patterns) must traverse the linear boundary (orange line around patterns)

We need a lot of $n \times n$-patterns ($\gg 2^n$) to prove non-soficness

- Can we construct an effective non-sofic shift with few patterns ?



Theorem 1[implicitly in Durand, Levin & Shen 2008]
For every sofic shift X, for every n, there exists an admissible $n \times n$- pattern P such that a program of size $O(n)$ prints P

Intuitively :

- The size of the program measures the information in P
- If every pattern contains too much information, the shift is not sofic

Formal statement involves Kolmogorov complexity

Theorem 1[implicitly in Durand, Levin & Shen 2008]

For every sofic shift X, for every n, there exists an admissible $n \times n$- pattern P such that a program of size $O(n)$ prints P

Theorem 2 [Rumyantsev & Ushakov, 2010]

There exists an effective shift such that for all admissible $n \times n$-patterns P, every program which print P have size $\Omega(n^2)$.

Corollary

Shift from Theorem 2 is effective and non-sofic

Theorem 1'

For every sofic shift X, for every n, there exists an admissible $n \times n$- pattern P such that a program of size $O(n)$ prints P in time $2^{O(n^2)}$.

Intuitively :

- P can be computed fast from a short description
- If no pattern can be computed fast from a short description, the shift is not sofic

Formal statement involves Time-bounded Kolmogorov complexity

Theorem 2' (main result)

There exists an effective shift such that :

(i) For all admissible $n \times n$-patterns P, every program which print P in time $2^{O(n^2)}$ have size $\Omega(n^{1.5})$.

(ii) For all admissible $n \times n$-patterns P, there exists a program of size $O(\log n)$ which prints P (very slowly)

Theorem 1'

For every sofic shift X, for every n, there exists an admissible $n \times n$- pattern P such that a program of size $O(n)$ prints P in time $2^{O(n^2)}$.

Theorem 2' (main result)

There exists an effective shift such that :

(i) For all admissible $n \times n$-patterns P, the programs which print P in time $2^{O(n^2)}$ have size $\Omega(n^{1.5})$.

(ii) ...

Fact 1

Shift from Theorem 2' is effective and non-sofic.

Theorem 2' (main result)

There exists an effective shift such that :

(i) ...

(ii) For all admissible $n \times n$-patterns P, there exists a program of size $O(\log n)$ which prints P (very slowly)

Fact 1

Shift from Theorem 2' is effective and non-sofic.

Fact 2

Shift from Theorem 2' has $\mathrm{poly}(n)$ $n \times n$-patterns

Corollary

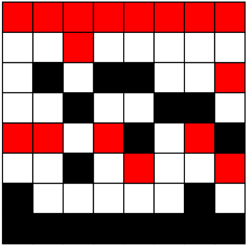
There exists an effective non-sofic shift with only poly(n) admissible $n \times n$-patterns

This is the non-technical version of our main result.

- A more general technique to prove non-soficness : see proceedings.
- We translate in the language of Kolmogorov complexity the proofs for most previously known examples and suggest several new ones.

Ex. The shift from Kaas & Madden :

$\Sigma = \{\square, \blacksquare, \blacksquare\}$



A forbidden pattern

Key points of the talk

- Unexpected use of time-bounded Kolmogorov complexity
- A new tool to prove the non-soficness of a shift
- Examples of effective non-sofic shifts, with few patterns

Thanks for your attention