Embedding computations in tilings (Part 2)

Andrei Romashchenko

the 2nd June 2016

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Set of Wang tiles: a set $\tau \subset C^4$

Tiling: a mapping $f : \mathbb{Z}^2 \to \tau$ that **respects the matching rules**

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(ii) all τ-tilings are aperiodic.

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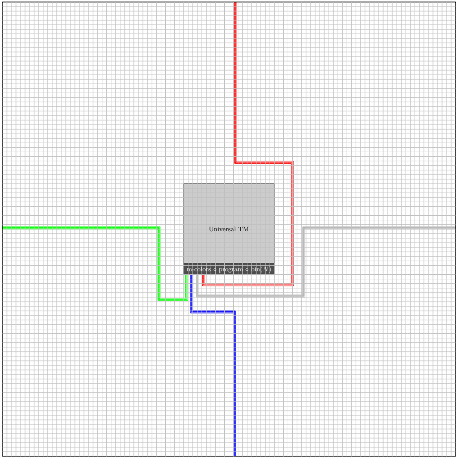
Funny, but... WHY???

A tile set that simulates itself:



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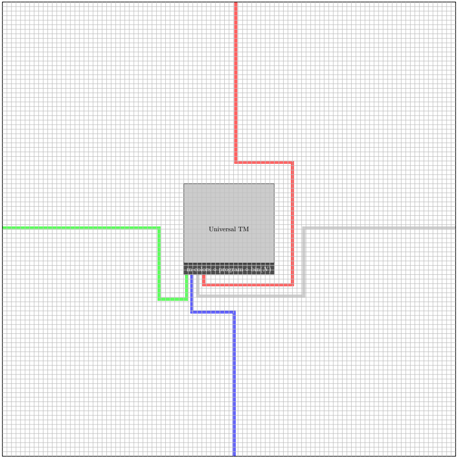

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- k = #[bits in a macro-color]
- m = [size of the computational zone]

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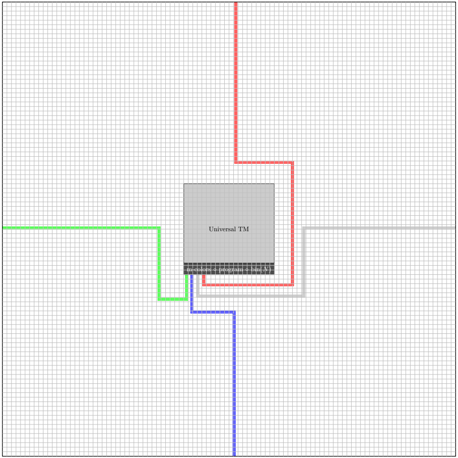

Parameters:

- N = zoom factor
- $k = \#$[bits in a macro-color] $:= 2\log N + O(1)$
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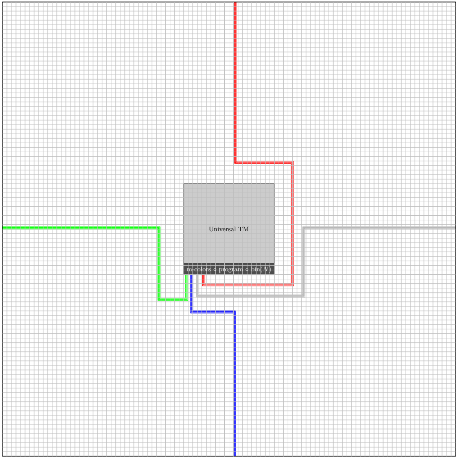

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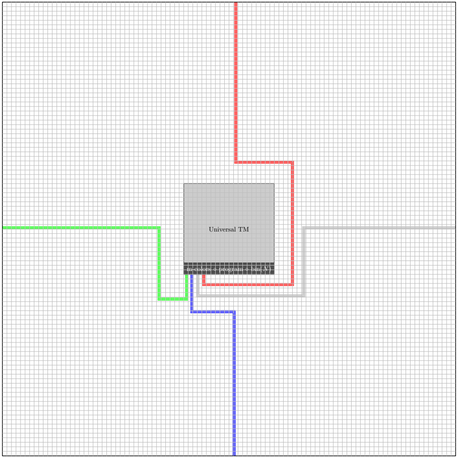
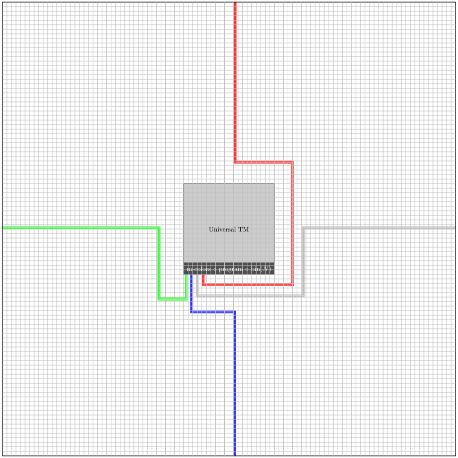
A tile set τ_N that simulates itself:



Parameters:

- N = zoom factor (works for all large enough N)
- $k = \#$[bits in a macro-color] $:= 2 \log N + O(1)$
- $m =$ [size of the computational zone]$:= \mathrm{poly}(\log N)$

A tile set τ_N that simulates itself with **variable** zoom :



- level 1 (macro-tiles): zoom=N,
- level 2 (macro-maro-tiles): zoom=$N+1$,
- level 3 (macro-maro-macro-tiles): zoom=$N+2$,
- . . .

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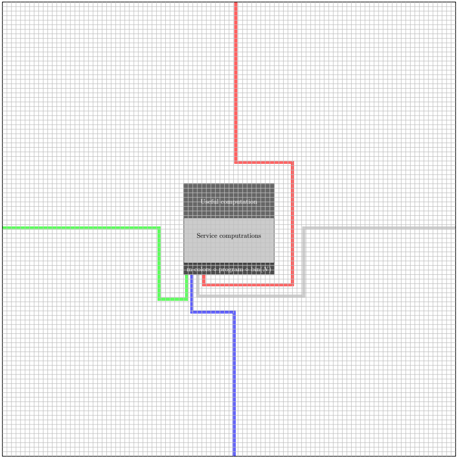

Useful computation = simulating machine π on available space and time

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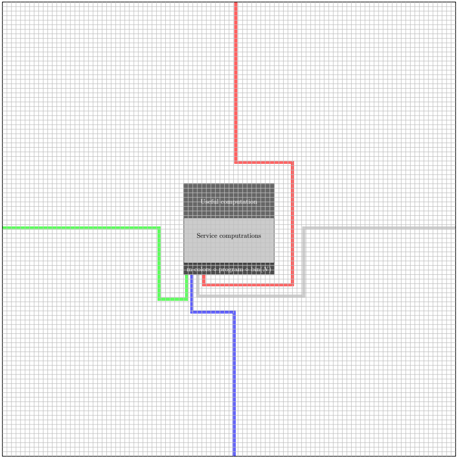

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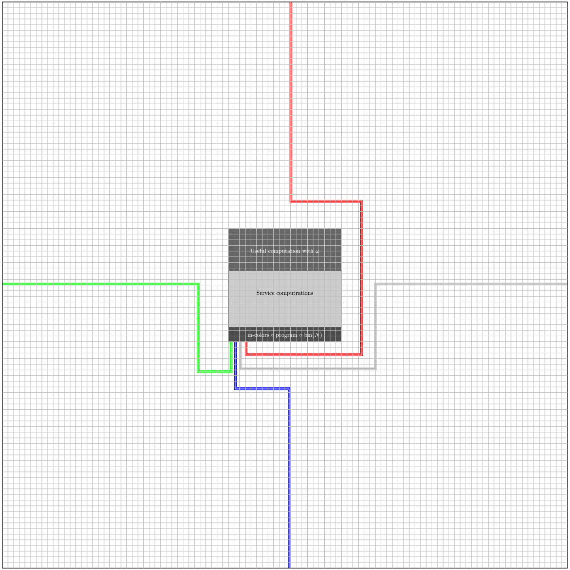
Theorem [Berger 66]. The tiling problem is undecidable (given a tile set we cannot decide algorithmically whether it can tile the plane).

a sequence embedded in a tiling:



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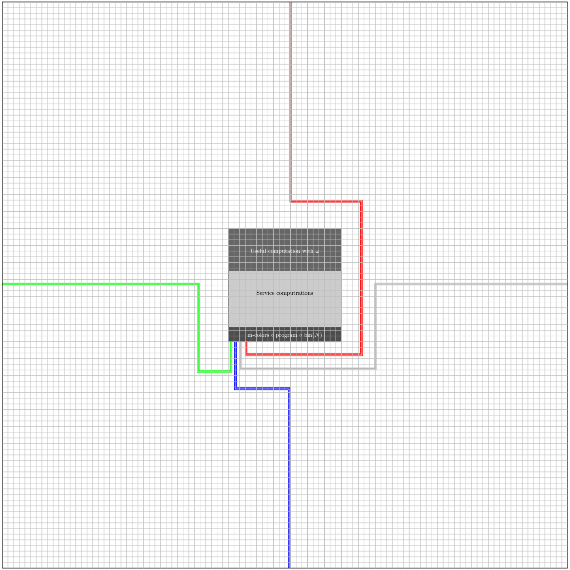

$\omega = \omega_0\omega_1 \dots \omega_n \dots$

N-macro-colors include the prefix $\omega_{[0:\log N]}$

Definition. $\omega = \omega_0\omega_1 \ldots \omega_n \ldots$ is a separator if

- $\omega_n = 0$ for every n s.t. the n-th Turing machine(n) $= 0$,
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Proof:

- embed an ω in our tiling
- **useful computation**: simulate in parallel n-th TM(n) and check that the embedded ω is a separator
- every (infinite) tiling must include an **incomputable** ω