Embedding computations in tilings
(Part 1: fixed point tilings)

Andrei Romashchenko

31 May 2016

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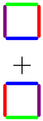

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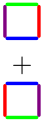
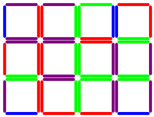


Example. A finite pattern from a valid tiling:



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Theorem. There exists a tile set τ such that
(i) τ-tilings exist, and
(ii) all τ-tilings are aperiodic.

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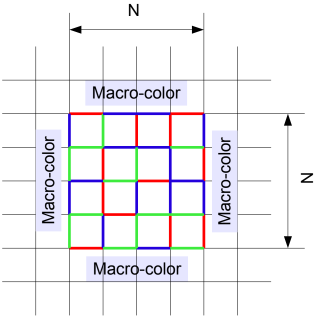
- observe that *every* self-similar tile set is aperiodic
- construct *some* self-similar tile set

Macro-tile:



an $N \times N$ square made of matching τ-tiles

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Definition 2. A tile set ρ is **simulated** by τ: there exists a family of τ-macro-tiles R such that

- R is *isomorphic* to ρ, and
- every τ-tiling can be *uniquely* split by an $N \times N$ grid into macro-tiles from R.

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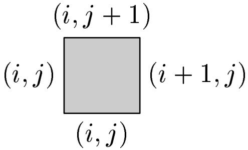

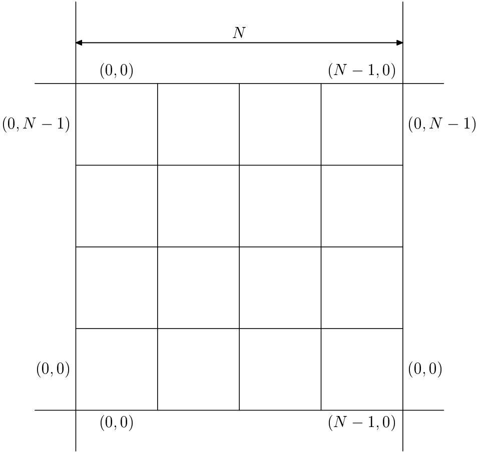

N
$(0,0)$
$(N-1,0)$
$(0,N-1)$
$(0,N-1)$
$(0,0)$
$(0,0)$
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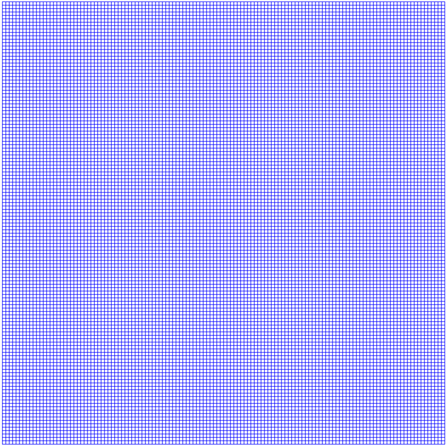

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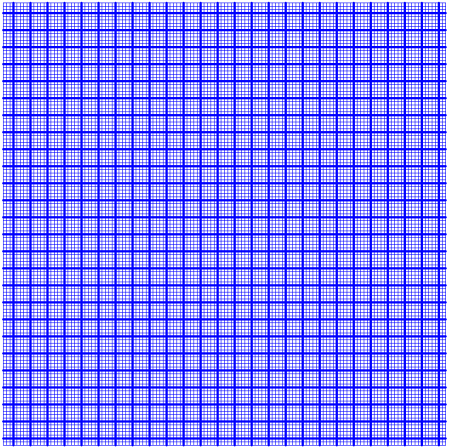

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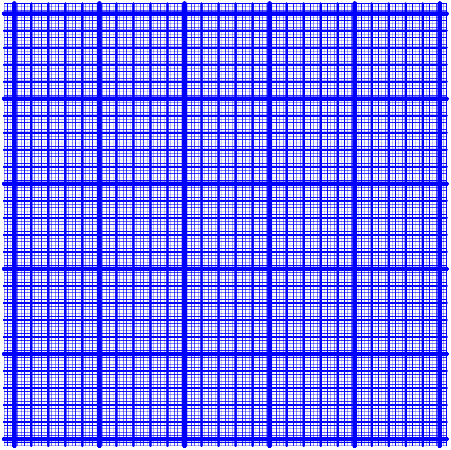

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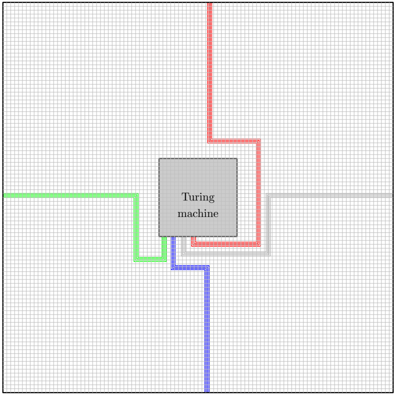
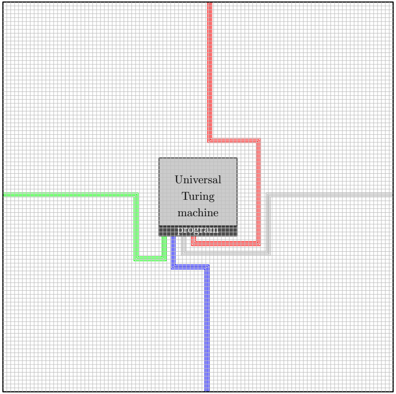
$\Downarrow$

a TM that accepts only 4-tuples of colors for the ρ-tiles

The scheme of implementation:



A more generic construction:
universal TM + program



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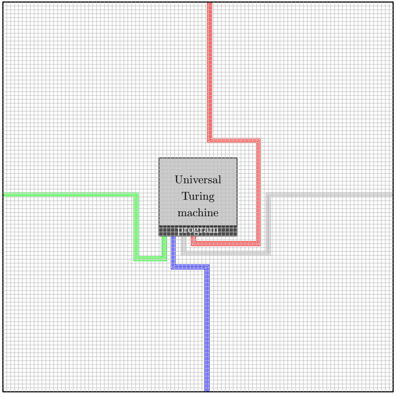
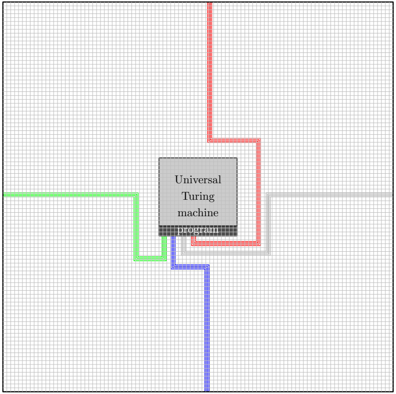


A fixed point: simulating tile set = simulated tile set

How to get **aperiodicity** + **quasiperiodicity** ?



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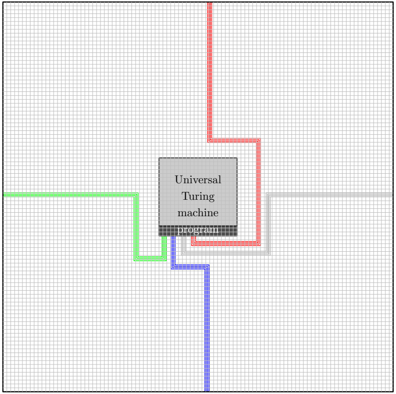
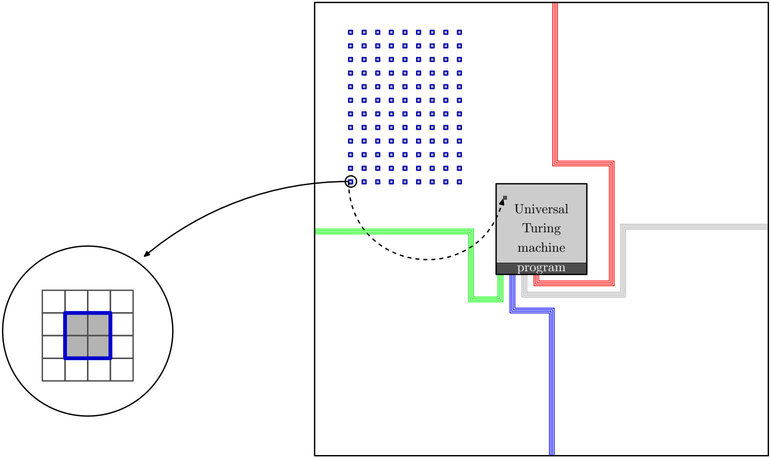


The problematic part is the computation zone...

Duplicate all 2×2 patterns that *may* appear in the computation zone!



A slot for a 2×2 patterns from the computation zone: